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Delay-Dependent Closed-Loop Stability of Linear Systems with Input Delay: An LMI Approach

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Abstract

This paper focuses on the problem of computing a control law which maximizes (in a *sub-optimal* sense) the delay of the closed-loop system for a class of linear systems with delayed input. The delay is assumed to be a continuous bounded time-varying function. The analysis is given using two different approaches: a Razumikhin based method and a frequency-filtering based method.

Keywords: Delay input; Closed-loop stability; Razumikhin approach; Filtering techniques.

1 Introduction

The control of dynamical systems including delayed states or delayed inputs is a problem of practical and theoretical interest since the existence of a delay in the closed-loop system may induce instability or bad performances (see, e.g. Malek-Zavarei and Jamshidi, [14]). An interesting technique for stabilizing such systems has been proposed in [1] by transforming the delay system into a linear finite-dimensional one using an appropriate *infinite-dimensional* controller. Other remarks and generalizations of this results could be found in [6] and [19].

The problem of the *existence* of *finite-dimensional controllers* for stabilizing classes of time-delay linear systems has been considered in [10, 13, 17]. Thus, Kamen *et al.* [10] prove that a stabilizable time-delay systems can always be stabilized by a finite-dimensional controller. Their technique based on the interpretation of the considered delay system as a linear system over an associated polynomial ring is better adapted to *delay-independent* closed-loop stability, which is relatively restrictive. A more general class of time-delay systems including the neutral one have been considered in Logemann [13]. The proposed conditions, which are necessary and sufficient, are based on a frequency-domain technique. Notice that all these results are relatively difficult to be checked for numerical example. Delay-independent closed-loop stability conditions have been also derived in [5] (a linear matrix inequality) or in [20] (a Riccati equation). A classification of memoryless controllers has been introduced in [17], function on the dependence or not of the closed-loop stability on the delay size and for both cases simple conditions expressed in terms of some appropriate properties of matrix pencils or linear matrix inequalities (LMI) have been derived.

In this paper, we consider a class of linear systems with delayed input including a continuous time-varying, but bounded delay. *Sufficient* conditions for *delay-dependent* closed-loop asymptotic stability are given in terms of some appropriate linear matrix inequalities (LMI). Furthermore, we propose a state feedback controller which maximizes the delay bound of the closed-loop system, by transforming the stability problem into a convex optimization one. The corresponding algorithm is of the type "convex / quasi-convex" similar to the one proposed in [15] or [16]. The approaches adopted here make use of the Lyapunov-Razumikhin function technique (the time-varying delay case) or of some frequency-filtering techniques (if the delay is constant). A good introduction to the LMI techniques could be found in Boyd *et al.* [2] and for Razumikhin techniques see, for instance, Hale and Lunel [8]. To the best authors' knowledge there does not exist in the literature any result concerning the construction of state feedback which maximizes the delay bound of the closed-loop system. Furthermore, all the results developed here could be extended to systems including uncertainty or several delayed inputs (see also the results proposed in [11, 12]).

The paper is organized as follows: the problem statement is presented in Section 2. The main results via the Razumikhin technique and via some filtering techniques are proposed in Section 3 and respectively in Section 4. Some concluding remarks end the paper.

2 Problem Statement

Consider the following delay system:

$$\dot{x}(t) \;=\; Ax(t) + Bu(t - \tau(t)), \qquad (1)$$

with an appropriate initial condition $(x(t_0), u(\cdot))$,

$$x(t_0) \in \mathbf{R}^n, \qquad u(\theta) = \phi(\theta) \quad \forall\, \theta \in \mathcal{E}_{t_0,\tau}, \qquad (2)$$

where $\phi : \mathcal{E}_{t_0,\tau} \mapsto \mathbf{R}^n$ is a continuous norm-bounded initial function (see also [4]) and

$$\mathcal{E}_{t_0,\tau} = \{t \in \mathbf{R} \quad : \quad t = \eta - \tau(\eta) \le t_0, \eta \ge t_0\},$$

with $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the input and $\tau(t) > 0$ is a continuous time-varying but bounded (i.e. there exists a $\bar{\tau}$ such that $\tau(t) < \bar{\tau}$) delay function. A and B are constant matrices of appropriate dimension.

We have the following *assumption*:

Assumption 1 *The pair (A, B) is stabilizable.*

Notice that this *Assumption* guarantees the existence of a controller

$$u(t) = Kx(t), \quad K \in \mathbf{R}^{m\times n}$$

such that the closed-loop system (1) free of delays is asymptotically stable.

Using a Datko [3] type argument, it follows that, in this case, the same controller guarantees the closed-loop stability for "sufficiently" small delays τ (see also [17]).

Furthermore, if A is an *unstable* matrix but satisfying *Assumption 1*, the the closed-loop system is always *delay-dependent* asymptotically stable. Indeed, since the Hurwitz stability of the matrix A is a *necessary* condition for *delay-independent* closed-loop stability (see Hale *et al.* [9]), the property follows (see, for instance, Niculescu [17]).

The problem that we consider in this paper consists in

Finding a finite-dimensional controller of the form

$$u(t) = Kx(t), \quad K \in \mathbf{R}^{m\times n}$$

which maximizes the delay bound of the closed-loop system.

We adopt two different approaches:

- the first one is based on a Razumikhin method and makes use of an appropriate Lyapunov-Razumikhin function;
- the second one is a frequency-filtering based method and makes use of an appropriate filter design for the computation of the delay bound.

Notice that if the Razumikhin approach allows a time-varying delay, the proposed frequency-filtering method needs a constant delay. Furthermore, the results developed in this note are *suboptimal*, but can be easily applied for numerical examples.

3 Razumikhin-based Approach

We have the following result:

Theorem 1 *Consider the system (1)-(2) satisfying Assumption 1. If there exist a symmetric and positive definite matrix $Q \in \mathbf{R}^{n\times n}$, a matrix $W \in \mathbf{R}^{m\times n}$ and the scalars β_1 and β_2 such that the following LMIs hold:*

$$\begin{bmatrix} \begin{pmatrix} \frac{1}{\tau^*}[QA^T + AQ+ \\ +BW + W^TB^T] \\ +(\beta_1+\beta_2)Q \end{pmatrix} & BW \\ W^TB^T & -\frac{1}{2}Q \end{bmatrix} < 0, \qquad (3)$$

$$-\beta_1 Q + AQA^T \le 0, \qquad (4)$$

$$\beta_2 \begin{bmatrix} -Q & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & BW \\ W^TB^T & -Q \end{bmatrix} \le 0, \qquad (5)$$

then the system (1)-(2) is closed-loop uniformly asymptotically stable via an input of the form

$$u(t) = Kx(t), \quad K \in \mathbf{R}^{m\times n},$$

for all the delays $\tau(t)$ satisfying

$$0 \le \tau(t) \le \tau^*.$$

Furthermore, the corresponding input is given by:

$$u(t) = WQ^{-1}x(t).$$

A sketch of the proof is given in *Appendix A*. The basic idea is to rewrite the closed-loop equation, which is a functional differential equation on a $\mathcal{E}_{t,\tau}$ delay set as a functional differential equation on a $\mathcal{E}_{t,2\tau}$ delay set and to use the Razumikhin method combined with some matrix inequalities for the "new" differential equation.

Next, notice that:

- for given W β_1 and β_2, the above optimization problem consists of minimizing a generalized eigenvalue problem which is a quasi-convex optimization problem (see [2] and the references therein);
- for a given $Q > 0$, the considered optimization problem consists of minimizing an eigenvalue problem which is a convex one.

Based on the above remarks, we propose the following "convex-quasiconvex" algorithm to find τ^* (see also [17] and the references therein):

Algorithm:
Initial Data: $Q_0 > 0$ and W_0 such that the following LMI holds[1]

$$Q_0A^T + AQ_0 + BW_0 + W_0^TB < 0, \tag{6}$$

Step 2: For $Q > 0$ given in the previous step, find β_{1s}, β_{2s} and W_s that solve the following convex optimization problem

$$\begin{cases} \max\limits_{W,\beta_1,\beta_2} \tau^*(W,\beta_1,\beta_2) & s.t. \\ (3)-(5) & \textit{hold for } Q > 0 \textit{ fixed.} \end{cases}$$

Step 3: For W, β_1 and β_2 given in the previous step, find $Q_s > 0$ that solve the following quasi-convex optimization problem

$$\begin{cases} \max\limits_{Q>0} \tau^*(Q) & s.t. \\ (3)-(5) & \textit{hold for } W,\ \beta_1 \textit{ and } \beta_2 \textit{ fixed.} \end{cases}$$

and return to step 2 until the convergence of τ^* is attained with a desired precision.

Remark 1 The *Assumption 1* guarantees the existence of two matrices $Q_0 = Q_0^T > 0, Q_0 \in \mathbf{R}^{n\times n}$ and $W_0 \in \mathbf{R}^{m\times n}$ satisfying the LMI (6).

Due to the form of (3)-(5), it follows that choosing β_{01}, β_{02} and τ_0^* such that:

$$\beta_{01} > \frac{\lambda_{max}\left(AQ_0A^T\right)}{\lambda_{min}(Q_0)},$$

$$\beta_{02} > \frac{\lambda_{max}\left(BW_0Q_0^{-1}W_0^TB^T\right)}{\lambda_{min}(Q_0)},$$

$$\frac{1}{\tau_0^*} > \frac{\lambda_{max}\left((\beta_{01}+\beta_{02})Q_0 + 2BW_0Q_0^{-1}W_0^TB^T\right)}{\lambda_{min}\left(-Q_0A^T - AQ_0 - BW_0 - W_0^TB\right)},$$

then the inequalities (3)-(5) are always satisfied.

[1] Choose β_{01}, β_{02} "great enough" such that (4) et (5) hold. The initial value of τ_0^* is chosen "small enough" such that the LMI (3) holds (see also Remark 1)

Due to the quasiconvexity (Step 3) and the convexity (Step 2) at each step of the algorithm, we have the following:

Proposition 1 *The above algorithm gives a suboptimal value of $\tau^*(Q^*,W^*,\beta_1^*,\beta_2^*)$ which guarantees that the closed-loop system (1)-(2) via the control law*

$$u(t) = W^*(Q^*)^{-1}x(t),$$

is uniformly asymptotically stable for all the continuous time-varying delays $\tau(t)$, satisfying

$$0 \le \tau(t) \le \tau^*(Q^*,W^*,\beta_1^*,\beta_2^*).$$

4 Frequency-filtering Approach

Consider now the case when the delay is constant $\tau(t) = \tau$. In this case the set $\mathcal{E}_{t,\tau}$ becomes the interval $[t-\tau, t]$. Let

$$f(s) = c_f(sI - a_f)^{-1}b_f + d_f \tag{7}$$

be any asymptotically stable scalar rational function with the following property:

$$|f(j\nu)| \ge |\frac{\sin(\nu)}{\nu}|, \qquad \forall\nu \in \mathbf{R}. \tag{8}$$

Note that $f(s)$ is in fact some kinds of filters which can be designed *a priori*. We can select higher order filters, therefore, a_f is not necessary a scalar.

Denote $A_f \in \mathbf{R}^{n_f\times n_f}$, $B_f \in \mathbf{R}^{n_f\times n}$, $C_f \in \mathbf{R}^{n\times n_f}$ and $D_f \in \mathbf{R}^{n\times n}$ diagonally stacked-up matrices of a_f, b_f, c_f and d_f respectively, i.e.,

$$\begin{aligned} F\left(\frac{s\tau}{2}\right) &= f\left(\frac{s\tau}{2}\right)I_n \\ &= \frac{2}{\tau}C_f\left(sI - \frac{2}{\tau}A_f\right)^{-1}B_f + D_f. \end{aligned} \tag{9}$$

Then we have the following theorem:

Theorem 2 *Consider the system (1)-(2) satisfying Assumption 1. If there exist two positive definite matrix $P_f \in \mathbf{R}^{n_f\times n_f}$ and $Q \in \mathbf{R}^{n\times n}$, a matrix $W \in \mathbf{R}^{m\times n}$ and two positive scalars β_1 and β_2 such that the following matrix inequality holds:*

$$\begin{bmatrix} \mathcal{P}_0(\tau^*) & \mathcal{P}_1(\beta_1) & \mathcal{P}_1(\beta_2) \\ \mathcal{P}_1(\beta_1)^T & -\frac{\beta_1}{\tau^*}Q & 0 \\ \mathcal{P}_1(\beta_2)^T & 0 & -\frac{\beta_2}{\tau^*}Q \end{bmatrix} < 0, \tag{10}$$

where

$$\mathcal{P}_0(\tau^*) = \begin{bmatrix} \frac{2}{\tau^*}(A_f^T P_f + P_f A_f) & \frac{2}{\tau^*} C_f^T W^T B^T \\ \frac{2}{\tau^*} BWC_f & \begin{pmatrix} QA^T + AQ \\ +BW + W^T B^T \end{pmatrix} \end{bmatrix},$$

$$\mathcal{P}_1(\beta) = \begin{bmatrix} P_f B_f \\ BWD_f + \beta QA^T \end{bmatrix},$$

then the system (1)-(2) is closed-loop asymptotically stable via an input of the form

$$u(t) = Kx(t), \quad K \in \mathbf{R}^{m\times n}$$

for all the delays τ satisfying

$$0 \le \tau \le \tau^*.$$

Furthermore, the corresponding input is given by: $u(t) = WQ^{-1}x(t)$.

A sketch of the proof is given in *Appendix B*.

Remark 2 Note that if we choose a zero order filter, i.e., $f(s) = 1$, then Theorem 2 reduces to a form similar to Theorem 1. Generally, a higher order filter will increase the suboptimal upper bound for the time-delay with the cost of computational complexity. A second order filter is usually sufficient for many applications.

Remark 3 Due to the similarity of the Theorems 1 and 2, the computation of the τ^* delay bound and of the corresponding control input may be done, in this case, via the same "convex / quasi-convex" optimization algorithm.

5 Concluding Remarks

This paper is devoted to the closed-loop stability analysis for a class of linear systems including a delayed input. Sufficient *delay-dependent* conditions are derived via two approaches: a Razumikhin approach (time-varying delay) and a frequency-filtering approach (constant delay). Notice that all these results can be extended to uncertain systems as well as to multiple delays case.

A. Proof of Theorem 1

We have the following *Lemma*:

Lemma A.1 *There exists a symmetric and positive-definite matrix $P \in \mathbf{R}^{n\times n}$ and a matrix $K \in \mathbf{R}^{m\times n}$ such that:*

$$\begin{aligned}\mathcal{M} = {} & (A+BK)^T P + P(A+BK) + \\ & +\tau^* \left[\beta_1^{-1} PBKAP^{-1}A^T K^T B^T P + \right. \\ & +(\beta_1+\beta_2)P + \\ & \left. +\beta_2^{-1} P(BK)^2 P^{-1} (K^T B^T)^2 P\right] < 0 \quad (11)\end{aligned}$$

if there exist the matrices Q and W satisfying (3)-(5).

Consider the following time-delay system

$$\dot{\xi}(t) = (A+BK)\xi(t) - BK \times$$
$$\times \int_{-\tau(t)}^{0} (A\xi(t+\theta) + BK\xi(t-\tau(t)+\theta))\, d\theta. \quad (12)$$

with the initial condition

$$\xi(t_0+\theta) = \bar{\phi}(\theta), \ \forall\, \theta \in \mathcal{E}_{t_0,2\tau}, \quad (13)$$

where $\bar{\phi} : \mathcal{E}_{t_0,2\tau} \mapsto \mathbf{R}^n$ is a continuous norm-bounded initial function and

$$\mathcal{E}_{t_0,2\tau} = \{t \in \mathbf{R} \ : \ t = \eta - 2\tau(\eta) \le t_0, \ \ \eta \ge t_0\},$$

Connection between the stability properties of this system and the original one can be found in [17].

Introduce now the following Lyapunov function

$$V(\xi) = \xi^T P \xi,$$

with $P = P^T > 0$ and $P \in \mathbf{R}^{n\times n}$, it follows

$$\alpha_1 \parallel \xi \parallel^2 \le V(\xi) \le \alpha_2 \parallel \xi \parallel^2 \quad (14)$$

(where $\alpha_1 = \lambda_{min}(P)$), $\alpha_2 = \lambda_{max}(P)$).

The time-derivative of $V(\xi(t))$ along the solutions of (12)-(13) is given by:

$$\begin{aligned}\dot{V}(\xi(t)) = {} & \xi^T(t)\left[(A+BK)^T P + P(A+BK)\right]\xi(t) \\ & -2\int_{-\tau(t)}^{0} \left[\xi^T(t) PBKA\xi(t+\theta)\right. \\ & \left. +\xi^T(t) P(BK)^2 \xi(t-\tau+\theta)\right] d\theta \quad (15)\end{aligned}$$

Following the Razumikhin Theorem (see Hale and Lunel [8]), we assume that for any $\delta > 1$, the following inequality holds

$$V(\xi(\eta)) < \delta V(\xi(t)), \quad t - 2\tau \le \eta \le t$$

then we have (after some simple algebraic manipulations):

$$\dot{V}(\xi(t)) \le -\xi^T(t)\mathcal{M}(\tau(t),\delta)\xi(t) \quad (16)$$

where $\mathcal{M}(\cdot,\cdot)$ is given by:

$$\begin{aligned}\mathcal{M}(\tau(t),\delta) = {} & (A+BK)^T P + P(A+BK) + \\ & +\tau(t)[\beta_1^{-1} PBKAP^{-1}A^T K^T B^T P + \\ & +(\beta_1+\beta_2)\delta P + \\ & +\beta_2^{-1} P(BK)^2 P^{-1}(K^T B^T)^2 P],\end{aligned}$$

for some positive scalars β_1 and β_2. Since

$$0 \le \tau(t) \le \tau^*,$$

we have from (16) that:

$$\dot{V}(\xi(t)) \le \xi^T(t)\mathcal{M}(\tau^*, \delta)\xi(t) \quad (17)$$

From *Lemma A.1*, it follows that if there exist $Q = Q^T > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and W satisfying the conditions (3)-(5), then

$$\mathcal{M}(\tau^*, \delta = 1) < 0. \quad (18)$$

Using the continuity property of the eigenvalues of $\mathcal{M}$ with respect to δ, then there exists a $\delta > 1$ sufficiently small such that (18) still holds. Thus, for a such δ, we have

$$\mathcal{M}(\tau^*, \delta) < 0, \quad (19)$$

and the conclusion follows, etc. ∇∇∇

B. Proof of Theorem 2

Let $u(t) = Kx(t)$, we have the following lemma:

Lemma B.1 (see also [7]) *The system (1)-(2) is asymptotically stable if $A + BK$ is stable and*

$$\begin{aligned} \mathcal{A}(j\omega, \tau) &= j\omega I - A - BK \\ &\quad -\tau\rho_1(j\omega\tau)BKA - \tau\rho_2(j\omega\tau)(BK)^2 \end{aligned} \quad (20)$$

is nonsingular for all $\omega \in \mathbf{R}$, where

$$\left\{ \begin{aligned} \rho_1(j\nu) &= -e^{-j\frac{\nu}{2}} \frac{\sin(\frac{\nu}{2})}{\frac{\nu}{2}} \\ \rho_2(j\nu) &= \rho_1(j\nu)e^{-j\nu}. \end{aligned} \right. \quad (21)$$

Generally, we can use (20) with loose bounds $|\rho_i(j\nu)| \le 1$, $i = 1, 2$ to derive delay-dependent results. To reduce the conservatism, we will overbound $\rho_i(t)$, $i = 1, 2$ with a better function in the sequel.

Take the controller gain matrix K as

$$K = WQ^{-1} \quad (22)$$

where W and Q are defined in Theorem 2. Then we can rewrite (20) as

$$\begin{aligned} \mathcal{A}(j\omega, \tau) &= j\omega I - A - BWQ^{-1} \\ -BW &\left\{ \frac{\sin \omega/2}{\omega/2}, \ldots, \frac{\sin \omega/2}{\omega/2} \right\} \\ \times &(e^{-j\omega\tau/2}\tau Q^{-1}A + e^{-j3\omega\tau/2}\tau Q^{-1}BWQ^{-1}). \end{aligned} \quad (23)$$

Consider

$$\begin{aligned} \bar{\mathcal{A}}(j\omega, \tau) &= j\omega I - A - BWQ^{-1} \\ &\quad -BWF\left(\frac{s\tau}{2}\right)(\delta_1(j\omega)\tau Q^{-1}A + \\ &\quad +\delta_2(j\omega)\tau Q^{-1}BWQ^{-1}). \end{aligned} \quad (24)$$

Obviously, $\mathcal{A}(j\omega, \tau)$ is nonsingular if $\bar{\mathcal{A}}(j\omega, \tau)$ is nonsingular for all $|\delta_i(j\omega)| \le 1, i = 1, 2$. In the sequel, we will consider $\bar{\mathcal{A}}(j\omega, \tau)$ instead.

A state-space realization for $\bar{\mathcal{A}}(j\omega, \tau)$ is given by the following augmented uncertain system:

$$\begin{aligned} \dot{\bar{x}}(t) &= \begin{bmatrix} \frac{\tau}{2}A_f & 0 \\ \frac{\tau}{2}BWC_f & A + BWQ^{-1} \end{bmatrix} \bar{x}(t) \\ &\quad + \begin{bmatrix} B_f \\ BWD_f \end{bmatrix} \xi(t) \end{aligned} \quad (25)$$

$$\begin{aligned} z(t) &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & Q^{-1}A \\ 0 & Q^{-1}BWQ^{-1} \end{bmatrix} \bar{x}(t) \end{aligned} \quad (26)$$

$$\xi = \Delta_1(z_1) + \Delta_2(z_2) \quad (27)$$

where $\Delta_1(\cdot)$ and $\Delta_2(\cdot)$ are causal and stable linear operators satisfying the following condition:

$$\begin{aligned} \xi(j\omega) &= \delta_1(j\omega)z_1(j\omega) + \delta_2(j\omega)z_2(j\omega), \\ &\quad |\delta_i(j\omega)| \le 1, \quad i = 1, 2 \end{aligned}$$

for any $z_1(\cdot)$ and $z_2(\cdot)$ in $\mathcal{L}_2^n[0, \infty)$.

It is then straightforward but tedious to show that (10) is a sufficient condition to guarantee that the augmented system (25)-(27) is asymptotically stable,which in turns means that $\bar{\mathcal{A}}(j\omega, \tau)$, and hence $\mathcal{A}(j\omega, \tau)$, is nonsingular. ∇∇∇

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