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ROBUST STABILITY ANALYSIS FOR TIME-DELAY SYSTEMS USING THE INTEGRAL QUADRATIC CONSTRAINT APPROACH*

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Abstract

Given a time-delay system, we are interested in finding new robust stability conditions for the system. We apply the integral quadratic constraint approach to obtain two results which allow us to test the robust stability using the linear matrix inequality technique. Both results give an estimate of the maximum time-delay which preserves robust stability. The first result is simpler to apply while the second one gives a less conservative robust stability condition.

1 Introduction

Consider a time-delay system with parametric uncertainty described by

$$\dot{x}(t) = A_0 x(t) + A_d x(t-\tau) \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state, τ is a constant time delay, A_0 and A_d are constant matrices.

The system above has been analyzed by many researchers in the past. Two types of robust stability conditions have been reported in the literature: the so-called *delay independent* conditions and *delay-dependent* conditions. In comparison, the delay independent conditions are simpler to apply, but the delay-dependent conditions are less conservative. With the recent advancement in convex optimization (see, e.g., [2]), the focus of the current research is towards finding less conservative delay-dependent conditions by allowing more complex convex optimization. See [10] for a review of robust stability results.

The goal of this paper is to provide new conditions under which the robust stability of the system (1) is guaranteed. Our work is based on two ingredients: 1) a sufficient condition for robust stability expressed in the frequency domain; and 2) the integral quadratic constraint (IQC) approach to robustness analysis. Two results are presented. Both results are expressed in terms of linear matrix inequalities (LMI), and they give an estimate of the maximum time-delay which preserves robust stability. The first result is simpler to apply while the second one gives a less conservative robust stability condition. We point out that the results in this paper generalize those in [10].

2 Preliminaries

Several preliminary results are required for robust stability analysis of (1).

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Lemma 1. *The system (1) is asymptotically stable if A is asymptotically stable and*

$$\mathcal{A}(j\omega,\tau) := j\omega I - A - \tau\rho_1(j\omega\tau)A_d A_0 - \tau\rho_2(j\omega\tau)A_d A_d \tag{2}$$

is nonsingular for all $\omega \in \mathbf{R}$, where

$$\rho_1(jv) = -e^{-jv/2}\frac{\sin(v/2)}{(v/2)}, \quad \rho_2(jv) = \rho_1(jv)e^{-jv}. \tag{3}$$

Proof. It is well-known that (1) is asymptotically stable if and only if

$$\hat{\mathcal{A}}(j\omega,\tau) = j\omega I - A_0 - A_d e^{-j\omega\tau}$$

is nonsingular for all $\omega \in \mathcal{R}$.

Suppose $\mathcal{A}(j\omega,\tau)$ is nonsingular, we need to show that $\hat{\mathcal{A}}(j\omega,\tau)$ is nonsingular. Let x be such that $\hat{\mathcal{A}}(j\omega,\tau)x = 0$. We need to show that $x = 0$. To see this, we note

$$\begin{aligned} 0 &= (j\omega I - A_0 - A_d e^{-j\omega\tau})x \\ &= (j\omega I - A - A_d(e^{-j\omega\tau} - 1))x \\ &= (j\omega I - A - \tau\rho_1(j\omega\tau)A_d j\omega)x \\ &= (j\omega I - A - \tau\rho_1(j\omega\tau)A_d(A + A_d e^{-j\omega\tau}))x \\ &= \mathcal{A}(j\omega,\tau)x \end{aligned} \tag{4}$$

So x must be zero due to the nonsingularity of $\mathcal{A}(j\omega,\tau)$. □

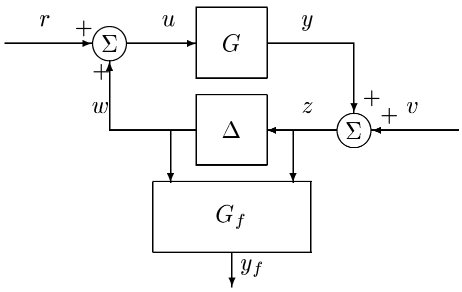

Figure 1: Interconnected Feedback System

Consider the interconnected system in Figure 1 which is also described by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ z(t) &= y(t) + v(t) \\ u(t) &= r(t) + w(t) \\ w(t) &= \Delta(z(t)) \end{aligned} \tag{5}$$

where $\Delta(\cdot) \in \underline{\Delta}$ which is a set of linear or nonlinear dynamic operators to be specified later. Denote

$$G(s) = C(sI - A)^{-1}B + D \tag{6}$$

and assume A to be asymptotically stable in the sequel.

The feedback block $\Delta(\cdot)$ is assumed to satisfy an IQC which is constructed via a *filter* given as follows:

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f u_f(t), \quad x_f(0) = 0 \\ y_f(t) &= C_f x_f(t) + D_f u_f(t) \\ u_f(t) &= \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{7}$$

where A_f is asymptotically stable. Denote the transfer function of the filter by

$$G_f(s) = C_f(sI - A_f)^{-1}B_f + D_f \tag{8}$$

The IQC used in this paper is then described by the following inequality:

$$\int_0^T y_f'(t)\tilde{\Phi}y_f(t)dt \geq 0, \quad \text{as } T \to \infty, \quad \forall\, \Delta \in \underline{\Delta} \tag{9}$$

where $\tilde{\Phi}$ is a constant symmetric matrix.

Remark 1. The definition above does not require $w \in \mathcal{L}_2[0,\infty)$. But if this is the case, then the IQC (7)-(9) becomes, following the Parseval Theorem,

$$\int_{-\infty}^{+\infty}[z^*(j\omega) \quad w^*(j\omega)]\Phi(j\omega)\begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix}d\omega \geq 0, \quad \forall\, \Delta \in \underline{\Delta} \tag{10}$$

where $z(j\omega), w(j\omega)$ are Fourier transforms of $z(t), w(t)$, respectively, and

$$\Phi(s) = G_f^*(s)\tilde{\Phi}G_f(s) \tag{11}$$

The following result serves the foundation of the IQC approach.

Theorem 1. **(The IQC Theorem)** *[15, 12, 11] Given a connected set of operators $\underline{\Delta}$, containing the zero operator, for the feedback block of the system (5), the system is absolutely stable if there exists some $\Phi(s)$ of the form (11) and a constant $\epsilon > 0$ such that both (9) and the following condition are satisfied:*

$$[G^*(j\omega) \quad I]\Phi(j\omega)\begin{bmatrix} G(j\omega) \\ I \end{bmatrix} + \epsilon I \leq 0, \quad \forall\, \omega \in (-\infty,\infty) \tag{12}$$

Further, for causal and asymptotically stable linear time-invariant (LTI) $\Delta(\cdot)$, (9) is equivalent to to the following:

$$[I \quad \Delta^*(j\omega)]\Phi(j\omega)\begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0, \quad \forall\, \omega \in (-\infty,\infty), \ \Delta \in \underline{\Delta} \tag{13}$$

That is, the system (5) is absolutely stable if there exists $\Phi(s)$ of the form (11) such that (12) and (13) hold.

Lemma 2. **(KYP Lemma)** *[1, 13] Given $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times k}$ and symmetric $\Omega \in \mathbf{R}^{(n+k)\times(n+k)}$, there exists a symmetric matrix $P \in \mathbf{R}^{n\times n}$ such that*

$$\begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix} + \Omega < 0 \tag{14}$$

if and only if there exists some constant $\epsilon > 0$ such that

$$[B^T((j\omega I - A)^{-1})^* \ I]\Omega \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} + \epsilon I \le 0,$$
$$\forall \ \omega \in (-\infty, \infty) \tag{15}$$

Further, if A is Hurwitz and the top-left $n \times n$ submatrix of Ω is positive semidefinite, then (14) implies $P > 0$.

3 Main Results

Consider the time-delay system in (1). Express

$$A_d = HE, \quad H \in \mathbf{R}^{n\times q}, \quad E \in \mathbf{R}^{q\times n} \tag{16}$$

where $q \le n$, and E is full rank. Define

$$B^T = \begin{bmatrix} H^T \\ H^T \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} EA_0 \\ EA_d \end{bmatrix},$$
$$C_\tau = \tau C, \quad D = 0, \tag{17}$$

and $\underline{\Delta}$ being the set of LTI operators with Fourier transform given by

$$\Delta(j\omega) = \lambda \ \mathrm{diag}\{\rho_1(j\omega\tau)I_q, \rho(j\omega\tau)I_q\} \tag{18}$$

for some $\lambda \in [0, \ 1]$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are defined in (3).

Using Lemma 1, we know that the system (1) is robustly stable if the following system is robustly stable:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_\tau x(t) + Du(t) \\ z(t) &= y(t) + v(t) \\ u(t) &= r(t) + w(t) \\ w(t) &= \Delta(z(t)) \end{aligned} \tag{19}$$

Following the IQC Theorem, we assert that system (1) is robustly stable if there exists some IQC, or equivalently, $\Phi(s)$ as in (11) such that (12) and (13) hold. Note that the notion of absolute stability coincides with the notion of robust stability for linear uncertain blocks Δ. In the rest of this section, we study two IQCs which give two robust stability conditions.

The first IQC is a simple constant D-scaling used in the analysis of structured singular value. More precisely, we take

$$G_f(s) = \ \mathrm{diag}\{I_{2q}, I_{2q}\} \tag{20}$$

and

$$\tilde{\Phi} = \tau^{-1} \ \mathrm{diag}\{\Lambda_1, \Lambda_2, -\Lambda_1, \Lambda_2\} \tag{21}$$

for some $q \times q$ symmetric and positive-definite $\Lambda_i, i = 1, 2,$ which are to be chosen. Denote

$$\Lambda = \ \mathrm{diag}\{\Lambda_1, \Lambda_2\} \tag{22}$$

The resulting IQC has

$$\Phi(s) = \tau^{-1} \ \mathrm{diag}\{\Lambda, -\Lambda\} \tag{23}$$

It is straightforward to verify that (13) holds because $\rho_i(\cdot)$ are contractive. The sufficient condition (12) for robust stability becomes

$$\begin{bmatrix} B^T((j\omega I - A)^{-1})^* & I_{2q} \end{bmatrix} \begin{bmatrix} \tau C^T\Lambda C & 0 \\ 0 & -\tau^{-1}\Lambda \end{bmatrix}$$
$$\times \begin{bmatrix} (j\omega I - A)^{-1}B \\ I_{2q} \end{bmatrix} + \epsilon I \le 0 \tag{24}$$

for all ω.

Using the KYP Lemma, the above is equivalent to the existence of $P = P^T > 0$ such that the following LMI holds:

$$\begin{bmatrix} \mathcal{P} & PH & PH \\ H^TP & -\tau^{-1}\Lambda_1 & 0 \\ H^TP & 0 & -\tau^{-1}\Lambda_2 \end{bmatrix} < 0$$

where

$$\mathcal{P} = A^TP + PA + \tau C_1^T\Lambda_1C_1 + \tau C_2^T\Lambda_2C_2.$$

Multiplying τ to the second and third row and column blocks, which does not alter the validity of the LMI, the above becomes

$$\Pi(\tau) = \begin{bmatrix} \mathcal{P} & \tau PH & \tau PH \\ \tau H^TP & -\tau\Lambda_1 & 0 \\ \tau H^TP & 0 & -\tau\Lambda_2 \end{bmatrix} < 0 \tag{25}$$

Note that $\Pi(\tau)$ is affine in P, Λ_1 and Λ_2.

We summarize the analysis above as follows:

Theorem 2. *The time-delay system (1) is robustly stable for all $0 \le \tau \le \bar{\tau}$ if there exist $n \times n$ symmetric and positive definite matrices Λ_1, Λ_2 and P such that the LMI*

$$\Pi(\bar{\tau}) < 0 \tag{26}$$

holds, where $\Pi(\tau)$ is defined in (25).

Proof. Suppose (26) holds. It follows from the analysis above that the system (1) is robustly stable for $\bar{\tau}$. The conclusion that the above also implies the robust stability for all $0 \le \tau \le \bar{\tau}$ follows from the fact that the $\Pi(\tau)$ is convex in τ. More precisely, $\Pi(\tau) < 0$ when τ is sufficiently small, due to (26) and $\Lambda_i > 0$. The rest follows from the convexity of $\Pi(\tau)$. □

The second IQC we use to study the robust stability of (1) will be more involved but give a less conservative condition. Let

$$f(s) = c_f(sI - a_f)^{-1}b_f + d_f \tag{27}$$

be any asymptotically stable SISO filter with the following property:

$$|f(jv)| \geq \left|\frac{\sin(v)}{v}\right|, \quad \forall\, v \in \mathbf{R} \tag{28}$$

Denote the diagonal transfer matrix

$$F(s) = f(s)I_{2q} = C_f(sI - A_f)^{-1}B_f + D_f \tag{29}$$

We will discuss how to choose $f(s)$ later.

Now define

$$G_f(s) = \text{diag}\{F(s\tau), I_{2q}\} \tag{30}$$

and $\tilde{\Phi}$ as in (21). This yields

$$y_f(s) = \begin{bmatrix} F(s\tau)z(s) \\ w(s) \end{bmatrix} \tag{31}$$

$$\Phi(s) = G_f^*(s)\tilde{\Phi}G_f(s) = \tau^{-1}\ \text{diag}\{F^*(s\tau)\Lambda F(s\tau), -\Lambda\} \tag{32}$$

Subsequently, condition (13) is automatically satisfied because

$$\begin{aligned}
&\tau[I_{2q}\quad \Delta^*(j\omega)]\Phi(j\omega)\begin{bmatrix} I_{2q} \\ \Delta(j\omega)\end{bmatrix} \\
&= F^*(j\omega\tau)\Lambda F(j\omega\tau) \\
&\quad -\lambda^2\text{diag}\{\rho_1^*(j\omega\tau)\rho_1(j\omega\tau)\Lambda_1, \rho_2^*(j\omega\tau)\rho_2(j\omega\tau)\Lambda_2\} \\
&= \Lambda^{1/2}(F^*(j\omega\tau)F(j\omega\tau) \\
&\quad -\lambda^2\text{diag}\{\rho_1^*(j\omega\tau)\rho_1(j\omega\tau), \rho_2^*(j\omega\tau)\rho_2(j\omega\tau)\})\Lambda^{1/2} \\
&\geq 0
\end{aligned}$$

Therefore, a sufficient condition for robust stability of system (1) is the condition (12) which, in our case, becomes

$$\begin{aligned} G_\tau^*(j\omega)F^*(j\omega\tau)\tau^{-1}\Lambda F(j\omega\tau)G_\tau(j\omega) - \tau^{-1}\Lambda + \epsilon I_{2q} \leq 0, \\ \forall\, \omega \in (-\infty, \infty) \end{aligned} \tag{33}$$

for some (sufficiently small) $\epsilon > 0$, where

$$G_\tau(s) = C_\tau(sI - A)^{-1}B.$$

Our next step is to convert the frequency domain condition (33) into the state space. To this end, we denote by $\bar{C}_\tau(sI - \bar{A}_\tau)^{-1}\bar{B}_\tau$ a state-space realization of $F(s\tau)G_\tau(s)$. Then, it is straightforward to verify that

$$\begin{aligned} \bar{A}_\tau &= \begin{bmatrix} \tau^{-1}A_f & B_fC \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \\ \bar{C}_\tau &= \begin{bmatrix} C_f & D_fC_\tau \end{bmatrix} \end{aligned} \tag{34}$$

Condition (33) can be rewritten as

$$\begin{aligned} &[\bar{B}^T((j\omega I - \bar{A}_\tau)^{-1})^*\ I]\ \text{diag}\{\tau^{-1}\bar{C}_\tau^T\Lambda\bar{C}_\tau, -\tau^{-1}\Lambda\} \\ &\times \begin{bmatrix} (j\omega I - \bar{A}_\tau)^{-1}\bar{B} \\ I \end{bmatrix} + \epsilon I \leq 0 \end{aligned}$$

for all $\omega \in (-\infty, \infty)$.

Applying the KYP Lemma, the above is equivalent to the existence of some $\bar{P} = \bar{P}^T > 0$ such that the following linear matrix inequality holds:

$$\bar{\Pi}(\tau) = \begin{bmatrix} \bar{A}_\tau^T\bar{P} + \bar{P}\bar{A}_\tau + \tau^{-1}\bar{C}_\tau^T\Lambda\bar{C}_\tau & \bar{P}\bar{B} \\ \bar{B}^T\bar{P} & -\tau^{-1}\Lambda \end{bmatrix} < 0 \tag{35}$$

The above analysis is summarized as follows:

Theorem 3. *The time-delay system (1) is robustly stable for all $\tau \leq \bar{\tau}$ if there exist an asymptotically stable filter $f(s)$, and symmetric and positive-definite matrices Λ and $\bar{P}$ such that the following LMI holds:*

$$\bar{\Pi}(\bar{\tau}) < 0 \tag{36}$$

where $\bar{\Pi}(\cdot)$ is defined in (35).

Proof. The proof is very similar to that of Theorem 2, so the details are omitted. The only step worth of discussion is the fact that $\bar{\Pi}(\bar{\tau}) < 0$ implies $\bar{\Pi}(\tau) < 0$ for all $0 < \tau \leq \bar{\tau}$. This step is a bit tedious but not too difficult to verify too. □

4 An Example

Before providing an example, we address the problem of finding a suitable filter $f(s)$. First, we note that $f(s)$ is a SISO transfer function, and that the constraint on $f(s)$ (28) is independent of the system (1). This means that once a "good" $f(s)$ is found, it can be applied to various time-delay systems of the form (1). The complexity of $f(s)$ is mainly determined by the degree of $f(s)$. A second order example is given below:

$$f(s) = \frac{2(s + 0.9)}{(s + 0.8)(s + 2.216)} \tag{37}$$

with its Bode plot given in Figure 2.

Also plotted in Figure 2 is $|\sin(\omega)/\omega|$ to justify (28).

Example: Consider the system (1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.25 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.85 \end{bmatrix} \tag{38}$$

Using Theorem 3 and the $f(s)$, the maximum τ is obtained to be $\tau_{\max} = 0.9848$.

Obviously, the conservatism of $\tau_{\max}$ depends on the filter $f(s)$. It is found in simulation that second order filters usually outperform first order ones. Also, higher order filters can be used to obtain slightly larger $\tau_{\max}$.

Using Theorem 2, the maximum τ is obtained to be $\tau_{\max} = 0.6417$.

As comparisons, we notice that the maximum τ using the results in [10, 9] is $\tau_{\max} = 0.58$ while the optimal τ for the system with the given parameters is $\tau_o = 1.54$[9].

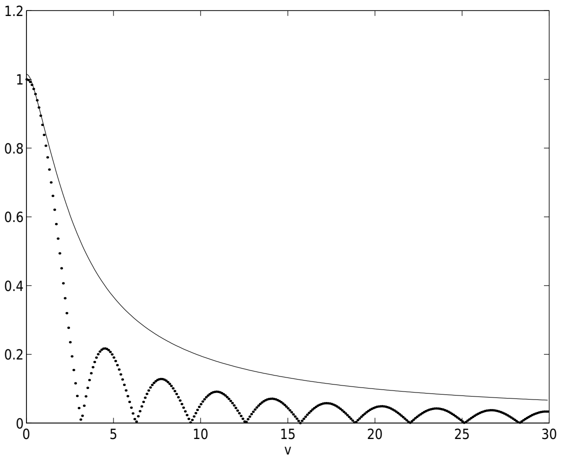

Figure 2: Example of $f(s)$. Solid line: $|f(jv)|$; Dotted line: $|\sin(v)/v|$

5 Conclusion

We have obtained two new robust stability conditions for time-delay systems by applying the IQC approach. These conditions are expressed in terms of LMIs and therefore easily solvable. Although a single delay is considered in this paper, we stress that an extension to multiple delays can be simply derived. Further, we point out that it is possible to interpret our robust stability conditions in the time-domain. Subsequently, these conditions can be used to treat systems with time-varying time-delays.

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