Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002

Regional Stability and Performance Analysis for a Class of Nonlinear Discrete-Time Systems[1]

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Abstract

This paper deals with the problem of regional stability and performance analysis for a class of nonlinear discrete-time systems with uncertain parameters. We use polynomial Lyapunov functions to derive stability conditions and performance criteria in terms of linear matrix inequalities (LMIs). Although the use of polynomial Lyapunov functions is common for continuous-time systems as a way to reduce the conservatism in analysis, we point out that direct generalization of such an approach to discrete-time systems leads to intractable solutions because it results in a large number of LMIs. We introduce a novel approach to reduce the computational complexity by generalizing a result of Oliveira *et. al.* on robust stability analysis for discrete-time systems with parameter uncertainties. We point out that the proposed method can lead to less conservative results when compared with results using quadratic Lyapunov functions.

1 Introduction

The last decade or so has witnessed active research work in the area of robust control of continuous-time nonlinear systems in the framework of linear matrix inequalities (LMIs). Design approaches range from using quadratic Lyapunov functions ([1, 2]) to those based on polynomial Lyapunov functions ([3, 4]). In general, non-quadratic Lyapunov functions are less conservative for dealing with uncertain and nonlinear systems than quadratic Lyapunov functions at the expense of extra computation [5]. However, most of the robust control results using non-quadratic Lyapunov functions are for continuous-time systems.

The fundamental difficulty with non-quadratic Lyapunov functions for discrete-time systems lies in the fact that the difference between the Lyapunov functions at time $k+1$ and k is highly nonlinear. To make this point clear, we consider the following system:

$$x(k+1) = A(x(k),\delta)x(k) \tag{1}$$

and the Lyapunov function $V(x,\delta) = x'P(x,\delta)x$, where δ represents (constant) uncertain parameters, and the matrices $A(x,\delta)$ and $P(x,\delta)$ depend on x and δ. The Lyapunov function difference is given by the following inequality (which we will refer to as a Lyapunov inequality):

$$\begin{aligned} V(x(k+1),\delta) - V(x(k),\delta) = {} & x'(k)A'(x(k),\delta) \\ & P(A(x(k),\delta)x(k),\delta)A(x(k),\delta)x(k) \\ & -x'(k)P(x(k),\delta)x(k) \end{aligned} \tag{2}$$

which is typically a highly nonlinear function of $x(k)$ and δ. In contrast, if we considered a similar continuous-time system

$$\dot{x}(t) = A(x(t),\delta)x(t) \tag{3}$$

and a similar Lyapunov function, we would have the derivative of the Lyapunov function given by the following Lyapunov inequality:

$$\begin{aligned} \dot{V}(x(t),\delta(t)) = {} & x'\Big(A'(x,\delta)P(x,\delta) + P(x,\delta)A(x,\delta) \\ & + \sum_{i=1}^{n} \frac{\partial P(x,\delta)}{\partial x_i} e_i' A(x,\delta)x\Big)x \end{aligned} \tag{4}$$

where x_i is the ith element of x and e_i is the ith column of an identity matrix. It is obvious from the above

[1]This work was partially supported by 'CAPES', Brazil, under grant BEX0784/00-1 and 'CNPq', Brazil, under grant 147055/99-7.

that the continuous-time case involves much less coupled nonlinear terms, especially when $P(x,\delta)$ is chosen to be in a simple form (e.g., affine in x and δ).

The approach used in this paper is motivated by the work of Oliveira *et. al.* [6] which proposed a new test of stability using LMIs for linear discrete-time systems with parameter uncertainties. This approach was extended to performance analysis [7]. In this approach, the system matrix and the Lyapunov matrix are assumed to be affine in uncertain parameters, i.e., $A(\delta)$ and $P(\delta)$ are used and they are affine in δ. The Lyapunov inequality is modified by introducing an auxiliary matrix [6]. The key feature of this auxiliary matrix is that it separates the system matrix $A(\delta)$ from the Lyapunov matrix $P(\delta)$, thus significantly reducing the nonlinearity. Further, the resulting inequality can be expressed as a linear matrix inequality which is affine in δ. This allows easy verification of robust stability and performance. Although one can think of many possible auxiliary matrices with the above feature, the particular one introduced in [6] appears to be excellent in terms of the conservatism it brings. In particular, if one resorts to a quadratic Lyapunov function, i.e., a constant P matrix, this auxiliary matrix does not bring any conservatism.

We point out that there are other approaches to robust stability and performance analysis for discrete-time systems. For example, the work of Iwasaki in [8] uses non-quadratic Lyapunov functions for analyzing the global and regional stability of a class of LTI systems with a nonlinear (or uncertain) feedback connection (Lur'e like systems) satisfying a sector bound condition; and Tuan *et. al.* [9] considers parameterized Lyapunov functions for nonlinear discrete $\mathcal{H}_\infty$ control of quasi-LPV systems, i.e. systems described by $x(k+1) = A(\theta(x(k))x(k)$ where the state-dependent parameter $\theta(x(k)) \in \Theta$ with Θ being a given polytope. Note that these works consider different modelling techniques in order to use the LMI tools developed for linear systems (or LPV systems). In spite of the fact that both overcome the problem caused by the term $A(x(k),\delta)'\mathcal{P}(A(x(k),\delta)A(x(k),\delta)$, they still have some shortcomings. Namely, [8] treats a restricted class of nonlinear systems; and the approach in [9] is computationally feasible only for systems with a small number of nonlinear terms.

The purpose of this paper is to devise a technique to analyze regional stability and performance for a class of nonlinear discrete-time systems with uncertain parameters. We employ polynomial Lyapunov functions and give stability and performance conditions in terms of linear matrix inequalities. But suitable Lyapunov inequalities will be used to simplify numerical computations, which is done by generalizing the work of [6]. We will consider the regional stability analysis problem first. This will be followed by output performance analysis, where the system has zero input but with a non-zero initial condition, and input-output performance analysis, where the system has zero initial condition but with a non-zero input. In this paper, we omitted proofs and numerical examples because of space limitation and they are available in the full version of this paper [10].

2 Problem Statement

Consider the following class of discrete-time nonlinear systems:

$$x_+ = A(x,\delta)x,\ x_+ = x(k+1),\ x = x(k) \quad (5)$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $\delta \in \Delta \subset \mathbb{R}^{n_\delta}$ denotes the vector of (constant) uncertain parameters, and the system matrix $A(x,\delta)$ is allowed to depend on x and δ. It is assumed in the sequel that Δ is a polytope and that $A(x,\delta)$ is a continuous function in $\mathbb{R}^{n_x} \times \Delta$.

The problem of concern in this paper is to determine a region in the state space in which robust stability and performance of system (5) is guaranteed. To this end, we first introduce the notion of domain of attraction.

Given a region $\mathcal{R} \subset \mathbb{R}^{n_x}$, we say $\mathcal{R}$ a *domain of attraction* for system (5) if for every $x(0) \in \mathcal{R}$ and $\delta \in \Delta$, the trajectory x remains in $\mathcal{R}$ for all $k \geq 0$ and approaches the origin as $k \to \infty$.

We have the following basic result:

Lemma 1 *Consider system (5). Let $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ be a given Lyapunov function candidate, where $\mathcal{P}(x,\delta)$ is a matrix function of (x,δ). Define a region in the state space as follows:*

$$\mathcal{X} \triangleq \{x \in \mathbb{R}^{n_x} : x'\mathcal{P}(x,\delta)x \leq 1,\ \forall\ \delta \in \Delta\} \quad (6)$$

Suppose there exist positive scalars $\epsilon_1, \epsilon_2, \epsilon_3$ such that

$$\epsilon_1 x'x \leq x'\mathcal{P}(x,\delta)x \leq \epsilon_2 x'x \quad (7)$$

$$x'\left(A'\mathcal{P}(x_+,\delta)A - \mathcal{P}(x,\delta)\right)x \leq -\epsilon_3 x'x \quad (8)$$

for all $(x,\delta) \in \mathcal{X} \times \Delta$, with $A = A(x,\delta)$. Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is a domain of attraction for system (5).

A possible approach to simplifying the product term $A(x,\delta)\mathcal{P}(x_+,\delta)A(x,\delta)$ is to use the idea of Schur complement, which converts (8) to the following condition:

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P} & A'\mathcal{P}_+ \\ \mathcal{P}_+A & -\mathcal{P}_+ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq -\epsilon_3 x'x \quad (9)$$

for all $(x,\delta,y) \in \mathcal{X} \times \Delta \times \mathbb{R}^{n_x}$, with $A = A(x,\delta), \mathcal{P} = \mathcal{P}(x,\delta), \mathcal{P}_+ = \mathcal{P}(x_+,\delta)$. The stability conditions (8) and (9) can be seen as equivalents by noticing that the maximizing y for the left hand side of (9) is given by $y = A(x,\delta)x$. With this choice of y, the two inequalities are the same.

The conditions (7) and (9) are still very complicated to check because of two problems: 1) Coupling of $A(x,\delta)$ and $\mathcal{P}(x_+,\delta)$ still gives high nonlinearity; 2) Checking the conditions over $\mathcal{X} \times \Delta$ is highly nontrivial. These are the problems we will address in the next section.

3 Preliminary Results

We give two results in this section. The first one, Lemma 2, is a nonlinear version of a result in [6] which aims to remove the coupling between the system and Lyapunov matrices. The second result, Lemma 3, gives a way to remove the nonlinear dependence on x in the conditions (7) and (9) by a relaxation technique.

Lemma 2 *Consider system (5) and $V(x,\delta)$ and $\mathcal{X}$ as defined in Lemma 1. Suppose (7) and the following inequality holds for some auxiliary matrix function $\mathcal{G}(x,\delta)$:*

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P} & A'\mathcal{G}' \\ \mathcal{G}A & \mathcal{P}_+ - \mathcal{G} - \mathcal{G}' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq -\epsilon_3 x' x \quad (10)$$

for all $(x,\delta,y) \in \mathcal{X} \times \Delta \times \mathbb{R}^{n_x}$, with $\mathcal{G} = \mathcal{G}(x,\delta)$. Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is a domain of attraction for system (5).

The conservativeness of Lemma 2 lies in the choice of the auxiliary matrix function $\mathcal{G}(x,\delta)$. Observe that we can recover Lemma 1 by considering $\mathcal{G}(x,\delta) = \mathcal{P}(x_+,\delta)$. But this choice of $\mathcal{G}(x,\delta)$ leads to a complicated condition. Consequently, we will have a compromise between the conservatism and complexity when choosing $\mathcal{G}(x,\delta)$.

We note that when $A(x,\delta), \mathcal{P}(x,\delta)$ and $\mathcal{G}(x,\delta)$ do not depend on x, the result above reduces to a result in [6].

Next, we aim to remove the dependence on x in conditions (7) and (10). To this end, we use a version of the well-known Finsler's lemma (see, e.g., [11]):

Lemma 3 *Consider the following nonlinear matrix inequality:*

$$\mathcal{T}(\xi) > 0,\ \mathcal{T}(\xi) = \mathcal{T}(\xi)',\ \forall\ \xi \in \mathcal{D} \qquad (11)$$

where $\xi \in \mathbb{R}^{n_\xi}$ denotes a generic parameter (that can represent the state or uncertainties), the matrix $\mathcal{T}(\xi) \in \mathbb{R}^{n_v \times n_v}$ is a nonlinear function of ξ and $\mathcal{D} \subset \mathbb{R}^{n_\xi}$ is a polytopic region with known vertices. Suppose $\mathcal{T}(\xi)$ can be decomposed as follows:

$$\mathcal{T}(\xi) = \mathcal{M}(\xi)' T \mathcal{M}(\xi) \qquad (12)$$

where $T \in \mathbb{R}^{m_v \times m_v}$ is a constant matrix, $\mathcal{M}(\xi) \in \mathbb{R}^{m_v \times n_v}$ is a nonlinear matrix function of ξ with the property that

$$\Xi(\xi)\mathcal{M}(\xi) = 0 \qquad (13)$$

for some $\Xi(\xi) \in \mathbb{R}^{m_\xi \times m_v}$ which is an affine matrix function of ξ. Then, (11) is satisfied if there exists a constant matrix L such that

$$T + L\Xi(\xi) + \Xi(\xi)'L' > 0 \qquad (14)$$

at all vertices of $\mathcal{D}$.

We point out that the decomposition conditions (12)-(13) are very general and can be satisfied for many nonlinear systems.

4 Stability Analysis

We are now ready to derive the main results of this paper. This section deals with the regional stability analysis problem, whereas the next two sections study performances.

In order to apply the results in the previous section, we need to re-parameterize the system model (5) and choose Lyapunov matrix function and the auxiliary matrix function accordingly. These are detailed below.

System Model Representation

We further assume that system (5) can be decomposed as follows:

$$\begin{aligned} x_+ &= A(x,\delta)x = \mathcal{A}\Pi(x,\delta)x \\ 0 &= \Omega(x,\delta)\Pi(x,\delta), Q\Pi(x,\delta) = I_{n_x} \end{aligned} \qquad (15)$$

where $\mathcal{A} \in \mathbb{R}^{n_x \times n_\pi}$ and $Q \in \mathbb{R}^{n_x \times n_\pi}$ are constant matrices, $\Omega(x,\delta)$ is an affine matrix function of (x,δ), and $\Pi(x,\delta)$ is a nonlinear matrix function in (x,δ).

We again point out that many nonlinear systems can be decomposed as above, see [10] for an illustrative example. However, the representation (15) of (5) is not unique and this fact may be a source of conservatism (See, e.g., [12]).

Lyapunov Function Candidate

With the decomposition of the system as in (15), we may choose the Lyapunov matrix function in the following form:

$$\mathcal{P}(x,\delta) = \begin{bmatrix} \Theta(x) \\ I_{n_x} \end{bmatrix}' P(\delta) \begin{bmatrix} \Theta(x) \\ I_{n_x} \end{bmatrix} \qquad (16)$$

where $P(\delta) = P(\delta)'$ is an affine matrix function of δ, and $\Theta(x) \in \mathbb{R}^{n_\theta \times n_x}$ is an affine matrix function of x, both to be determined.

Observe from lemma 2 that we need to compute the following matrix:

$$\mathcal{P}(x_+, \delta) = \begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix}' P(\delta) \begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix}.$$

To this end, we require the following constraints on $\Theta(x)$ and $\Theta(x_+)$:

$$\begin{bmatrix} \Theta(x) \\ I_{n_x} \end{bmatrix} = F\Pi(x,\delta) = \begin{bmatrix} F_1 \\ Q \end{bmatrix} \Pi(x,\delta) \quad (17)$$

$$\begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix} = H\Pi(x,\delta) = \begin{bmatrix} H_1 \\ Q \end{bmatrix} \Pi(x,\delta) \quad (18)$$

where $F_1, H_1 \in \mathbb{R}^{n_\theta \times n_\pi}$ are constant matrices.

Auxiliary Matrix Function $\mathcal{G}(x, \delta)$

We choose the auxiliary matrix function $\mathcal{G}(x, \delta)$ to be of the following form:

$$\mathcal{G}(x,\delta) = \Pi'(x,\delta)G(\delta) \quad (19)$$

where $G(\delta) \in \mathbb{R}^{n_\pi \times n_x}$ is an affine matrix function of δ to be determined.

Estimating the Domain of Attraction

With the decomposition of the system model and constraints on Lyapunov matrix function and the auxiliary matrix function, we can rewrite inequality (10) as follows:

$$\begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix}' \begin{bmatrix} \begin{pmatrix} -F'PF \\ +\epsilon_3 Q'Q \end{pmatrix} & A'G' \\ GA & \begin{pmatrix} H'PH \\ -GQ - Q'G' \end{pmatrix} \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix} \leq 0, \quad (20)$$

for all $(x, \delta, y) \in \mathcal{X} \times \Delta \times \mathbb{R}^{n_x}$, where $P = P(\delta), G = G(\delta)$, $\sigma_a = \Pi(x,\delta)x$ and $\sigma_b = \Pi(x,\delta)y$.

In order to apply Lemma 3, we also need a polytopic bounding set $\hat{\mathcal{X}}$ for $\mathcal{X}$. We will require (20) to hold for all $x \in \hat{\mathcal{X}}$ instead of $\mathcal{X}$. Hence, we want to choose $\hat{\mathcal{X}}$ to be reasonably close to $\mathcal{X}$ to reduce conservatism but having a small number of vertices so the resulting conditions are easy to check. A possible way to achieve a good compromise is to define the shape of the bounding set and use a parameter to control its size. This parameter can then be adjusted through iterations to obtain an optimal size. But for the discussion in the sequel, we assume that the bounding set $\hat{\mathcal{X}}$ is given.

Without loss of generality, we assume that the bounding set is represented in terms of the following constraints:

$$\hat{\mathcal{X}} = \left\{ x : a_j' x \leq 1,\ j = 1, \ldots, n_e \right\} \quad (21)$$

where $a_j \in \mathbb{R}^{n_x}$ are given vectors associated with the n_e edges of $\hat{\mathcal{X}}$.

Using the $\mathcal{S}$-procedure (See, e.g. [11, Sections 2.6 and 5.2]), the condition $\mathcal{X} \subset \hat{\mathcal{X}}$ is satisfied if the following inequality is satisfied for all j:

$$2\left(1 - a_j' x\right) + x'\mathcal{P}(x,\delta)x - 1 \geq 0 \quad (22)$$

Taking into account the structure of $\mathcal{P}(x,\delta)$ in (16), we can rewrite (22) as follows:

$$\begin{bmatrix} 1 \\ \Theta x \\ x \end{bmatrix}' \begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & P(\delta) \end{bmatrix} \begin{bmatrix} 1 \\ \Theta x \\ x \end{bmatrix} \geq 0 \quad (23)$$

for $j = 1, \ldots, n_e$, where $\Theta = \Theta(x)$. In order to ensure that the Lyapunov matrix function $\mathcal{P}(x,\delta)$ in (16) is positive definite for all $x \in \hat{\mathcal{X}}$, we apply lemma 3 and obtain the following condition:

$$P(\delta) + L\Psi_1(x) + L'\Psi_1'(x) > 0, \quad \forall x \in \hat{\mathcal{X}}, \delta \in \Delta \quad (24)$$

where L is a free matrix to be determined and

$$\Psi_1(x) = \begin{bmatrix} I_{n_\xi} & -\Theta(x) \end{bmatrix} \quad (25)$$

In order to maximize the volume of $\mathcal{X}$, we normally approximate it by minimizing the trace of the Lyapunov matrix. However, $\mathcal{P}(x,\delta)$ is a nonlinear function of (x,δ) that leads to a non-convex condition. To overcome this problem, we will approximate the volume maximization by

$$\min \max_{x \in \hat{\mathcal{X}}, \delta \in \Delta} \text{trace}\, (P(\delta) + L\Psi_1(x) + L'\Psi_1'(x)) \quad (26)$$

Now, with above analysis we can state the following theorem which gives a convex solution to the regional stability problem for system (5) in terms of LMIs.

Theorem 1 *Consider the system (5) as decomposed in (15). Let $\Theta(x)$ be a given affine matrix function of x satisfying (17) and the Lyapunov matrix function $\mathcal{P}(x,\delta)$ be in the form of (16). Let $\hat{\mathcal{X}}$ be a given bounding set as in (23). Define $\Psi_1(x)$ as in (25) and*

$$\Psi_2(x,\delta) = \begin{bmatrix} \Omega(x,\delta) & 0 \\ 0 & \Omega(x,\delta) \end{bmatrix}. \quad (27)$$

Suppose there exist affine matrices $G(\delta)$ and $P(\delta)$ and constant matrices L, N and $M_j, j = 1, \cdots, n_e$ solving the following linear matrix inequalities at all vertices of

$\hat{\mathcal{X}} \times \Delta$:

$$\min \eta \quad \text{subject to:}$$

$$\eta - \text{trace}\left(P + L\Psi_1 + \Psi_1' L'\right) \geq 0 \tag{28}$$

$$P + L\Psi_1 + \Psi_1' L' > 0 \tag{29}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & \left(P + M_j\Psi_1 + \Psi_1' M_j'\right) \end{bmatrix} \geq 0, \ \forall\ j \tag{30}$$

$$\begin{bmatrix} -F'PF & \mathcal{A}'G' \\ G\mathcal{A} & H'PH - GQ - Q'G' \end{bmatrix} + N\Psi_2 + \Psi_2' N' < 0 \tag{31}$$

where $P = P(\delta), G = G(\delta), \Psi_1 = \Psi_1(x)$ and $\Psi_2 = \Psi_2(x,\delta)$. Then, $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is a domain of attraction for system (5).

5 Output Performance Analysis

In this section, we consider the analysis problem where we require the region $\mathcal{X}$ to satisfy certain output performance in addition to robust stability. More specifically, we add to the system (5) the following output signal:

$$z(k) = e(x(k), \delta) \tag{32}$$

where $z \in \mathbb{R}^{n_z}$ and $e(x, \delta)$ can be decomposed as follows:

$$e(x, \delta) = E\Pi(x, \delta)x \tag{33}$$

where E is a constant matrix and $\Pi(x, \delta)$ is as in (15).

Given a region $\mathcal{R} \subset \mathbb{R}^{n_x}$ and a level of performance $\lambda > 0$, we say $\mathcal{R}$ a *domain of output performance* (with level λ) if $\mathcal{R}$ is a domain of attraction and in addition, $\|z(k)\|_2^2 < \lambda$ holds for all trajectories of $x(k)$ and all $\delta \in \Delta$, provided $x(0) \in \mathcal{R}$.

To accommodate the extra performance requirement, we need to return to Lemma 1 and modify (8) to the following :

$$x'\left(A'\mathcal{P}_+ A - \mathcal{P}\right)x + \lambda^{-1} z' z \leq -\epsilon_3 x' x, \tag{34}$$

for all $(x, \delta) \in \mathcal{X} \times \Delta$, where $A = A(x,\delta), \mathcal{P} = \mathcal{P}(x,\delta)$ and $\mathcal{P}_+ = \mathcal{P}(x_+, \delta)$. To see this, we note that the condition above (together with other conditions in Lemma 1) implies that $\mathcal{X}$ is a domain of attraction and $\|z\|_2^2 < \lambda V(x(0), \delta) \leq \lambda$ for all $x(0) \in \mathcal{X}$ and $\delta \in \Delta$, since $V(x(0), \delta) \leq 1$.

The modification above leads to further modification in Theorem 1, i.e., we need to change (31) to the following:

$$\begin{bmatrix} \lambda^{-1}E'E - F'PF & \mathcal{A}'G' \\ G\mathcal{A} & H'PH - GQ - Q'G' \end{bmatrix} + N\Psi_2 + \Psi_2' N' < 0 \tag{35}$$

Theorem 2 *Consider system (5) and (32) and a given level of performance $\lambda > 0$. Let $\Theta(x)$, $\mathcal{P}(x,\delta)$, $\hat{\mathcal{X}}$, $\Psi_1(x)$ and $\Psi_2(x,\delta)$ be the same as in Theorem 1. Suppose there exist affine matrices $G(\delta)$ and $P(\delta)$ and constant matrices L, N and $M_j, j = 1, \cdots, n_e$ solving the linear matrix inequalities (28)-(30) and (35) at all vertices of $\hat{\mathcal{X}} \times \Delta$. Then, $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is a domain of output performance with level λ.*

6 Input-Output Performance Analysis

Consider the following system:

$$\begin{aligned} x_+ &= A(x(k),\delta)x(k) + b(x(k),\delta)w(k), \\ z(k) &= e(x(k),\delta)x(k) + d(x(k),\delta)w(k) \end{aligned} \tag{36}$$

where $x(0) = 0$, $w \in \mathcal{W}$ is the input disturbance signal, $\mathcal{W} = \{w : w \in \mathcal{L}_2^{n_w}, \|w\|_2 \leq 1\}$, $\delta \in \Delta$, and $b(x,\delta), d(x,\delta)$ are nonlinear functions of x and δ with appropriate dimensions.

Given a region $\mathcal{R} \subset \mathbb{R}^{n_x}$ and a performance level $\gamma > 0$, we say $\mathcal{R}$ a *domain of $\mathcal{L}_2$ performance* (with level γ) if for any $w \in \mathcal{W}$ and $x(0) = 0$, the trajectory $x(k)$ stays in $\mathcal{R}$ at all $k \geq 0$, $x(k) \to 0$ as $k \to \infty$, and $\|z\|_2 \leq \gamma\|w\|_2$.

Consider the region $\mathcal{X}_c$ with the following definition

$$\mathcal{X}_c \triangleq \left\{x : x \in \mathbb{R}^{n_x},\ x'\mathcal{P}(x,\delta)x \leq c,\ \delta \in \Delta\right\} \tag{37}$$

To ensure that $\mathcal{X}_c$ is a domain of $\mathcal{L}_2$ performance, we again modify (8) but to the following:

$$V(x_+, \delta) - V(x,\delta) + \gamma^{-2} z' z - w' w \leq \epsilon_3 x' x \tag{38}$$

for all $\delta \in \Delta$ and $w \in \mathcal{W}$, with the additional constraint $\mathcal{X}_c \subset \hat{\mathcal{X}}$. To see this condition (along with other conditions in Lemma 1) guarantees that $\mathcal{X}_c$ is a domain of $\mathcal{L}_2$ performance, we add up the inequality (38) from $k = 0$ to N to obtain the following:

$$V(x(N+1), \delta) + \textstyle\sum_{k=0}^{N} \left(\gamma^{-2} z'(k)z(k) + \epsilon_3 x'(k)x(k)\right) \leq \sum_{k=0}^{N} w'(k)w(k) \leq 1$$

This implies that $V(x(k), \delta) \leq c < 1$ for all $k \geq 0$, i.e., $x(k) \in \mathcal{X}_c$. This also implies that $\|z\|_2 \leq \gamma\|w\|_2$. Further, $x(k) \to 0$ as $k \to \infty$ because of (38) and the fact that $w(k) \to 0$ as $k \to \infty$.

As in the previous sections, we need to decompose the system (36). Let $A(x,\delta)$ and $e(x,\delta)$ be decomposed as in (15) and (33). In addition, we require

$$\begin{aligned} & b(x,\delta) = B\Phi(x,\delta),\ d(x,\delta) = D\Phi(x,\delta), \\ & \Lambda(x,\delta)\Phi(x,\delta) = 0,\ Q_2\Phi(x,\delta) = I_{n_w} \end{aligned} \tag{39}$$

where $B \in \mathbb{R}^{n_z \times n_\phi}$, $D \in \mathbb{R}^{n_z \times n_\phi}$ and $Q_2 \in \mathbb{R}^{n_w \times n_\phi}$ are constant matrices, $\Lambda(x,\delta) \in \mathbb{R}^{n_\lambda \times n_\phi}$ is an affine matrix function of (x,δ) and $\Phi(x,\delta) \in \mathbb{R}^{n_\phi \times n_w}$ is a nonlinear matrix function of (x,δ).

Also, we need to modify (18) to the following:

$$\begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix} = \left(H + \sum_{i=1}^{n_w} w_i Ji \right) \Pi(x,\delta) \qquad (40)$$

where w_i is the ith element of w and J_i (for $i = 1, \ldots, n_w$) are constant matrices.

We then have the following result:

Theorem 3 *Consider system (36) with the decompositions as given above. Let $\gamma > 0$ be a given level of $\mathcal{L}_2$ performance. Let $\Theta(x)$, $\mathcal{P}(x,\delta)$, $\hat{\mathcal{X}}$ and $\Psi_1(x)$ be the same as in Theorem 1. Define*

$$\Psi_3(x,\delta) = \text{diag}\ \{\Omega, \Omega, \Lambda, \Omega, \ldots, \Omega\} \qquad (41)$$

$$\begin{bmatrix} c^{-1} & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & \left(P + M_j \Psi_1 + \Psi_1' M_j' \right) \end{bmatrix} > 0,\ \forall\ j \qquad (42)$$

$$\begin{bmatrix} \Gamma_{11} & A'G' & \Gamma_{13} & 0 \\ GA & \Gamma_{22} & GB & \Gamma_{24} \\ \Gamma_{13}' & B'G' & \Gamma_{33} & 0 \\ 0 & \Gamma_{24}' & 0 & \Gamma_{44} \end{bmatrix} + N\Psi_3 + \Psi_3' N' < 0 \qquad (43)$$

where $\Gamma_{11} = -F'PF + \gamma^{-2}E'E$, $\Gamma_{13} = \gamma^{-2}E'D$, $\Gamma_{22} = H'PH - Q'G' - GQ + \sum_{i=1}^{n_w} S_i$, $\Gamma_{24} = [\ H'PJ_1 + R_1 \ \cdots \ H'PJ_{n_w} + R_{n_w}\ ]$, $\Gamma_{33} = -Q_2'Q_2 + \gamma^{-2}D'D$, $\Gamma_{44} = [\Upsilon_{ij}] - diag\{S_i\}$, $\Upsilon_{ij} = J_i'PJ_j$ $(i,j = 1,\ldots,n_w)$, $P = P(x,\delta)$, $G = G(x,\delta)$, $\Omega = \Omega(x,\delta)$, $\Lambda = \Lambda(x,\delta)$, $\Psi_1 = \Psi_1(x)$ *and* $\Psi_3 = \Psi_3(x,\delta)$. *Suppose there exist a scalar c, affine matrices $G(\delta)$ and $P(\delta)$, and constant matrices $S_i > 0$, $R_i = -R_i'$, $i = 1,\ldots,n_w$, N and $M_j, j = 1, \cdots, n_e$ solving the following optimization problem at all vertices of $\hat{\mathcal{X}} \times \Delta$.*

$$\max\ c \text{ subject to: (42) and (43).}$$

Then, $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{X}_c$ and $\mathcal{X}_c$ is a domain of $\mathcal{L}_2$ performance with level γ.

7 Concluding Remarks

This paper has generalized the result of [6] to deal with the problem of regional stability and performance analysis for a class of nonlinear uncertain discrete-time systems. We have used polynomial Lyapunov functions to reduce the conservatism in analysis. In order to make the computations feasible, we have applied a decomposition technique to both the nonlinear system and the Lyapunov function. We have considered three analysis problems: domain of attraction, domain of output performance, and domain of $\mathcal{L}_2$ performance. Future research will be concentrated on extending the proposed technique to control design problems for nonlinear uncertain discrete-time systems.

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