Robust Analysis and Control for a Class of Uncertain Nonlinear Discrete-Time Systems*

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Abstract

This paper deals with the problem of robust analysis and control of a class of nonlinear discrete-time systems with (constant) uncertain parameters. We use polynomial Lyapunov functions to derive stability conditions in terms of linear matrix inequalities (LMIs). Although the use of polynomial Lyapunov functions is common for continuous-time systems as a way to reduce the conservatism in analysis, we point out that direct generalization of such approach to discrete-time systems leads to intractable solutions since it results in large LMI dimensions. We introduce a novel approach to reduce the computational complexity by generalizing a result of *Oliveira et. al.* (1999) on robust stability analysis for discrete-time systems with polytopic uncertainties. We then extend this approach to the synthesis problem by considering parameter-dependent Lyapunov functions and nonlinear (state- and parameter-dependent) multipliers. Numerical examples illustrate the approach and show its potential as tool of analysis and control of nonlinear discrete-time systems.

Keywords: convex optimization, discrete-time nonlinear systems, robust control, Lyapunov theory.

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1 Introduction

The last decade or so has witnessed active research work in the area of robust control of continuous-time nonlinear systems in the framework of linear matrix inequalities (LMIs). Design approaches range from using quadratic Lyapunov functions ([1, 2] to those based on polynomial Lyapunov functions ([3, 4]). In general, non-quadratic Lyapunov functions are less conservative for dealing with uncertain and nonlinear systems than the quadratic ones at the expense of extra computation [4]. However, most of the robust control results using non-quadratic Lyapunov functions are for continuous-time systems.

The fundamental difficulty with non-quadratic Lyapunov functions for discrete-time systems lies in the fact that the difference between the Lyapunov functions at time $k+1$ and k is highly nonlinear. To make this point clear, we consider the following system:

$$x(k+1) = A(x(k), \delta)x(k)$$

and the Lyapunov function $V(x, \delta) = x^{'}P(x, \delta)x$, where δ represents (constant) uncertain parameters, and the matrices $A(x, \delta)$ and $P(x, \delta)$ depend on x and δ. The Lyapunov function difference is given by the following inequality (which we will refer to as a Lyapunov inequality):

$$\begin{aligned} V((k+1), \delta) - V(x(k), \delta) &= x(k)^{'}A(x(k), \delta)^{'}P(A(x(k), \delta)x(k), \delta)A(x(k), \delta)x(k) \\ &- x(k)^{'}P(x(k), \delta)x(k) < 0 \end{aligned}$$

which is typically a highly nonlinear function[1] of $x(k)$ and δ. In contrast, if we considered a similar continuous-time system

$$\dot{x}(t) = A(x(t), \delta)x(t)$$

and a similar Lyapunov function, we would have the time-derivative of the Lyapunov function given by the following Lyapunov inequality:

$$\dot{V}(x(t), \delta) = x(t)^{'}\left(A(x(t), \delta)P(x(t), \delta) + P(x(t), \delta)A(x(t), \delta) + \dot{P}(x(t), \delta)\right)x(t) < 0$$

It is obvious from the above that the continuous-time case involve much less nonlinearity, especially when $P(x(t), \delta)$ is chosen to be in a simple form (e.g. affine in $x(t)$ and δ).

The approach used in this paper is motivated by the work of *Oliveira et. al.* [5] which proposed a new test of stability using LMIs for linear discrete-time systems with polytopic uncertainties (this approach was also extended to performance analysis [6]). In this approach,

[1]We mean by a highly nonlinear function the huge number of product terms involving $x(k)$ and δ that appears in the Lyapunov inequality.

the Lyapunov inequality is modified by introducing an auxiliary matrix [5]. The key feature of this auxiliary matrix is that it separates the system matrix $A(\delta)$ from the Lyapunov matrix $P(\delta)$, thus significantly reducing the nonlinearity. Further, the resulting inequality can be expressed as an LMI which is affine in δ. Although, one can think of many possible auxiliary matrices with the above feature, the particular one introduced in [5] appears to be excellent in terms of the conservatism it brings. In particular, if one resorts to a quadratic Lyapunov function, i.e. a constant P matrix, this auxiliary matrix does not bring any conservatism.

We point out that there are other approaches to robust stability and performance for discrete-time systems. For example, the work of *Iwasaki* in [7] uses non-quadratic Lyapunov functions for analyzing the global and regional stability of a class of LTI (linear time-invariant) systems with a nonlinear (or uncertain) feedback connection (Lur'e like systems) satisfying a sector bound condition; and *Tuan et. al* in [8] considers parameterized Lyapunov functions for nonlinear $\mathcal{H}_\infty$ control of quasi-LPV (linear parameter-varying) discrete-time systems, i.e. systems described by $x(k+1) = A(\theta(x(k)))x(k)$ where the state-dependent parameter $\theta(x(k)) \in \Theta$ with Θ being a given polytope. Note that these works consider different modelling techniques in order to use the LMI tools developed for linear (or LPV) systems. In spite of the fact that both overcome the problem caused by the term $A(x(k),\delta)'P(A(x(k),\delta)x(k),\delta)A(x(k),\delta)$, they still have some shortcomings. Namely, [7] treats a restricted class of nonlinear systems; and the approach in [8] is computationally feasible only for systems with a small number of nonlinear terms.

The purpose of this paper is devise a technique to analyze regional stability and design robust controllers for a class of nonlinear discrete-time systems with time-invariant uncertain parameters. We use polynomial Lyapunov functions and give stability conditions in terms of LMIs. However, suitable Lyapunov inequalities will be used to simplify the numerical computations which will be done by generalizing the work of [5]. We will firstly consider the regional stability analysis problem and then use this result for control synthesis. Examples will be used to compare the proposed approach with the approach based on a quadratic Lyapunov function.

The notation used is standard. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, $0_{n\times m}$ is the $n \times m$ matrix of zeros and 0_n is the $n \times n$ matrix of zeros. For a real matrix S, the notation $S > 0$ means that S is symmetric and positive definite and S' is its transpose. Let $\mathcal{F}_1 \subseteq \mathbb{R}^{n_1}$ and $\mathcal{F}_2 \subseteq \mathbb{R}^{n_2}$ be two polytopes. Then, the notation $\mathcal{F}_1 \times \mathcal{F}_2$ represents a meta-polytope of dimension $n_1 + n_2$ obtained by the cartesian product, and $\mathcal{V}(\mathcal{F}_1)$ is the set of all vertices of the polytope $\mathcal{F}_1$. To simplify the notation, the argument k will be often omitted as well as matrix and vector dimensions whenever they can be determined from the context.

2 Problem Statement

Consider the following class of discrete-time nonlinear systems:

$$x_+ = f(x(k),\delta) = A(x(k),\delta)x,\ \ x_+ = x(k+1) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $\delta \in \mathbb{R}^l$ denotes the vector of constant uncertain parameters, and the system matrix $A(x(k),\delta)$ is allowed to depend on x and δ. It is assumed in the sequel that $\mathcal{D}$ is a polytope and that $A(x,\delta)$ is a continuous function in $\mathbb{R}^n \times \mathcal{D}$.

The problem of concern in this paper is of two fold: (i) to determine a region in the state-space in which the robust stability of system (1) is guaranteed, and (ii) to design a robust controller such that the domain of attraction is maximized. To this end, we first introduce the notion of domain of attraction (DOA).

Given a region $\mathcal{R} \subset \mathbb{R}^n$, we say that $\mathcal{R}$ is a DOA for system (1) if for every $x(0) \in \mathcal{R}$ and $\delta \in \mathcal{D}$, the trajectory $x(k)$ remains in $\mathcal{R}$ for all $k \geq 0$ and approaches the origin as $k \to \infty$.

From the Lyapunov theory, we have the following result:

Lemma 1 *Consider system (1). Let $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ be a given Lyapunov function candidate, where $\mathcal{P}(x,\delta)$ is a matrix function of (x,δ). Define a region in the state-space as follows:*

$$\mathcal{X} \triangleq \{x : x \in \mathbb{R}^n,\ x'\mathcal{P}(x,\delta)x \leq 1,\ \forall\ \delta \in \mathcal{D}\} \tag{2}$$

Suppose there exists positive scalars ϵ_1, ϵ_2 and ϵ_3 such that:

$$\epsilon_1 x'x \leq x'\mathcal{P}(x,\delta)x \leq \epsilon_2 x'x,\ \ \forall\ x \in \mathcal{X},\ \ \delta \in \mathcal{D} \tag{3}$$

$$x'\Big(A(x,\delta)'\mathcal{P}(x_+,\delta)A(x,\delta) - \mathcal{P}(x,\delta)\Big)x \leq -\epsilon_3 x'x,\ \ \forall\ x \in \mathcal{X},\ \ \delta \in \mathcal{D} \tag{4}$$

where x_+ is as defined in (1). Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is an estimate of the DOA.

A possible approach to simplifying the product term $A(x,\delta)'\mathcal{P}(x_+,\delta)A(x,\delta)$ is to use the idea of Schur complement, which converts (4) to the following condition:

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P}(x,\delta) & A(x,\delta)'\mathcal{P}(x_+,\delta) \\ \mathcal{P}(x_+,\delta)A(x,\delta) & -\mathcal{P}(x_+,\delta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq -\epsilon_3 x'x, \tag{5}$$

for all $x \in \mathcal{X}$, $y \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$.

The conditions (4) and (5) can be seen as equivalents by noticing that the maximization of y for left-hand side of (5) is given by $y = A(x,\delta)x$. With this choice of y, the two inequalities are the same.

The conditions (3) and (5) are still very complicated to check because of two problems: (1) coupling of $A(x,\delta)$ and $\mathcal{P}(x_+,\delta)$ still gives high nonlinearity; (2) checking the conditions over $\mathcal{X}\times\mathcal{D}$ is highly nontrivial. These are the problems we will address in the next section.

3 Preliminary Results

We give two results in this section. The first one, Lemma 2, is a nonlinear version of the result in [5, Theorem 1] which will allow to remove the coupling between the system and the Lyapunov matrices. The second one, Lemma 3, give a way to remove the nonlinear dependence on x in the conditions of Lemma 1 by a relaxation technique.

Lemma 2 *Consider system (1) and $V(x,\delta)$ and $\mathcal{X}$ as defined in Lemma 1. Suppose (3) and the following inequality holds for some auxiliary matrix function $\mathcal{G}(x,\delta)$:*

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P}(x,\delta) & A(x,\delta)'\mathcal{G}(x,\delta)' \\ \mathcal{G}(x,\delta)A(x,\delta) & \mathcal{P}(x_+,\delta)-\mathcal{G}(x,\delta)-\mathcal{G}(x,\delta)' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le -\epsilon_3 x'x \tag{6}$$

Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is an estimate of DOA.

Remark 1 The conservativeness of Lemma 2 lies in the choice of the auxiliary matrix $\mathcal{G}(x,\delta)$. Observe that we can recover Lemma 1 by considering $\mathcal{G}(x,\delta)=\mathcal{P}(x_+,\delta)$. But this choice of $\mathcal{G}(x,\delta)$ leads to complicated conditions. Consequently, we will have a compromise between the conservatism and complexity when choosing $\mathcal{G}(x,\delta)$.

Note that when $A(x,\delta)$, $\mathcal{P}(x,\delta)$ and $\mathcal{G}(x,\delta)$ do not depend on x, the result above reduces to the original one in [5].

Next, we aim to remove the dependence on x in conditions (3) and (6). To this end, we use a version of the well-known Finsler's Lemma (see, e.g. [9]).

Lemma 3 *Consider the following nonlinear matrix inequality:*

$$\mathcal{T}(\xi) > 0,\ \mathcal{T}(\xi)=\mathcal{T}(\xi)',\ \forall\ \xi\in\mathcal{E} \tag{7}$$

where $\xi\in\mathbb{R}^{n_\xi}$ denotes a generic parameter (that can represent the state and/or uncertainties), the matrix $\mathcal{T}(\xi)\in\mathbb{R}^{n_t\times n_t}$ is a nonlinear function of ξ and $\mathcal{E}\subset\mathbb{R}^{n_\xi}$ is a polytopic region with known vertices. Suppose $\mathcal{T}(\xi)$ can be decomposed as follows:

$$\mathcal{T}(\xi)=\begin{bmatrix} I_{n_t} \\ \mathcal{M}(\xi) \end{bmatrix}' T \begin{bmatrix} I_{n_t} \\ \mathcal{M}(\xi) \end{bmatrix} \tag{8}$$

where $T \in \mathbb{R}^{(n_t+m_t)\times(n_t+m_t)}$ *is a constant matrix,* $\mathcal{M}(\xi) \in \mathbb{R}^{m_t\times n_t}$ *is a nonlinear matrix function of* ξ *with the property that*

$$\Xi_1(\xi) + \Xi_2(\xi)\mathcal{M}(\xi) = 0 \tag{9}$$

for some matrices $\Xi_1(\xi) \in \mathbb{R}^{m_\xi\times n_t}$, $\Xi_2(\xi) \in \mathbb{R}^{m_\xi\times m_t}$ *which are affine functions of* ξ *with* $\Xi_2(\xi)$ *having column full rank for all* ξ *of interest. Then, (7) is satisfied if there exists a constant matrix* L *such that*

$$T + L\Xi + \Xi' L' > 0,\ \xi \in \mathcal{V}(\mathcal{E}) \tag{10}$$

where $\mathcal{V}(\mathcal{E})$ *is the set of all vertices of* $\mathcal{E}$ *and* $\Xi = [\ \Xi_1(\xi) \quad \Xi_2(\xi)\ ]$.

We point out that the decomposition conditions (8) and (9) are very general and can be satisfied for many nonlinear matrix inequalities. The following example demonstrates this point.

Example 1 *Consider the following nonlinear matrix inequality (NLMI):*

$$\begin{bmatrix} (1+\xi_1^2\xi_2^2) & (\xi_1\xi_2^2 - \xi_1) \\ (\xi_1\xi_2^2 - \xi_1) & (1+\xi_2^2) \end{bmatrix} > 0,\ \forall\ \xi \in \mathcal{E} \triangleq \{\xi : |\xi_i| \le \alpha,\ i = 1,2\} \tag{11}$$

where α *is a positive scalar. The problem of interest in this example is to determine the maximum* α *such that the NLMI in (11) is satisfied for all* $\xi \in \mathcal{E}$. *To this end, let us consider the following auxiliary matrices:*

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \mathcal{M}(\xi) = \begin{bmatrix} \xi_1 & 0 \\ \xi_1\xi_2 & 0 \\ 0 & \xi_2 \end{bmatrix} \ \textit{and}\ \Xi = \begin{bmatrix} \xi_1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \xi_2 & -1 & 0 \\ 0 & \xi_2 & 0 & 0 & -1 \end{bmatrix}.$$

Then, applying Lemma 3 to above via any available LMI solver, we conclude that (11) is satisfied for all $\xi \in \mathcal{E}$ *when* $\alpha < 1$.

Conservativeness of NLMIs

The use of standard LMI techniques for testing nonlinear systems can be quite conservative [3]. To make this point clear, consider the condition $\xi'\mathcal{T}(\xi)\xi > 0$. This condition[2] may be

[2]Observe for this particular case that $n_t = n_\xi$.

tested by applying Lemma 3 to $\mathcal{T}(\xi) > 0$ (see Example 1). If there is a solution to (10) for all $\xi \in \mathcal{V}(\mathcal{E})$, then the following is satisfied:

$$v'\mathcal{T}(\xi)v > 0, \ \forall\, \xi \in \mathcal{E}, v \in \mathbb{R}^{n_\xi}.$$

Notice that we are not taking into account the dependence between ξ and $\mathcal{T}(\xi)$, i.e. we are testing the condition for all $v \in \mathbb{R}^{n_\xi}$ instead of the original case in which $v = \xi$. *Trofino* in [3, Lemma 2.1] proposed a solution to this problem in terms of the notion of Linear Annihilators[3].

Basically, the Trofino's approach consists of adding a free multiplier associated with the constraint $\mathcal{N}(\xi)\xi = 0$ to the state-dependent LMI reducing its conservatism. In this paper, we will consider the following Linear Annihilator:

$$\mathcal{N}(\xi) = \begin{bmatrix} \xi_2 & -\xi_1 & 0 & \cdots & \cdots & 0 \\ 0 & \xi_3 & -\xi_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \xi_{n_\xi} & -\xi_{n_\xi - 1} \end{bmatrix} \in \mathbb{R}^{(n_\xi - 1)\times n_\xi} \tag{12}$$

With above definition, we can modify the condition (10) to the following:

$$T + L_1\Xi + \Xi' L_1' + L_2\mathcal{N}(\xi)Q_m + Q_m'\mathcal{N}(\xi)' L_2' > 0, \ \forall\, \xi \in \mathcal{E} \tag{13}$$

where $Q_m = [\ I_{n_\xi} \quad 0_{n_\xi \times m_t}\ ]$ and L_1, L_2 are constant matrices to be determined. Note when we pre- and post-multiply the above LMI by $[\ \xi' \quad \xi'\mathcal{M}(\xi)'\ ]$ and its transpose, respectively, we get $\xi'\mathcal{T}(\xi)\xi > 0$ which is the original expression to be tested.

Example 2 *Consider the following condition:*

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}' \begin{bmatrix} (1+\xi_1^2\xi_2^2) & (\xi_1\xi_2^2 - \xi_1) \\ (\xi_1\xi_2^2 - \xi_1) & (1+\xi_2^2) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} > 0, \ \forall\, \xi \in \mathcal{E} \triangleq \{\xi : |\xi_i| \leq \alpha, \ i = 1,2\} \tag{14}$$

Applying Lemma 3 with (13), we get an optimal $\alpha = 1.1$ *illustrating that the use of Linear Annihilators can lead to less conservative results.*

4 Stability Analysis

We are now ready to derive the main result of this paper. This section deal with the regional stability analysis problem, whereas the next section studies the control problem.

[3]A matrix $\mathcal{N}(\xi)$ is called a Linear Annihilator of ξ if it is a linear function of ξ and $\mathcal{N}(\xi)\xi = 0$

In order to apply the results in the previous section, we need to re-parameterize the system model (1) and define the structure of the Lyapunov matrix $\mathcal{P}(x,\delta)$ and the auxiliary matrix $\mathcal{G}(x,\delta)$ accordingly. These are detailed bellow.

4.1 System Model Representation

We further assume that system (1) can be decomposed as follows:

$$\left\{\begin{array}{rcl} x_+ &=& (A_0 + A_1\Pi_1(x,\delta))\,x, \\ 0 &=& \Omega_0(x,\delta) + \Omega_1(x,\delta)\Pi_1(x,\delta) \end{array}\right. \tag{15}$$

where $A_0 \in \mathbb{R}^{n\times n}$ and $A_1 \in \mathbb{R}^{n\times m}$ are constant matrices; $\Pi_1(x,\delta) \in \mathbb{R}^{m\times n}$ is a nonlinear matrix function of (x,δ); and $\Omega_0(x,\delta) \in \mathbb{R}^{p\times n}$, $\Omega_1(x,\delta) \in \mathbb{R}^{p\times m}$ are affine matrix functions of (x,δ) with $\Omega_1(x,\delta)$ having column full rank for all x and δ of interest. For simplicity of notation, we may hereafter represent the matrices $\Pi_1(x,\delta)$, $\Omega_0(x,\delta)$ and $\Omega_1(x,\delta)$ without their respective dependence on (x,δ) and the system (15) may be also described in the following compact form:

$$x_+ = \mathcal{A}\Pi x,\ \Omega\Pi = 0$$

where

$$\mathcal{A} = \left[\begin{array}{cc} A_0 & A_1 \end{array}\right],\ \Pi = \left[\begin{array}{c} I_n \\ \Pi_1(x,\delta) \end{array}\right] \ \text{and} \ \ \Omega = \left[\begin{array}{cc} \Omega_0(x,\delta) & \Omega_1(x,\delta) \end{array}\right]. \tag{16}$$

Note that the choice of $\mathcal{A}$, Π and Ω is not unique and there is no a systematic way to compute them. In this paper, we will use this degree of freedom in order to parameterize the Lyapunov matrix in terms of the above nonlinear decomposition in order to test the conditions of Lemma 2 via an optimization problem over a set of LMIs.

Remark 2 In fact, a wrong choice of $\mathcal{A}$, Π and Ω may lead to a poor estimate of the DOA and even fail to provide regional stability (see an example of conservativeness for continuous-time systems in [10, Example 3.4]). The conservativeness of choosing these matrices is partly reduced by the inclusion of the constraint $\mathcal{N}(x)x = 0$ in the stability conditions, similarly to [10, Lemma 3.1]. This technique is used in Theorem 1 of this section.

4.2 Lyapunov Function Candidate

Consider the following Lyapunov matrix:

$$\mathcal{P}(x,\delta) = \left[\begin{array}{c} \Theta(x) \\ I_n \end{array}\right]' P(\delta) \left[\begin{array}{c} \Theta(x) \\ I_n \end{array}\right] \tag{17}$$

where $P(\delta) = P(\delta)'$ is an affine matrix functions of δ, and $\Theta(x) \in \mathbb{R}^{q\times n}$ is a given matrix function of x.

Observe from Lemma 2 that we need to compute the following matrix:

$$\mathcal{P}(x_+, \delta) = \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix}' P(\delta) \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix}.$$

To this end, we require the following constraint on $\Theta(x)$:

$$\begin{bmatrix} \Theta(x) \\ I_n \end{bmatrix} = F\Pi = \begin{bmatrix} F_1 \\ Q \end{bmatrix} \Pi \quad \text{and} \quad \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix} = H\Pi = \begin{bmatrix} H_1 \\ Q \end{bmatrix} \Pi \tag{18}$$

where $F_1, H_1 \in \mathbb{R}^{q\times m}$ are constant matrices, Π is the same matrix defined in (16) and

$$Q = \begin{bmatrix} I_n & 0_{n\times m} \end{bmatrix} \tag{19}$$

To illustrate the above system and Lyapunov matrix parameterization, let us consider the following example.

Example 3 *Consider the following nonlinear discrete-time system:*

$$x_+ = \begin{bmatrix} 0 & (0.005 + 1.99x_1) \\ (0.005 - 1.99x_1) & (0.25 + 0.5x_1) \end{bmatrix} x, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{20}$$

Also, define the Lyapunov function candidate by choosing

$$\Theta(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix} \tag{21}$$

Using the nonlinear decomposition in (15), we get that:

$$\begin{cases} x_+ = \left(\begin{bmatrix} 0 & 0.005 \\ 00.005 & 0.25 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1.99 & 0 & 0 & 0 \\ -1.99 & 0 & 0 & 0.5 & 0 & 0 & 0 \end{bmatrix} \Pi_1 \right) x \\ \begin{bmatrix} 0_{7\times 2} \end{bmatrix} = \Omega_0 + \Omega_1 \, \Pi_1 \end{cases} \tag{22}$$

and

$$\begin{bmatrix} \Theta(x) \\ I_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ Q \end{bmatrix} \begin{bmatrix} I_2 \\ \Pi_1 \end{bmatrix}, \; \begin{bmatrix} \Theta(x_+) \\ I_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ Q \end{bmatrix} \begin{bmatrix} I_2 \\ \Pi_1 \end{bmatrix}. \tag{23}$$

Where Ω_0, Ω_1, Π_1, F_1, H_1 and Q are given by:

$$
\begin{aligned}
F_1 &= \begin{bmatrix} 0_{1\times 2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; Q \;=\; \begin{bmatrix} I_2 \\ 0_{2\times 7} \end{bmatrix}, \\
H_1 &= \begin{bmatrix} 0_{1\times 3} & 0.005 & 1.99 & 0 & 0 & 0 & 0 \\ 0_{1\times 3} & 0 & 0 & 0 & 0.005 & 0 & 1.99 \\ 0_{1\times 3} & 0 & 0 & 0.005 & 0.25 & -1.99 & 0.5 \end{bmatrix}, \\
\Omega_0 &= \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ 0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \Omega_1 \;=\; \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & -1 \end{bmatrix}, \\
\Pi_1 &= \begin{bmatrix} x_1 & x_2 & x_1x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_1^2 & x_1x_2 \end{bmatrix}'.
\end{aligned}
$$

4.3 Matrix $\mathcal{G}(x,\delta)$

We choose the auxiliary matrix function $\mathcal{G}(x,\delta)$ to be of the following form:

$$\mathcal{G}(x,\delta) = \Pi' G(\delta) \tag{24}$$

where $G(\delta) \in \mathbb{R}^{m\times n}$ is an affine matrix function of δ to be determined.

With the above choice of $\mathcal{G}(x,\delta)$, we may have a certain degree of conservatism (see Remark 1). Nevertheless, it can lead to a convex characterization of Lemma 1 as we will see in the following.

4.4 Estimating the Domain of Attraction

With the above definitions, we can rewrite the inequality (6) as follows:

$$
\begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix}' \begin{bmatrix} \begin{pmatrix} -F'P(\delta)F+ \\ L_0\mathcal{N}(x)Q + Q'\mathcal{N}(x)'L_0' \end{pmatrix} & \mathcal{A}'G(\delta)' + Q'\mathcal{N}(x)'L_0' \\ G(\delta)\mathcal{A} + L_0\mathcal{N}(x)Q & H'P(\delta)H - G(\delta)Q - Q'G(\delta)' \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix} < 0
$$
$$\forall\, x \in \mathcal{X},\;\; y \in \mathbb{R}^n,\;\; \delta \in \mathcal{D} \tag{25}$$

where L_0 is a free multiplier associated with the constraint $\mathcal{N}(x)x = 0$ and

$$\begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix} = \begin{bmatrix} \Pi x \\ \Pi y \end{bmatrix}.$$

In order to apply Lemma 2 in a numerically tractable manner, we also need a polytopic bounding set $\hat{\mathcal{X}}$ for $\mathcal{X}$. In this way, we will require (25) to hold for all $\hat{\mathcal{X}}$ instead of $\mathcal{X}$. Hence, we want tho choose $\mathcal{X}$ to be reasonably close to $\mathcal{X}$ to reduce the conservativeness but having a small number of vertices so the resulting conditions are easy to check. A possible way to achieve a good compromise is to define the shape of the bounding set and use a parameter to control its size. This parameter can be then adjusted through iterations to obtain an optimal size. But for the discussion in the sequel, we assume that the bounding set $\hat{\mathcal{X}}$ is given.

Without loss of generality, we assume that the bounding set is represented in terms of the following constrains:

$$\hat{\mathcal{X}} = \left\{ x : a_j' x \leq 1,\ j = 1, \ldots, n_e \right\} \tag{26}$$

where $a_j \in \mathbb{R}^{n_x}$ are given vectors associated with the n_e edges of $\hat{\mathcal{X}}$.

Using the $\mathcal{S}$-procedure (see e.g. sections 2.6 and 5.2 of [9]), the condition $\mathcal{X} \subset \hat{\mathcal{X}}$ is satisfied if the following inequality is satisfied for all j:

$$2\left(1 - a_j' x\right) + x' \mathcal{P}(x, \delta) x - 1 \geq 0 \tag{27}$$

Taking into account the structure of $\mathcal{P}(x, \delta)$ in (17), we can rewrite (27) as follows:

$$\begin{bmatrix} 1 \\ \Theta(x)x \\ x \end{bmatrix}' \begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & P(\delta) \end{bmatrix} \begin{bmatrix} 1 \\ \Theta(x)x \\ x \end{bmatrix} \geq 0,\ \forall\ j \tag{28}$$

In order to ensure that the Lyapunov matrix function $\mathcal{P}(x, \delta)$ in (17) is positive definite for all $x \in \hat{\mathcal{X}}$, we apply Lemma 3 and obtain the following condition:

$$P(\delta) + L_1 \Psi_1(x) + \Psi_1(x)' L_1' > 0,\ \forall\ x \in \mathcal{V}(\hat{\mathcal{X}}),\ \delta \in \mathcal{V}(\mathcal{D}) \tag{29}$$

where L_1 is a free matrix to be determined and

$$\Psi_1(x) = \begin{bmatrix} I_q & -\Theta(x) \end{bmatrix}. \tag{30}$$

In order to maximize the volume of $\mathcal{X}$, we normally approximate it by minimizing the trace of the Lyapunov matrix. However, $\mathcal{P}(x, \delta)$ is a nonlinear function of (x, δ) that leads to a non-convex condition. To overcome this problem, we will approximate the volume maximization by

$$\min \max_{\substack{x \in \mathcal{V}(\hat{\mathcal{X}}) \\ \delta \in \mathcal{V}(\mathcal{D})}} \mathsf{trace}\,(P(\delta) + L_1\Psi_1(x) + \Psi(x)'L_1') \tag{31}$$

Now, with above analysis we can state the following theorem which gives a convex solution to the regional stability problem for system (1) in terms of LMIs.

Theorem 1 *Consider the nonlinear discrete-time system (1) as decomposed in (15), the notation in (16) and the matrices $\mathcal{N}(\cdot), Q$ as defined in (12) and (19), respectively. Let $\Theta(x)$ be a given affine matrix function of x satisfying (18) and the Lyapunov matrix $\mathcal{P}(x,\delta)$ be in the form of (17). Let $\hat{\mathcal{X}}$ and $\mathcal{D}$ be given polytopes. Define $\Psi_1(x)$ as in (30) and*

$$\Psi_2(x,\delta) = diag\{\Omega, \Omega\}. \tag{32}$$

Suppose there exist affine matrices $G(\delta)$, $P(\delta)$; constant matrices L_0, L_1, N and M_j ($j = 1,\ldots,n_e$); and a positive scalar η solving the following optimization problem where the LMIs are constructed at $\mathcal{V}(\hat{\mathcal{X}}) \times \mathcal{V}(\mathcal{D})$.

$$\min \quad \eta \quad \textit{subject to:}$$

$$\eta - \mathsf{trace}\left(P(\delta) + L_1\Psi_1(x) + \Psi_1(x)'L_1'\right) \geq 0 \tag{33}$$

$$P(\delta) + L_1\Psi_1(x) + \Psi_1(x)'L_1' > 0 \tag{34}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & \left(P(\delta) + M_j\Psi_1(x) + \Psi_1(x)'M_j'\right) \end{bmatrix} \geq 0, \; j = 1,\ldots,n_e \tag{35}$$

$$\begin{bmatrix} \begin{pmatrix} -F'P(\delta)F + \\ L_0\mathcal{N}(x)Q + Q'\mathcal{N}(x)'L_0' \end{pmatrix} & \mathcal{A}'G(\delta)' + Q'\mathcal{N}(x)'L_0' \\ G(\delta)\mathcal{A} + L_0\mathcal{N}(x)Q & H'P(\delta)H - G(\delta)Q - Q'G(\delta)' \end{bmatrix} + \\ +N\Psi_2(x,\delta) + \Psi_2(x,\delta)'N' < 0 \tag{36}$$

Then, $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is an estimate of DOA, where $\mathcal{X}$ is given by (2).

PROOF. Suppose there is a solution to Theorem 1. Then, by convexity, the LMIs (33)-(36) are satisfied for all $x \in \hat{\mathcal{X}}$ and $\delta \in \mathcal{D}$. In the sequel, this proof will be carried out in the following steps for readability:

Step 1: Denote the LMI (34) as $Y_1 > 0$ and define $U_1 = [\; 0_{n\times q} \quad I_n \;]$. Since (33) is strict, the condition $Y_1 - \epsilon_1 U_1' U_1 \geq 0$ is still satisfied, provided that ϵ_1 is sufficiently small. Pre- and post-multiplying it by $[\; x'\Theta(x)' \quad x' \;]$ and its transpose, respectively, yields:

$$V(x,\delta) = x'\mathcal{P}(x,\delta)x \geq \epsilon_1 x'x, \; \forall\, x \in \hat{\mathcal{X}}, \; \delta \in \mathcal{D} \tag{37}$$

Note that the entries of $P(\delta)$ and $\Phi(x)$ are bounded in $\mathcal{D}$ and $\hat{\mathcal{X}}$, respectively. Then there is a sufficient large ϵ_2 such that

$$V(x,\delta) = x'\mathcal{P}(x,\delta)x \leq \epsilon_2 x'x, \ \forall\ x \in \hat{\mathcal{X}}, \ \delta \in \mathcal{D} \tag{38}$$

Step 2: Consider LMI (36). For simplicity, denote (36) by $Y_2 < 0$ and define:

$$U_2 = \begin{bmatrix} Q'Q & 0 \\ 0 & 0_{m\times m} \end{bmatrix}$$

Similarly, $Y_2 + \epsilon_3 U_2 \leq 0$ for some sufficient small $\epsilon_3 > 0$. Pre- and post-multiplying it by $[\ x'\Pi(x,\delta)' \quad y'\Pi(x,\delta)'\ ]$ and its transpose, respectively, leads to (6) for all $x \in \hat{\mathcal{X}}$ and $\delta \in \mathcal{D}$.

Step 3: Consider the set of LMIs in (35). Pre- and post-multiplying each of them by $[\ 1 \quad x'\Theta(x)' \quad x'\ ]$ and its transpose, respectively, yields $1 - a_j'x + x'\mathcal{P}(x,\delta)x \geq 0$ for all $x \in \hat{\mathcal{X}}$ and $\delta \in \mathcal{D}$. From section 5.2 of [9], this implies that $\mathcal{X} = \{x : V(x,\delta) \leq 1\} \subset \hat{\mathcal{X}}$ and the rest of this proof follows from Lemma 2. □

To illustrate the result above, we give the following example.

Example 4 *Consider system (20) and take $\Theta(x)$ as in (21). Let $\hat{\mathcal{X}}$ be the polytope with the vertices given by:*

$$\left\{ \begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} -a \\ a \end{bmatrix}, \begin{bmatrix} -a \\ -a \end{bmatrix}, \begin{bmatrix} a \\ -a \end{bmatrix} \right\} \tag{39}$$

where a is a positive scalar to be tuned in order to minimize η in Theorem 1.

Figure 1 shows the computed estimates of DOA for system (20) by considering the following Lyapunov functions: (a) *a quadratic one represented by $\mathcal{X}_0$ (obtained at the optimal $a = 0.50$), and* (b) *the polynomial function represented by $\mathcal{X}$ (for an optimal $a = 0.68$). For the purpose of comparison, the initial conditions of some unstable trajectories are also shown in the figure. Observe that polynomial Lyapunov function has led to a less conservative estimate of DOA, thus justifying the required extra computation.*

5 Control Design

The LMI methods for control synthesis of discrete-time LPV systems use the dual version of Lemma 1 (i.e. stability conditions in terms of the Lyapunov matrix inverse) and then parameterize the control gain leading to convex conditions (See, e.g. [11]). However, an extension of this technique to deal with nonlinear systems yields non-convex conditions since it appears in the stabilization matrix inequalities the vector x_+ which is a function of

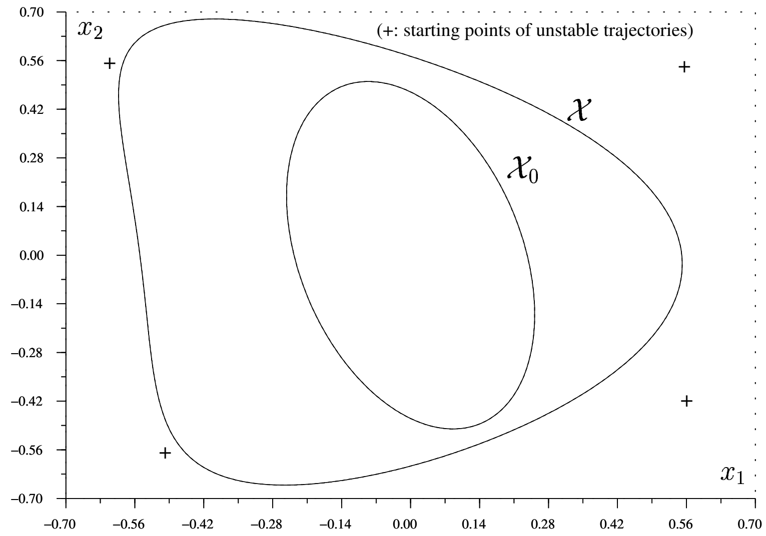

Figure 1: Estimates of the domain of attraction for system (20).

the control matrix loosing the convexity. To make this point clear, consider in the following the dual version of Lemma 2.

Consider the following nonlinear system:

$$x_+ = A(x,\delta)x + B(x,\delta)u,\ u = K(x,\delta)x \tag{40}$$

where $u \in \mathbb{R}^r$ is the control input, $B(x,\delta) \in \mathbb{R}^{n\times r}$ and $K(x,\delta) \in \mathbb{R}^{r\times n}$ are nonlinear matrix functions of (x,δ). Now, we are ready to state the following basic result for control design (a nonlinear version of [11, Theorem 1]).

Lemma 4 *Consider system (40), $V(x,\delta) = x'(\mathcal{Y}(x,\delta))^{-1}x$ and $\mathcal{X} = \{x : x \in \mathbb{R}^n, V(x,\delta) \le 1,\ \forall\ \delta \in \mathcal{D}\}$. Suppose the following inequalities are satisfied for $\mathcal{Y}(x,\delta)$ and some auxiliary matrix functions $\mathcal{G}(x,\delta)$ and $K(x,\delta)$ of appropriate dimensions:*

$$w'\mathcal{Y}(x,\delta)w > 0,\ \forall\ x \in \mathcal{X},\ w \in \mathbb{R}^n,\ \delta \in \mathcal{D} \tag{41}$$

$$\begin{bmatrix} z \\ w \end{bmatrix}' \begin{bmatrix} \mathcal{Y}(x,\delta) - \mathcal{G}(x,\delta) - \mathcal{G}(x,\delta)' & \begin{pmatrix} A(x,\delta)\mathcal{G}(x,\delta)+ \\ B(x,\delta)K(x,\delta)\mathcal{G}(x,\delta) \end{pmatrix} \\ \begin{pmatrix} \mathcal{G}(x,\delta)'A(x,\delta)'+ \\ \mathcal{G}(x,\delta)'K(x,\delta)'B(x,\delta)' \end{pmatrix} & -\mathcal{Y}(x_+,\delta) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0,$$
$$\forall\ x \in \mathcal{X},\ z \in \mathbb{R}^n,\ w \in \mathbb{R}^n,\ \delta \in \mathcal{D} \tag{42}$$

Then, the following holds: (i) the closed-loop system with $u = K(x,\delta)x$ is asymptotically stable; (ii) $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$; and (iii) $\mathcal{X}$ is an estimate of the closed-loop DOA.

Remark 3 When using the dual version of Lemma 2 for control purposes, we cannot apply the definition of Linear Annihilators (since there is no a linear coupling between the vectors w and x as in the primal version). In this way, we are adding some conservativeness to the control design problem.

Observe from above that the matrix $\mathcal{Y}(x_+, \delta)$ is a nonlinear function of x_+, i.e. it is a function of the control matrix $K(x, \delta)$, not allowing a parameterization of the nonlinear matrix inequality as defined in the analysis case. A possible solution could be decoupling the vectors x and x_+ by considering that x_+ as a parameter that belongs to a known polytope turning the vector x_+ independent of x as in the quasi-LPV representation (see e.g. [12, 8]). On the one hand we are considering polynomial Lyapunov functions for control purposes, on the other hand we are adding a certain degree of conservativeness since we are not taking into the account the system dynamics. To overcome this problem, we will use in this paper a parameter-dependent Lyapunov matrix of the form $\mathcal{Y}(x, \delta) = Y(\delta)$ leading to the following lemma.

Lemma 5 *Consider system (40), $V(x, \delta) = x'(Y(\delta))^{-1}x$ and $\mathcal{X} = \{x : x \in \mathbb{R}^n, V(x, \delta) \leq 1, \ \forall \ \delta \in \mathcal{D}\}$. Suppose the following inequalities are satisfied for $Y(\delta)$ and some auxiliary matrix functions $\mathcal{G}(x, \delta)$ and $\mathcal{Z}(x, \delta)$ of appropriate dimensions:*

$$\begin{bmatrix} z \\ w \end{bmatrix}' \begin{bmatrix} Y(\delta) - \mathcal{G}(x,\delta) - \mathcal{G}(x,\delta)' & \begin{pmatrix} A(x,\delta)\mathcal{G}(x,\delta)+ \\ B(x,\delta)\mathcal{Z}(x,\delta) \end{pmatrix} \\ \begin{pmatrix} \mathcal{G}(x,\delta)'A(x,\delta)'+ \\ \mathcal{Z}(x,\delta)'B(x,\delta)' \end{pmatrix} & -Y(\delta) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0,$$

$$\forall \ x \in \mathcal{X}, \ \ z \in \mathbb{R}^n, \ \ w \in \mathbb{R}^n, \ \ \delta \in \mathcal{D} \qquad (43)$$

Then, the following holds: (i) the closed-loop system with $K(x, \delta) = \mathcal{Z}(x, \delta)(\mathcal{G}(x, \delta))^{-1}$ is asymptotically stable; (ii) $V(x, \delta)$ is a Lyapunov function in $\mathcal{X}$; and (iii) $\mathcal{X}$ is an estimate of the closed-loop DOA.

PROOF. From Schur complement, we get $w'Y(\delta)w > 0$ for all $w \in \mathbb{R}^n$ and all $\delta \in \mathcal{D}$ which implies (41) with $\mathcal{Y}(x, \delta) = Y(\delta)$. Let $\mathcal{Z}(x, \delta)$ be given by $\mathcal{Z}(x, \delta) = K(x, \delta)\mathcal{G}(x, \delta)$. Also, notice from (43) that $\mathcal{G}(x, \delta) + \mathcal{G}(x, \delta)' - Y(\delta) > 0$ and thus $\mathcal{G}(x, \delta)$ is non-singular. Finally, the rest of this proof follows from Lemma 4. □

Similarly to the analysis problem, we can transform the stabilization conditions in Lemma 5 into convex ones. To this end, consider that system (40) can be rewritten as follows:

$$\left\{ \begin{array}{rcl} x_+ & = & \left(A_0 + \tilde{\Pi}_1(x,\delta)'\tilde{A}_1\right)x + \left(B_0 + \tilde{\Pi}_1(x,\delta)'B_1\right)u, \\ 0 & = & \tilde{\Omega}_1(x,\delta) + \tilde{\Omega}_2(x,\delta)\tilde{\Pi}_1(x,\delta) \end{array} \right. \qquad (44)$$

where $A_0 \in \mathbb{R}^{n\times n}, \tilde{A}_1 \in \mathbb{R}^{\tilde{m}\times n}, B_0 \in \mathbb{R}^{n\times r}, B_1 \in \mathbb{R}^{\tilde{m}\times r}$ are constant matrices; $\tilde{\Pi}_1(x,\delta) \in \mathbb{R}^{\tilde{m}\times n}$ is a nonlinear matrix function of (x,δ); and $\tilde{\Omega}_1(x,\delta) \in \mathbb{R}^{\tilde{p}\times n}, \tilde{\Omega}_2(x,\delta) \in \mathbb{R}^{\tilde{p}\times \tilde{m}}$ are affine matrix function of (x,δ) with $\tilde{\Omega}_2(x,\delta)$ having column full rank for all x and δ of interest. Similarly to Section 4, we may represent system (44) in the following compact form:

$$x_+ = \tilde{\Pi}' \left(\tilde{\mathcal{A}}x + \mathcal{B}u\right), \ \tilde{\Omega}\tilde{\Pi} = 0$$

where the matrices $\tilde{\mathcal{A}}, \mathcal{B}, \tilde{\Omega}$ and $\tilde{\Pi}$ are given by

$$\tilde{\mathcal{A}} = \begin{bmatrix} A_0 \\ \tilde{A}_1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_1(x,\delta) & \tilde{\Omega}_2(x,\delta) \end{bmatrix}, \tilde{\Pi} = \begin{bmatrix} I_n \\ \tilde{\Pi}_1(x,\delta) \end{bmatrix}. \tag{45}$$

Also, consider the following form for the matrices $\mathcal{G}(x,\delta)$ and $\mathcal{Z}(x,\delta)$:

$$\mathcal{G}(x,\delta) = G(\delta)\Pi(x,\delta) \quad \text{and} \quad \mathcal{Z}(x,\delta) = Z(\delta)\Pi(x,\delta) \tag{46}$$

where $G(\delta) \in \mathbb{R}^{n\times\tilde{m}}$ and $Z(\delta) \in \mathbb{R}^{r\times\tilde{m}}$ are affine matrix of δ to be determined. In addition, define the following matrix:

$$\tilde{Q} = \begin{bmatrix} I_n & 0_{n\times\tilde{m}} \end{bmatrix} \tag{47}$$

Thus, we can state the following convex characterization of Lemma 5.

Theorem 2 *Consider the nonlinear discrete-time system (40) as decomposed in (44), the notation (45) and the matrix $\tilde{Q}$ as defined in (47). Let $\hat{\mathcal{X}}$ and $\mathcal{D}$ be given polytopes. Suppose there exist affine matrices $Y(\delta), G(\delta), Z(\delta)$, and constant matrices L_a, L_b solving the following optimization problem where the LMIs are constructed at $\mathcal{V}(\hat{\mathcal{X}}) \times \mathcal{V}(\mathcal{D})$.*

$$\max \ \ \mathsf{trace}\{Y(\delta)\} \quad \textit{subject to:}$$

$$1 - a_j' Y(\delta) a_j \geq 0, \ j = 1, \ldots, n_e \tag{48}$$

$$\begin{bmatrix} \begin{pmatrix} \tilde{Q}'Y(\delta)\tilde{Q} - \tilde{Q}'G(\delta) - G(\delta)'\tilde{Q} + \\ L_a\tilde{\Omega} + \tilde{\Omega}L_a' \end{pmatrix} & G(\delta)'\tilde{\mathcal{A}}' + Z(\delta)'\mathcal{B}' \\ \tilde{\mathcal{A}}G(\delta) + \mathcal{B}Z(\delta) & \begin{pmatrix} -\tilde{Q}'Y(\delta)\tilde{Q} + \\ L_b\tilde{\Omega} + \tilde{\Omega}'L_b' \end{pmatrix} \end{bmatrix} < 0 \tag{49}$$

Then, the following holds: (i) the closed-loop system with $K(x,\delta) = \mathcal{Z}(x,\delta)(\mathcal{G}(x,\delta))^{-1}$ is asymptotically stable; (ii) $V(x,\delta) = x'(Y(\delta))^{-1}x$ is a Lyapunov function in $\mathcal{X}$; and (iii) $\mathcal{X} = \{x : V(x,\delta) \leq 1, \ \forall \ \delta \in \mathcal{D}\}$ is an estimate of the closed-loop DOA.

PROOF. Pre- and post-multiplying LMI (49) by $\begin{bmatrix} z'\tilde{\Pi}(x,\delta)' & w'\tilde{\Pi}(x,\delta)' \end{bmatrix}$ and its transpose, respectively, leads to (43). From [9], the inequality (48) implies $\mathcal{X} \subset \hat{\mathcal{X}}$ for all $\delta \in \mathcal{D}$. Thus, system (40) with $u = \mathcal{Z}(x,\delta)(\mathcal{G}(x,\delta))^{-1}x$ is asymptotically stable in $\mathcal{X}$. □

To illustrate the application of Theorem 2, we propose the following example:

Example 5 *Consider the following time-invariant system:*

$$x_+ = \begin{bmatrix} 0 & 0.1 + x_1 x_2 \\ 0.5 + 0.1\delta & 0.1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{50}$$

where $x \in \hat{\mathcal{X}} = \{x : |x_i| \leq 1, \ i = 1, 2\}$ *and* $\delta \in \mathcal{D} = \{\delta : |\delta| \leq 1\}$.

Note that system (50) can be decomposed as in (44) with:

$$A_0 = \begin{bmatrix} 0.0 & 0.1 \\ 0.5 & 0.1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 1.0 \\ 1.0 & 0.0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{\Pi}_1(x,\delta) = \begin{bmatrix} x_1 & 0 \\ x_1 x_2 & 0 \\ 0 & \delta \end{bmatrix}, \ \tilde{\Omega}_1(x,\delta) = \begin{bmatrix} x_1 & 0 \\ 0 & 0 \\ 0 & \delta \end{bmatrix}, \ \tilde{\Omega}_2(x,\delta) = \begin{bmatrix} -1 & 0 & 0 \\ x_2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Figures 2 and 3 show respectively the estimate of the closed-loop DOA and state trajectories of system (50) by applying Theorem 2 for an initial condition $x_0 = [\ -0.5 \quad 0.5\ ]'$.

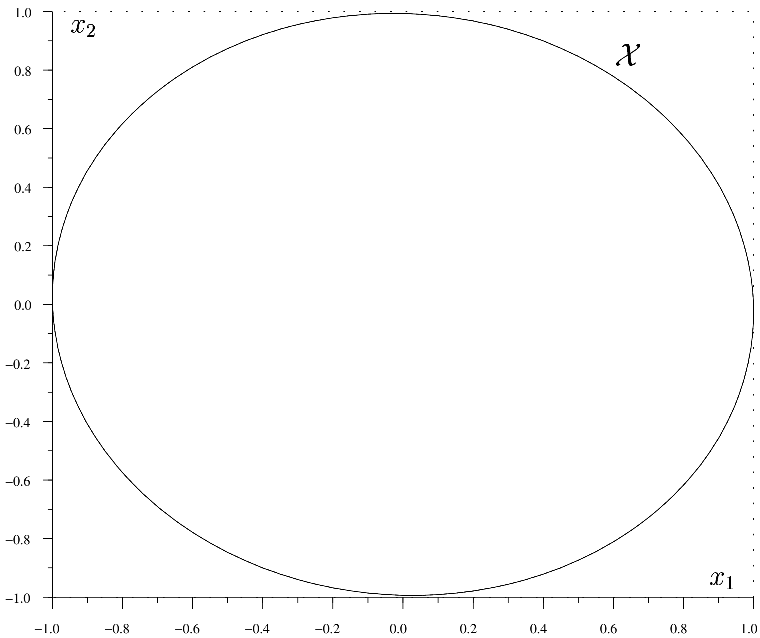

Figure 2: Estimate of the closed-loop DOA.

We pointed out that using Lemma 4 with $x_+ \in \hat{\mathcal{X}}$ *and a polynomial Lyapunov function led approximately to the same estimate of DOA as the proposed approach (with extra computations), where we have used:*

$$\mathcal{Y}(x,\delta) = \begin{bmatrix} \tilde{\Theta}(x) \\ I_2 \end{bmatrix}' Y(\delta) \begin{bmatrix} \tilde{\Theta}(x) \\ I_2 \end{bmatrix} \quad \text{and} \quad \tilde{\Theta}(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}.$$

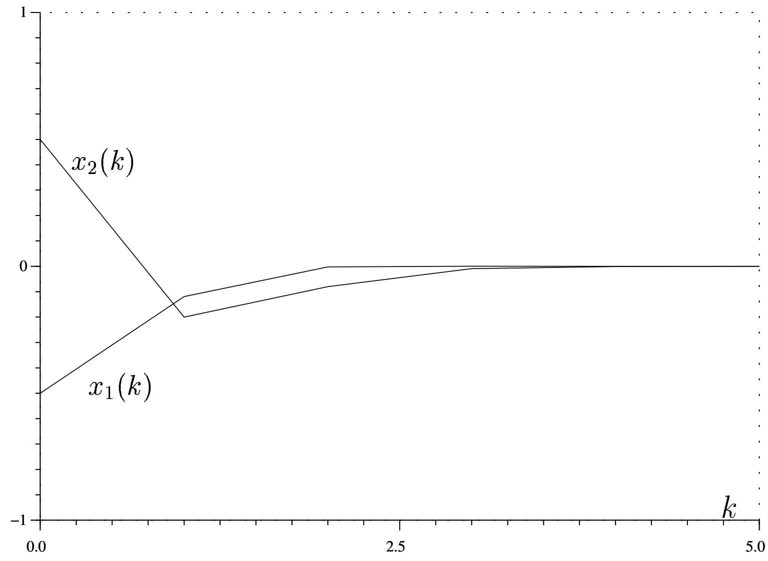

Figure 3: Closed-loop state trajectories for $x(0) = x_0$.

6 Concluding remarks

This paper has generalized the result of [5] to deal with the problem of regional stability and control of nonlinear uncertain discrete-time systems. For the analysis case, we have used a polynomial Lyapunov function to reduce the conservativeness and applied a decomposition technique to both the nonlinear system and the Lyapunov function in order to make the computations feasible. We then extended these results for designing stabilizing controllers by considering parameter-dependent Lyapunov matrices and a nonlinear (state- and parameter-dependent) multiplier overcoming the non-convex synthesis conditions of polynomial Lyapunov functions. Numerical examples have been used to illustrate the advantages of the proposed approach. Future research will be concentrate on extending the proposed technique for performance analysis (and control) of nonlinear uncertain discrete-time systems.

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