Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

Dyadic Factorization of All Pass IIR Linear Time Varying Analysis Banks in Multirate Signal Processing

Chris Schwarz[1] Soura Dasgupta[2] Minyue Fu[3]

Abstract

In this paper, we consider a factorization scheme for linear time varying (LTV) IIR all pass systems to be used specifically in the analysis bank of multirate subband coders. The factorization scheme is based on a certain base dyadic structure. Such LTV analysis banks are required to be square systems. It is known that linear time invariant (LTI) square all pass systems admit such dyadic factorizations and that a limited class of FIR LTV all pass square systems have a specialized dyadic factorization. We extend these results to all pass IIR LTV square systems that have uniformly completely observable and uniformly completely controllable realizations.

1 Introduction

Motivated by Linear Time Varying (LTV) Filter Bank (FB) theory as applied to subband coding and other multirate signal processing problems, this paper considers the factorization of M-input M-output, LTV all pass operators.

In recent years there has been considerable interest in the theory and design ot LTVFB [1, 2]. Two requirements often imposed on FB design are that the analysis bank (AB) be collectively all pass (individual analysis filters of course will not be all pass), and that the overall FB have the perfect reconstruction (PR) property.

More precisely [3], it is known that a LTVFB can be represented as in figure 1, where q^{-1} is the unit delay operator, k is the time index, $u(k)$ is the FB input, $\hat{u}(k)$ is the output, $E(k, q^{-1})$ is the M-input M-output ($M \times M$) type I polyphase matrix of the AB, and $R(q^{-1}, k)$ is the $M \times M$ type II polyphase matrix of the synthesis bank (SB). For the purposes of this paper, $E(k, q^{-1})$ and $R(q^{-1}, k)$ can be treated as two $M \times M$ matrix LTV systems, the presence of k indicating their time varying nature. Then the all pass requirement on the AB boils down to the requirement that with $E(k, q^{-1})$ at initial rest, for all square summable $u_i(k)$,

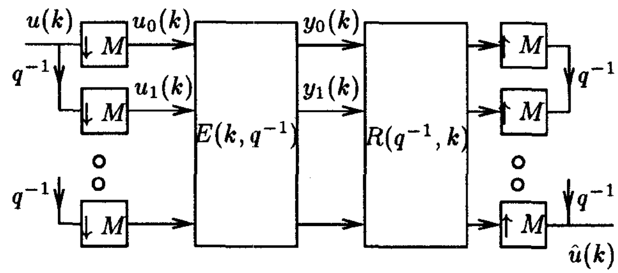

Figure 1: A PR filter bank

$$\sum_{k=-\infty}^{\infty} \sum_{i=0}^{M-1} |u_i(k)|^2 = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{M-1} |y_i(k)|^2 \qquad (1)$$

The PR requirement boils down to

$$R(q^{-1}, k)E(k, q^{-1}) = I \qquad (2)$$

Generally $E(k, q^{-1})$ is causal, and under the all pass requirement, $R(q^{-1}, k)$ anticausal. See [4] for details on how an anticausal $R(q^{-1}, k)$ can be implemented through the transmission of judiciously chosen samples of the states of the AB .

An attractive property of linear time invariant (LTI) all pass, $M \times M$ systems is that they admit rather elegant dyadic-based realizations. One such is given below. In figure 2 we have what we call the base dyadic cascade, with H_0 an $M \times M$ *unitary matrix*, i.e.

$$H_0^\dagger H_0 = I \qquad (3)$$

'†' denoting the conjugate transpose, $H_i(q^{-1})$, as in figure 3 with $U(k) = [u_0(k), \ldots, u_{M-1}(k)]'$, $Y(k) = [y_0(k), \ldots, y_{M-1}(k)]'$, $v_i^\dagger v_i = I$, v_i, $M \times 1$, and α_i a scalar obeying $|\alpha_i| < 1$. Note, henceforth the absence of the time index k from an operator such as $H_i(q^{-1})$ will denote its time invariance.

There is another dyadic based factorization involving Householder matrices given in [5]. Details on these factorizations will be presented later.

[1]Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grant ECS-9350346.
[2]Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grant ECS-9350346.
[3]Department of Electrical and Computer Engineering, The University of Newcastle, 2308, Australia. Was on leave to The University of Iowa, Iowa City, IA-52242, USA when this work was completed. Supported in part by NSF grant ECS-9350346.

0-7803-3970-8/97 $10.00 © 1997 IEEE

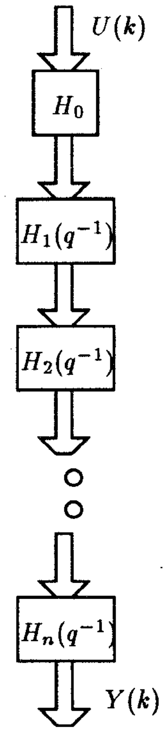

Figure 2: A dyadic cascade

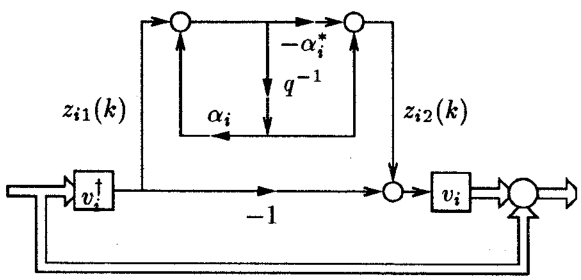

Figure 3: The base dyadic structure

The main issue addressed in this paper is as follows. Given an $M \times M$, causal, all pass, IIR $E(k, q^{-1})$, does $E(k, q^{-1})$ admit similar factorizations? To avoid structural variability, we will require the number of delays, and hence the number of blocks in figure 2 to be time invariant.

The only LTV result of this type that we are aware of is in [3]. It concerns FIR blocks as in figure 3 with $v_i^\dagger$ now time varying, hence denoted $v_i^\dagger(k)$, and obeying

$$v_i^\dagger(k) v_i(k) = 1 \quad \forall\, k, \text{ and } \alpha_i = 0. \tag{4}$$

Essentially, [3] demonstrates that certain types of FIR, LTV all pass $M \times M$ systems can be so factorized, but that even within the FIR LTV all pass system, this class could be quite narrow. In the $M = 1$ case for example only trivial systems admit such a factorization.

The main result of this paper is to demonstrate that, under mild assumptions, all $M \times M$ IIR LTV all pass systems can be implemented as slight variations of the dyadic implementations of figures 2 and 3. We also show that implementations that are analogs of the other two dyadic based LTI factorizations given in [5], also follow. An interesting aside to these results is that while certain FIR LTV all pass $M \times M$ systems do not have a factorization as in [3], they can be implemented through the factorization given here.

Section 2 gives certain preliminary facts concerning LTV systems in general and all pass LTV systems in particular. Section 3 concerns certain observability/controllability properties. Section 4 states the main results. Section 5 concludes.

2 Preliminaries

This section contains certain preliminary definitions concerning LTV systems. The analysis in this paper is concerned with State Variable Realizations (SVR). A p-input M-output $M \times p$ system, with $p \times 1$ input vector $U(k)$ and $M \times 1$ output vector $Y(k)$ is said to have an SVR, $\{A(k), B(k), C(k), D(k)\}$, with $n \times 1$ state vector $x(k)$, if one has for all k,

$$\begin{bmatrix} x(k+1) \\ Y(k) \end{bmatrix} = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix} \begin{bmatrix} x(k) \\ U(k) \end{bmatrix} \tag{1}$$

where $A(k)$ is $n \times n$, $B(k)$ is $n \times p$, $C(k)$ is $M \times n$, $D(k)$ is $M \times p$ and the *realization matrix* is denoted by

$$\Sigma(k) = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix}. \tag{2}$$

We call the realization (1) real if $\Sigma(k)$ is real for all k. This system is said to be exponentially asymptotically stable (eas) if $A(k)$, $B(k)$, $C(k)$, $D(k)$ are bounded and for zero $U(k)$ there exist some positive constant, c, and some constant, δ, with $0 < \delta < 1$, such that for all initial times, k_0, the state obeys

$$||x(k)|| \leq c||x(k_0)||\delta^{k-k_0}. \quad (3)$$

A linear system with input $U(k)$ and $Y(k)$ is all pass if for all $U(k) \in l_2$,

$$\sum_{k=-\infty}^{\infty} Y^{\dagger}(k)Y(k) = \sum_{k=-\infty}^{\infty} U^{\dagger}(k)U(k) \quad (4)$$

whenever the system is at initial rest.

We also need the concepts of uniform completely controllability and observability. To this end we first define the state transition matrix

$$\Phi(k, k_0) = A(k-1)A(k-2)\cdots A(k_0) \quad (5)$$

with

$$\Phi(k, k) = I. \quad (6)$$

The system described in (1) is called *uniformly completely controllable* (ucc) if there exist positive K_1, β_1, and β_2 such that for all times, k,

$$\beta_1 I \leq \sum_{i=k}^{k+K_1} \Phi(k+K_1, i)B(i-1)B^{\dagger}(i-1)\Phi^{\dagger}(k+K_1, i) \leq \beta_2 I. \quad (7)$$

The same system is called *uniformly completely observable* (uco) if there exist positive K_2, β_3, and β_4 such that for all times, k,

$$\beta_3 I \leq \sum_{i=k}^{k+K_2} \Phi^{\dagger}(i, k)C^{\dagger}(i)C(i)\Phi(i, k) \leq \beta_4 I. \quad (8)$$

Essentially, uco and ucc generalize the concepts of observability and controllability to the LTV case. In the sequel (1) is called a *minimal* realization of the system it describes if (1) is uco and ucc and $\Sigma(k)$ is bounded. Further if a system has a minimal realization as in (1), it is said to have McMillan degree n.

Observe that the McMillan degree of a system, should it exist, measures the minimum number of delays that are needed to implement the system. It is moreover, time invariant. As opposed to this one has the *instantaneous degree* of (1). Suppose, with zero initial conditions the input output description exemplified by (1) is $Y(k) = H(k, q^{-1})U(k)$ with $H(k, q^{-1})$ rational for each k. Then the instantaneous degree $\rho(k)$ at time k is the McMillan degree of the transfer function $H(k, q^{-1})$ frozen at time k. Clearly $\rho(k)$ need not be time invariant, even if the McMillan degree as defined above exists.

Consider for example the LTV 1×1 system,

$$Y(k) = \begin{cases} U(k) & k \text{ even} \\ -U(k-2) & k \text{ odd} \end{cases} \quad (9)$$

This system has

$$\rho(k) = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases} \quad (10)$$

Yet it has the uco, ucc realization

$$\begin{aligned} x(k+1) &= -\alpha(k)x(k) + \sqrt{1-\alpha^2(k)}U(k) \quad (11) \\ Y(k) &= \sqrt{1-\alpha^2}x(k) + \alpha(k)U(k) \quad (12) \\ \alpha(k) &= \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \quad (13) \end{aligned}$$

Thus, its McMillan degree is 1. A few features about this example are noteworthy. First in this case the McMillan degree is actually *smaller* than the largest instantaneous degree. Second, despite the fact that this system is FIR at every instant, the frozen SVR in (11,12), is *not*. Thus the minimal realization of an LTV FIR system may well have frozen IIR values. Moreover the mere inspection of the input output operator may lead to erroneous conclusions about the minimum number of delays required to implement an LTV operator. Further, results to be presented later will show that this system is all pass. For such an all pass FIR system to have a factorization as in figures 2, 3 with (4) holding, [3] shows that it is necessary that $\rho(k+1) \geq \rho(k)$. Thus this system cannot have such a factorization. Yet, it will have the factorization to be presented in this paper.

Next we present the LTV version of the discrete time All Pass Lemma. We have not found this result in the open literature, though we have been informed by A. VanderVeen, that such a Lemma does indeed appear in his PhD thesis [7]. Henceforth if (1) is uco, ucc or eas, we will say that $\Sigma(k)$ is uco,ucc or eas respectively. All realizations that obey all three of these properties and have a bounded realization $\Sigma(k)$, will be called compatible.

Theorem 2.1 *Suppose the LTV system (1) is compatible. Then this system is all pass iff there exists an $n \times n$ matrix $P(k)$, scalar β_5, $\beta_6 > 0$, such that for all k,*

$$\beta_5 I \leq P^{\dagger}(k) = P(k) \leq \beta_6 I \quad (14)$$

and

$$\Sigma^{\dagger}(k) \begin{bmatrix} P(k+1) & 0 \\ 0 & I \end{bmatrix} \Sigma(k) = \begin{bmatrix} P(k) & 0 \\ 0 & I \end{bmatrix}. \quad (15)$$

Further if (1) is real, then $P(k)$ is real for all k

3 SVR's of Cascade Blocks

The basic factorization to be undertaken in this paper is as depicted in figures 2 and 3. It then becomes important to consider how the SVR of the overall system in figure 2 relates to the SVR's of the cascade components. In addition, we also need to study the uco, ucc and unitary nature of the individual blocks. To this end we will consider two distinct settings, respectively depicted in figures 4 and 5.

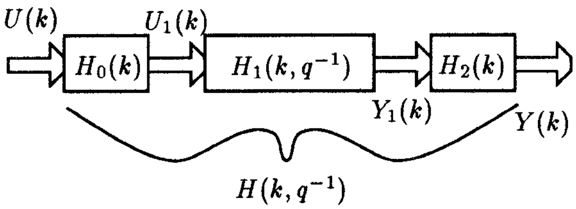

Figure 4: A cascade of dynamic and static systems

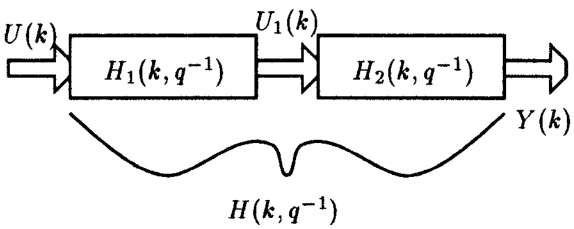

Figure 5: A dynamic cascade

The first Lemma concerns the SVR of the setting in figure 4. In the sequel dynamic operators such as $H_i(k,q^{-1})$ will have SVR's $\{A_i(k), B_i(k), C_i(k), D_i(k)\}$. Related quantities such as realization and state transition matrices will be denoted by $\Sigma_i(k)$ and $\Phi_i(k)$ respectively. Throughout we will assume that there exists M_i such that for all k

$$\|\Sigma_i(k)\| \leq M_i \tag{1}$$

Lemma 3.1 *Consider the LTV system in figure 4, with $H(k,q^{-1})$ and $H_1(k,q^{-1})$ $M \times M$, LTV operators, and $H_0(k)$ and $H_2(k)$ $M \times M$ matrices. Then an SVR of $H(k,q^{-1})$ is given by*

$$\{A_1(k), B_1(k)H_0(k), H_2(k)C_1(k), H_2(k)D_1(k)H_0(k)\} \tag{2}$$

i.e.

$$\Sigma(k) = \begin{bmatrix} I & 0 \\ 0 & H_2(k) \end{bmatrix} \Sigma_1(k) \begin{bmatrix} I & 0 \\ 0 & H_0(k) \end{bmatrix} \tag{3}$$

The next Lemma is with respect to figure 5.

Lemma 3.2 *In figure 5 suppose both $H_1(k,q^{-1})$, $H_2(k,q^{-1})$ are $M \times M$, LTV dynamical systems. Then the SVR of $H(k,q^{-1})$ is as below*

$$\{A(k), B(k), C(k), D(k)\} = \left\{ \begin{bmatrix} A_1(k) & 0 \\ B_2(k)C_1(k) & A_2(k) \end{bmatrix}, \begin{bmatrix} B_1(k) \\ B_2(k)D_1(k) \end{bmatrix}, \begin{bmatrix} D_2(k)C_1(k) & C_2(k) \end{bmatrix}, D_2(k)D_1(k) \right\} \tag{4}$$

i.e.

$$\Sigma(k) = \left[\begin{array}{cc|c} A_1(k) & 0 & B_1(k) \\ B_2(k)C_1(k) & A_2(k) & B_2(k)D_1(k) \\ \hline D_2(k)C_1(k) & C_2(k) & D_2(k)D_1(k) \end{array} \right] \tag{5}$$

We now turn to such properties of the SVR's corresponding to the settings depicted in figure 4 and 5 as uco, ucc and unitariness.

Lemma 3.3 *Under the hypothesis of Lemma 3.1, assume that for all k*

$$H_0^{\dagger}(k)H_0(k) = H_2^{\dagger}(k)H_2(k) = I. \tag{6}$$

Then with $\Sigma(k)$ as in Lemma 3.1, $\Sigma_1(k)$ is uco, ucc and unitary for all k, iff so is $\Sigma(k)$.

We conclude with the setting of figure 5.

Lemma 3.4 *Suppose the hypothesis of Lemma 3.2 holds, $\Sigma(k)$, $\Sigma_1(k)$, and $\Sigma_2(k)$ are as in that Lemma, and obey the relationships in (4, 5), and $\Sigma_2(k)$ is unitary for all k. Then:*

(i) $\Sigma_1(k)$ is unitary iff $\Sigma(k)$ is unitary.

(ii) $\Sigma_1(k)$ and $\Sigma_2(k)$ are both compatible if $\Sigma(k)$ is compatible.

4 Main Results

In this section we provide the statements of the main results of this paper. In Section 4.1 we talk about the base dyadic factorization as well as explaining the significance of some of the underlying assumptions. Section 4.2 describes the second dyadic structure. Section 4.3 discusses the anticausal inverses of these structures.

4.1 The Base Dyadic Structure

In this section we consider factorizations of square $M \times M$ LTV all pass systems of the form described in Definition 4.1

Definition 4.1 *This LTV $M \times M$ operator has a base dyadic factorization if it is as in figure 6 where $H_0(k)$ is an $M \times M$ matrix that obeys for all k*

$$H_0^{\dagger}(k)H_0(k) = I \tag{1}$$

and each $H_i(k,q^{-1})$ is an $M \times M$ LTV operator of the form in figure 7, with $v_i(k)$ $M \times 1$ vectors obeying

$$v_i^\dagger(k)v_i(k) = 1 \quad \forall\, k \tag{2}$$

and $\alpha_i(k)$, $\hat{\alpha}_i(k)$, scalars obeying

$$|\alpha_i(k)| \leq 1 \quad \forall\, k \tag{3}$$

and

$$\hat{\alpha}_i(k) = \sqrt{1-|\alpha_i(k)|^2} \quad \forall\, k. \tag{4}$$

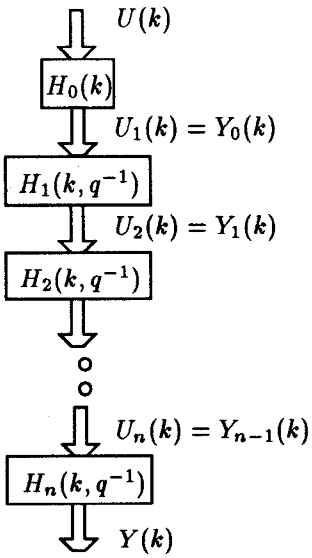

Figure 6: The base dyadic cascade

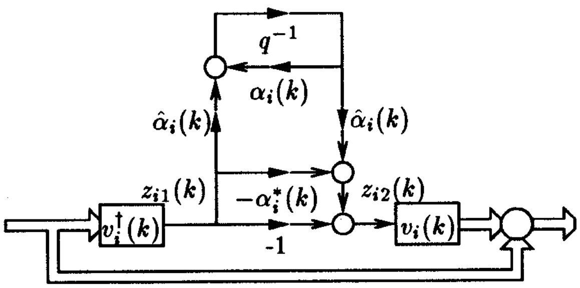

Figure 7: A degree one LTV dyadic structure

Observe the difference between the structure in figures 6 and 7, and that in figures 2, 3. The need for this somewhat different version of the base dyadic structure can be understood in the following terms. While the SISO LTV structure relating $z_{i1}(k)$ and $z_{i2}(k)$ is all pass as long as (3,4) hold, the block relating $z_{i1}(k)$ and $z_{i2}(k)$ in figure 3 will not in general be all pass, even under (3), [3]. Note the structure in figure 7 is often referred to as the normalized version of that in figure 3; that in figure 3 is all pass under (3) if $\alpha_i(k)$ is a constant. We can now state the main result of this subsection.

Theorem 4.1 *Suppose an all pass $M \times M$ LTV system has McMillan degree n and a compatible realization. Then it has a base dyadic implementation described in Definition 4.1, with n as in figure 6, and with each $H_i(k,q^{-1})$ as in figure 6, 7 having McMillan degree 1. Furthermore, if this system has a real compatible realization then the $H_0(k)$ in figure 6 and $v_i(k)$ and $\alpha_i(k)$ in figure 7 are real for all k. Finally, all systems admitting an implementation of the form in Definition 4.1 are all pass.*

The need for the uco/ucc assumption is to guarantee eas of the implementation. Without eas, these implementations will be potentially nonrobust even to minor implementation errors. It is well known that the all pass assumption together with the uco/ucc requirement suffices to guarantee eas [6].

Finally, consider the all pass system in 9, which as noted earlier, despite being FIR, does not have the FIR based dyadic implementation given in [3]. Yet, from (11,12), observe that this system is a special case of the structure in figure 7 ($U(k), Y(k)$ are 1×1. $v_i(k) = 1\ \forall\ k$). Thus, this system has the dyadic implementation depicted in figures 6 and 7

4.2 Householder Structure

In this subsection we present a variation of the Householder matrix based factorization presented in [5]. A Householder matrix is a unitary matrix that can be expressed for some column vector w,

$$I - 2ww^\dagger,\ w^\dagger w = 1. \tag{5}$$

Then the structure is as defined in Definition 4.2.

Definition 4.2 *This structure has a Householder factorization if it is as in Figure 8 with $G_0(k)$ $M \times M$, and obeying*

$$G_0^\dagger(k)G_0(k) = I; \quad \forall\, k \tag{6}$$

for suitable $M \times 1$ vectors $w_i(k)$, $W_i(k)$ is

$$W_i(k) = I - w_i(k)w_i^\dagger(k),\ w_i^\dagger(k)w_i(k) = 1, \quad \forall\, k, \tag{7}$$

the $M \times M$ operator

$$\Lambda_i(k,q^{-1}) = \begin{bmatrix} I_{M-1} & 0 \\ 0 & \lambda_i(k,q^{-1}) \end{bmatrix} \tag{8}$$

with $\lambda_i(k,q^{-1})$ the same as the block relating z_{i1} and z_{i2} in figure 7, and with $\alpha_i(k)$, $\hat{\alpha}_i(k)$ as in (3,4).

Then the main result of the subsection is as follows.

Theorem 4.2 *Suppose an all pass $M \times M$, LTV system has McMillan degree n with a compatible realization. Then it has a Householder based implementation*

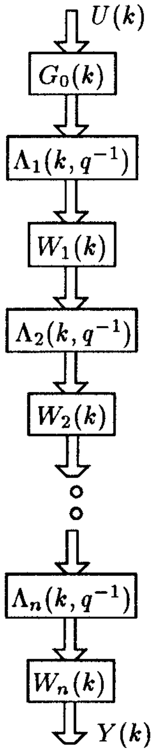

Figure 8: A householder structure

described in Definition 4.2, with n as in figure 8, with each block in, $\Lambda_i(k, q^{-1})$ block having McMillan degree 1. Furthermore, if this system has a real compatible realization, then the $G_0(k)$ and $W_i(k)$ in figure 8 and $\alpha_i(k)$ (see defenition 4.2) are real for all k. Finally, all systems admitting an implementation of the form in Definition 4.2 are all pass.

As with Theorem 4.2, the uco, ucc nature of each block $\Lambda_i(k, q^{-1})$ guarantees that the LTV system in figure 8 is uco and ucc. Further, the corresponding structure in [5] is identical to that in figure 8, with the $\Lambda_i(k, q^{-1})$ replaced by blocks of the form relating z_{i1} and z_{i2} in figure 3

4.3 The Anticausal Inverse

We can now turn to the question of a realization of the synthesis bank $R(q^{-1}, k)$ in figure 1, when the analysis bank is all pass and has a McMillan degree. To achieve the PR requirement, we wish $R(q^{-1}, k)$ to be the anticausal inverse of $E(k, q^{-1})$. Given that $E(k, q^{-1})$ will have either of the two implementations given in Sections 4.1 and 4.2, the following fact readily follows from [4]: That in each case, the corresponding $R(q^{-1}, k)$ can be implemented by reversing the order of the blocks in figure 4 or 8 as the case may be, and by replacing q^{-1} by q. The resulting realization of $R(q^{-1}, k)$ is also all pass and is eas, uco and ucc.

5 Conclusion

We have developed a factorization scheme for LTV IIR square all pass systems which are often used in LTV filter banks. This scheme is based on a simple dyadic structure and is guaranteed to exist if the system is ucc, uco, and eas. Moreover, certain FIR systems which cannot be realized with the dyadic structure in [3] do have realizations of the type presented here. Computation simplification may be achieved by implementing a Householder variation of the dyadic structure; and in both cases, an anticausal inverse may be used in the synthesis bank of the LTVFB.

References

[1] J.L. Arrowood Jr. and M.J.T. Smith, "Exact reconstruction analysis/synthesis filter banks with time-varying filters", *Proc. ICASSP*, April 1993.

[2] S. Phoong and P.P. Vaidyanathan, "Factorizability of lossless time-varying filters and filter banks", *CalTech. Report*, April 1995.

[3] S. Phoong and P.P. Vaidyanathan, "A polyphase approach to time-varying filter banks", *Proc. ICASSP*, May 1996.

[4] P.P. Vaidyanathan and T. Chen, "Role of anticausal inverses in multirate filter banks - part I: system-theoretic fundamentals", *IEEE Trans. on Signal Processing*, V. 43, pp 1090-1102, May 1995.

[5] P.P. Vaidyanathan, *Multirate systems and filter banks*, Prentice Hall, 1993.

[6] B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Prentice Hall, 1979.

[7] A.J. Van der Veen, "Time-varying system theory and computational modelling: realization, approximation, and factorization", PhD Thesis, Delft University of Technology, Delft, 1993.