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Discrete-Time Convex Direction for Matrices *

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The notion of convex direction has often been used in the study of robust Hurwitz/Schur stability of polynomials and Hurwitz stability of matrices. In this paper, a notion of matrix convex direction is introduced for discrete-time systems. Namely, a matrix D is called a discrete-time matrix convex direction if for any Schur stable matrix A, stability of $A + D$ implies that of $A + \mu D$ for all $0 \leqslant \mu \leqslant 1$. We provide a complete characterization of all such convex direction.

Keywords: Robust stability; Discrete-time systems; Convex directions

1. Introduction and Preliminaries

This paper is concerned with the robust Schur-stability of a segment of matrices of the the following form:

$$A(\mu) = A + \mu D,\ \mu \in [0, 1] \tag{1}$$

where A is a nominal matrix and D represents the direction of the segment. An important robust stability problem is the following: under what conditions does the stability of the extreme members (i.e. A and $A + D$) of the segment implies the stability of the whole segment?

The notion of convex direction is first proposed by Rantzer [1] to study the extreme point property for a segment of polynomials, and it is defined as follows: a polynomial $p(s)$ is called a convex direction if for any $p_0(s)$ with $\deg(p_0(s)) > \deg(p(s))$, the stability of both $p_0(s)$ and $p_0(s) + p(s)$ implies that of every $p_0(s) + \mu p(s)$, $0 \leqslant \mu \leqslant 1$. It is known that many existing extreme point results can be interpreted and unified using convex direction; see [2,3] for details. In [1], a necessary and sufficient condition, called phase growth condition, is provided for testing convex directions. In [2], a simple sufficient condition for convex directions, called alternating Hurwitz minor condition (AHMC), is given using the Hurwitz matrix corresponding to the coefficients of $p(s)$. In [4], a finite algorithm for testing convex directions is presented in terms of Routh tables. Note that, in the polynomial case, there is no significant difference between testing continuous-time convex directions and testing discrete-time convex directions. This is due to the fact that the discrete-time problem can be converted into a continuous-time one or vice versa by using, e.g., bilinear transformation. A remarkable feature of the bilinear transformation is that the transformed family of polynomials remains linear in μ. By introducing the notion of convex direction in the state space, Kokame et al. [5] provide a simple characterization of all matrix convex directions. Although a bilinear transformation can be used to convert Schur stability to Hurwitz stability, it will destroy the linearity in μ. To be more precise, a bilinear transformation for the family of matrices is given by $A(\mu) \rightarrow (I - A(\mu))^{-1}(I + A(\mu))$ and the result is no longer linear in μ. Hence, the continuous- and discrete-time cases have to be treated separately. We shall provide a complete characterization of all discrete-time matrix convex directions. It turns out that convex directions are restricted to only a few special forms.

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2. Main Result

The main result is simply stated as follows.

Theorem 1. A matrix D is a discrete-time convex direction if and only if it is similar[1] to one of the following matrices:

$$\text{(i)} \quad D_1 = \alpha I_n \tag{2}$$

$$\text{(ii)} \quad D_2 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \; D_3 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

$$\text{(iii)} \quad D_4 = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

where I_n is an identity matrix in $\mathcal{R}^{n\times n}$ and α is a real scalar such that there exists at least one A for which both A and $A + D$ are Schur stable.

We break the proof into two parts. The sufficiency is given in Section 3, and necessity in Section 4. In the sequel, we denote

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{5}$$

and

$$\begin{aligned} p(s,\mu) &= \det[sI - A - \mu D] \\ &= s^n + e_{n-1}s^{n-1} + \dots + e_1 s + e_0 \end{aligned} \tag{6}$$

3. Proof of Sufficiency

(i) To show that αI_n is a convex direction, we need to show that the Schur stability of $A \in \mathcal{R}^{n\times n}$ and $A + \alpha I_n$ implies that of $A + \mu\alpha I_n$, $\forall \mu \in (0,1)$, $\forall \alpha \in \mathcal{R}$. Assume that s^* is an eigenvalue of A, then $s^* + \mu\alpha$ is an eigenvalue of $A + \mu\alpha I_n$. From the convexity of the unit circle, we can conclude that if s^* and $s^* + \alpha$ are in the unit circle, so is $s^* + \mu\alpha$, $\forall \mu \in (0,1)$;
(ii) The characteristic polynomials of $A_2 + \mu D_2$ and $A_3 + \mu D_3$ are given by

$$\begin{aligned} p_2(s,\mu) &= s^2 + e_1(\mu)s + e_0(\mu) \\ &= s^2 + (-a_{11} - a_{22})s + (-a_{21}\mu\alpha \\ &\quad + (a_{11}a_{22} - a_{12}a_{21})) \end{aligned} \tag{7}$$

and

$$\begin{aligned} p_3(s,\mu) &= s^2 + e_1(\mu)s + e_0(\mu) \\ &= s^2 + (-a_{11} - a_{22} - \mu\alpha)s \\ &\quad + ((a_{11}a_{22} - a_{12}a_{21}) + \mu(a_{22}\alpha)) \end{aligned} \tag{8}$$

respectively. The stability constraints for a matrix in $\mathcal{R}^{2\times 2}$, are as follows:

$$\begin{aligned} e_0(\mu) - e_1(\mu) + 1 &> 0 \\ e_0(\mu) + e_1(\mu) + 1 &> 0 \\ e_0(\mu) &< 1 \end{aligned} \tag{9}$$

Since for D_2 and D_3 both $e_i(\mu)$, $i = 0, 1$, and the constraints in (9) are linear in μ, (9) holds in such case for all $\mu \in [0,1]$ if it does for $\mu \in \{0,1\}$. That is, D_2 and D_3 are both convex directions.
(iii) The discrete-time stability constraints for matrix in $\mathcal{R}^{3\times 3}$ are as follows:

$$\begin{aligned} e_0(\mu) + e_1(\mu) + e_2(\mu) + 1 &> 0 & (a) \\ e_0(\mu) - e_1(\mu) + e_2(\mu) - 1 &< 0 & (b) \\ -1 < e_0(\mu) &< 1 & (c) \\ e_0^2(\mu) - 1 - e_0(\mu)e_2(\mu) + e_1(\mu) &< 0 & (d) \end{aligned} \tag{10}$$

It is straightforward to verify that $e_2(\mu)$ is a constant and

$$\begin{aligned} e_1(\mu) &= e_1(0) - \alpha\mu a_{21} \\ e_0(\mu) &= e_0(0) + \alpha\mu(a_{21}a_{33} - a_{23}a_{31}) \end{aligned} \tag{11}$$

which are linear in μ.
Because (10d) is a quadratic function with leading coefficient being positive and (10a)–(10c) are all linear with respect to μ, we conclude that D_3 is a convex direction.

Finally, we need this lemma:

Lemma 1. If $D \in \mathcal{R}^{n\times n}$ is a convex direction, so is $T^{-1}DT$, where T is a real nonsingular matrix.

Proof. Suppose A and $A + T^{-1}DT$ are stable matrices. We need to show that $A + \mu T^{-1}DT$ is stable for all $\mu \in [0,1]$. Obviously, TAT^{-1} and $TAT^{-1} + D$ are stable. Hence, $TAT^{-1} + \mu D$ is stable for all $\mu \in [0,1]$. Equivalently, $A + \mu T^{-1}DT$ is stable for all $\mu \in [0,1]$.

4. Proof of Necessity

To show that the conditions in Theorem 1 are necessary, we need to exclude all other matrices from the set of convex directions, which is straightforward but unfortunately rather tedious. To begin with, we need the following result.

[1]A matrix X is said to be similar to matrix Y if there exists a nonsingular matrix T such that $T^{-1}XT = Y$.

Lemma 2. For a block triangular matrix:

$$D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}$$

where D_{11} and D_{22} are square matrices, if D is a convex direction, so are both D_{11} and D_{22}.
Proof of Lemma 2: It is obvious by choosing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{12}$$

Now we proceed to prove the necessity of Theorem 1. Suppose D is a convex direction. It is obvious that there must be at least one A such that both A and $A + D$ are stable. To show that D must be in one of the forms (i)–(iii), we consider the use of similarity transformations. Subsequently we only need to treat matrices in Jordan canonical form. In the following, we shall use ϵ_1, ϵ_2 and ϵ_3 to denote sufficiently small positive real numbers. In particular, when we say $1 - 2\epsilon_1 - \epsilon_2 > 0$, we mean that there exist sufficiently small $\epsilon_1, \epsilon_2 > 0$ such that the above is true. Four major cases are considered, depending on the dimensions of D.

Case 1. $D \in \mathcal{R}^{2\times 2}$. We need to consider three types of matrices:

$$D_5 = \operatorname{diag}\{x_1, x_2\},\ x_1 \neq x_2,\ x_1 x_2 \neq 0,$$

$$D_6 = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix},\ x \neq 0$$

and

$$D_7 = \begin{bmatrix} x & y \\ -y & x \end{bmatrix},\ y \neq 0$$

An immediate application of Lemma 2 to the above matrices reveal that we must have $|x_1| < 2$, $|x_2| < 2$ for D_5 and $|x| < 2$ for D_6 . This restriction on the size of the diagonal elements shall be used throughout the proof.

Case 1.1. $D_5 = \operatorname{diag}\{x_1, x_2\},\ x_1 \neq x_2,$

$$\begin{aligned} e_1(\mu) &= -a_{11} - a_{22} - \mu(x_1 + x_2) \\ e_0(\mu) &= (a_{22} + \mu x_2)(a_{11} + \mu x_1) - a_{12}a_{21} \end{aligned} \tag{13}$$

and (9) becomes

$$\begin{aligned} &\mu^2 x_1 x_2 + \mu(a_{11}x_2 + a_{22}x_1 + x_1 + x_2) \\ &+ (a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22} + 1) > 0 \\ &\mu^2 x_1 x_2 + \mu(a_{11}x_2 + a_{22}x_1 - x_1 - x_2) \\ &+ (a_{11}a_{22} - a_{12}a_{21} - a_{11} - a_{22} + 1) > 0 \\ &\mu^2 x_1 x_2 + \mu(a_{11}x_2 + a_{22}x_1) \\ &+ (a_{11}a_{22} - a_{12}a_{21}) < 1 \end{aligned} \tag{14}$$

From the symmetry of x_1 and x_2 in (14), we need to consider only two subcases: (i) $x_1 > 0, x_2 > 0$ and (ii) $x_1 < 0 < x_2$. The counterexamples below show that D_5 is not a convex direction. We only list the counterexamples here and the reader can consult the Appendix for details.

Counterexample for subcase (i-1): $x_1 > 0, x_2 > 0$ and $0 < x_1 + x_2 < 2$.

$$\begin{aligned} a_{11} &= \frac{3\epsilon_1 x_1 + 2(x_1^2 - x_1 x_2 + 2x_2)}{4(x_2 - x_1)} \\ a_{22} &= \frac{3\epsilon_1 x_2 - 2(x_1(x_2 - 2) - x_2^2)}{4(x_1 - x_2)} \\ a_{12} &= 1 \\ a_{21} &= -\frac{9\epsilon_1^2 x_1 x_2 + 8\epsilon_1(2x_1^2 - x_1 x_2 + 2x_2^2) - 4x_1 x_2(x_1^2 - 2x_1 x_2 + x_2^2 - 4)}{16(x_1 - x_2)^2} \end{aligned} \tag{15}$$

Counterexample for subcase (i-2): $x_1 > 0, x_2 > 0$ and $2 < x_1 + x_2 < 4$.

$$\begin{aligned} a_{11} &= \frac{8 - 4x_1 + 2\epsilon_1 x_1 - 2x_1(\epsilon_2 + x_1) + x_1 x_2(8\epsilon_3 + x_1 - 4\epsilon_3 x_1)}{4(x_1 - x_2)} \\ a_{22} &= \frac{-8 + 2x_2(2 - \epsilon_1) + x_2(2\epsilon_2 - 8\epsilon_3 x_1 + 2x_2 - x_1 x_2 + 4\epsilon_3 x_1 x_2}{4(x_1 - x_2)} \\ a_{12} &= 1 \\ a_{21} &= a_{11}a_{22} - \epsilon_1 + 1 + (a_{11} + a_{22}) \end{aligned} \tag{16}$$

Counterexample for subcase (ii-1): $x_1 < 0 < x_2$ and $0 < x_1 + x_2$.

$$\begin{aligned} a_{11} &= \frac{-2\epsilon_1 + 2\epsilon_2 + 2x_1 - x_1^2 + x_1 x_2}{2(x_1 - x_2)} \\ a_{22} &= \frac{2\epsilon_1 - 2\epsilon_2 - 2x_2 - x_1 x_2 + x_2^2}{2(x_1 - x_2)} \\ a_{12} &= 1 \\ a_{21} &= a_{11}a_{22} - (1 - \epsilon_1) \end{aligned} \tag{17}$$

Counterexample for subcase (ii-2): $x_1 < 0 < x_2$ and $x_1 + x_2 < 0$.

$$\begin{aligned} a_{11} &= \frac{x_1(\epsilon_1 + \epsilon_2 + 2\epsilon_3 x_2(x_1 + 1) - x_2 - 2)}{x_2 - x_1} \\ a_{22} &= \frac{x_2(\epsilon_1 + \epsilon_2 + 2\epsilon_3 x_1(x_2 + 1) - x_1 - 2)}{x_1 - x_2} \\ a_{12} &= 1 \\ a_{21} &= a_{11}a_{22} + 1 - a_{11} - a_{22} - \epsilon_1 \end{aligned} \tag{18}$$

Case 1.2. $D_6 = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix},\ x \neq 0$. We have

$$\begin{aligned} e_0(\mu) &= \mu^2 x^2 + \mu(a_{11}x + a_{22}x - a_{21}) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21}) \\ e_1(\mu) &= -a_{11} - a_{22} - 2\mu x \end{aligned} \tag{19}$$

and (9) reduces to

$$\begin{aligned} &\mu^2x^2 + \mu(a_{11}x + a_{22}x - a_{21} - 2x) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21} - a_{11} - a_{22}) + 1 > 0 \\ &\mu^2x^2 + \mu(a_{11}x + a_{22}x - a_{21} + 2x) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22}) + 1 > 0 \\ &\mu^2x^2 + \mu(a_{11}x + a_{22}x - a_{21}) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21}) < 1 \end{aligned} \tag{20}$$

We only have to construct counterexamples for the case $x > 0$. For $x < 0$, we simply change the signs of a_{11} and a_{22}.

Counterexample for subcase (i): $0 < x < 1$.

$$\begin{aligned} a_{11} &= -\frac{4x + 2 - \epsilon_1 + \epsilon_2}{2}, \\ a_{22} &= 1 \\ a_{21} &= -\frac{x(2x - \epsilon_1 + \epsilon_2 - 4)}{2} \\ a_{12} &= \frac{2(4x + \epsilon_2)}{x(2x - \epsilon_1 + \epsilon_2 - 4)} \end{aligned} \tag{21}$$

Counterexample for subcase (ii): $1 < x < 2$.

$$\begin{aligned} a_{11} &= \epsilon_3x^2 - 3 \\ a_{22} &= 1 \\ a_{12} &= \frac{-2\epsilon_3x^2 + \epsilon_1 + 4}{-\epsilon_3x^3 - x^2(-2\epsilon_3 + 1) + 4x - \epsilon_1 + \epsilon_2 - 4} \\ a_{21} &= \epsilon_3x^3 + x^2(-2\epsilon_3 + 1) - 4x + \epsilon_1 - \epsilon_2 + 4 \end{aligned} \tag{22}$$

Case 1.3. $D_7 = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, $y \neq 0$. Now, we have

$$\begin{aligned} e_1(\mu) &= -a_{11} - a_{22} - 2\mu x \\ e_0(\mu) &= \mu^2(x^2 + y^2) + \mu(x(a_{11} + a_{22}) \\ &\quad + y(a_{12} - a_{21})) + (a_{11}a_{22} - a_{12}a_{21}) \end{aligned} \tag{23}$$

and the stability constraints (9) become

$$\begin{aligned} &\mu^2(x^2 + y^2) + \mu(x(a_{11} + a_{22}) + y(a_{12} - a_{21}) \\ &\quad - 2x) + (a_{11}a_{22} - a_{12}a_{21} - a_{11} - a_{22} + 1) > 0 \\ &\mu^2(x^2 + y^2) + \mu(x(a_{11} + a_{22}) + y(a_{12} - a_{21}) \\ &\quad + 2x) + (a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22} + 1) > 0 \\ &\mu^2(x^2 + y^2) + \mu(x(a_{11} + a_{22}) + y(a_{12} - a_{21})) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21}) < 1 \end{aligned} \tag{24}$$

By substituting $\mu = 0$ and 1 into (24), we find that

$$|2x + a_{11} + a_{22}| < 2 \text{ and } |a_{11} + a_{22}| < 2 \tag{25}$$

which in turn shows that $|x| < 2$. Also, if (24) is violated for x, a_{11} and a_{22}, it remains so for $-x, -a_{11}$ and $-a_{22}$. Again, we only need to consider $x > 0$.

Counterexample for subcase (i): $0 < x < 1$.

$$\begin{aligned} a_{11} + a_{22} &= -2x \\ a_{12} - a_{21} &= \frac{x^2 - 2x - y^2 - \epsilon_1 + \epsilon_2}{y} \\ a_{11}a_{22} - a_{12}a_{21} &= 2x + \epsilon_1 - 1 \end{aligned} \tag{26}$$

The equations above have at least the following solution:

$$\begin{aligned} a_{11} &= -x \\ a_{22} &= -x \\ a_{12} &= \frac{-\epsilon_1 + \epsilon_2 - 2x + x^2 - y^2 - \sqrt{O(\epsilon_1, \epsilon_2) + (x^2 + y^2)((2 - x)^2 + y^2)}}{2y} \\ a_{21} &= \frac{\epsilon_1 - \epsilon_2 + 2x - x^2 + y^2 - \sqrt{4(1 - \epsilon_1 - 2x + x^2)y^2 + (-\epsilon_1 + \epsilon_2 - 2x + x^2 - y^2)^2}}{2y} \end{aligned} \tag{27}$$

where $O(\epsilon_1, \epsilon_2) = (\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_2 + 4x - 2x^2) - 2y^2 (\epsilon_1 + \epsilon_2)$.

Counterexample for subcase (ii): $1 < x < 2$.

$$\begin{aligned} a_{11} + a_{22} &= \epsilon_3(x^2 + y^2) - 2 \\ a_{12} - a_{21} &= \frac{-\epsilon_3x^3 - x^2(-2\epsilon_3 + 1) + x(-\epsilon_3y^2 + 4) - y^2(-2\epsilon_3 + 1) - \epsilon_1 + \epsilon_2 - 4}{y} \\ a_{11}a_{22} - a_{12}a_{21} &= -\epsilon_3(x^2 + y^2) + \epsilon_1 + 1 \end{aligned} \tag{28}$$

The equations above also have at least the following solution:

$$\begin{aligned} a_{11} &= -1 \\ a_{22} &= -1 + \epsilon_3(x^2 + y^2) \\ a_{12} &= \frac{O_1(\epsilon_1, \epsilon_2, \epsilon_3) + (4x - 4 - x^2 - y^2) - \sqrt{((2 - x)^2 + y^2)^2 + O_2(\epsilon_1, \epsilon_2, \epsilon_3)}}{2y} \\ a_{21} &= \frac{O_3(\epsilon_1, \epsilon_2, \epsilon_3) - \sqrt{(-(2 - x)^2 - y^2 + (\epsilon_2 - \epsilon_1) + \epsilon_3(2x^2 - x^3 + 2y^2 - xy^2))^2}}{2y} \end{aligned} \tag{29}$$

where

$$\begin{aligned} O_1(\epsilon_1, \epsilon_2, \epsilon_3) &= -\epsilon_1 + \epsilon_2 + \epsilon_3(2x^2 - x^3 + 2y^2 \\ &\quad - xy^2) \\ O_2(\epsilon_1, \epsilon_2, \epsilon_3) &= (\epsilon_1 - \epsilon_2)(8 + \epsilon_1 - \epsilon_2 - 8x \\ &\quad + 2x^2) - 2y^2(\epsilon_1 + \epsilon_2) + \\ &\quad 2\epsilon_3(x - 2)(x^2 + y^2)(4 + \epsilon_1 - \epsilon_2 - 4x + x^2 \\ &\quad + y^2) + \epsilon_3^2(x - 2)^2(x^2 + y^2)^2 \\ O_3(\epsilon_1, \epsilon_2, \epsilon_3) &= (\epsilon_1 - \epsilon_2) + \epsilon_3(-2x^2 + x^3 - 2y^2 \\ &\quad + xy^2) \end{aligned} \tag{30}$$

Case 2. $D \in \mathcal{R}^{3\times3}$. The discrete-time stability constraints for third order polynomials are given by (10).

The possible Jordan forms in $\mathcal{R}^{3\times3}$ are as follows:

$$\begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & y \end{bmatrix}, \begin{bmatrix} y & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & y & 0 \\ -y & x_1 & 0 \\ 0 & 0 & x_2 \end{bmatrix}, \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & y \\ 0 & -y & x_2 \end{bmatrix} \tag{31}$$

We first notice the fact that for a block diagonal matrix, exchange two of its non-zero blocks will not affect its convex direction property. Then, by applying Lemma 2 and results in $\mathcal{R}^{2\times 2}$, it is revealed that the only possible candidates for discrete-time convex directions in $\mathcal{R}^{3\times 3}$ are αI_3 and the following:

$$D_8 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_9 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{bmatrix},$$

$$D_{10} = \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad D_{11} = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

Case 2.1. $D = D_8$. We have the following counterexample:

$$A = A_8 = \begin{bmatrix} -0.9890029 & -0.999938 & 6.25\times 10^{-5} \\ 4 & 0 & 0 \\ 63996 & -4 & 0 \end{bmatrix} \tag{33}$$

Case 2.2. $D = D_9$. We can consider two cases with the observation that a necessary condition is $|x| < 2$. We need to show that D_9 is not a convex direction when $x \neq 0$.

Counterexample for subcase (i): $(0 < x < 2)$.

$$\begin{aligned}
a_{11} &= -\frac{x^2 + 3x(\epsilon_1 - 3) + \epsilon_1^2 - 7(\epsilon_1 - 2)}{x + 2\epsilon_1 - 5} \\
a_{12} &= -\frac{x^4 + 3x^3(2\epsilon_1 - 5) + x^2(11\epsilon_1^2 - 53\epsilon_1 + 69) + x(6\epsilon_1^3 - 38\epsilon_1^2 + 92\epsilon_1 - 89) + \epsilon_1^4 - 6\epsilon_1^3 + 11\epsilon_1^2 - 5\epsilon_1 + 6}{x(x + 2\epsilon_1 - 5)^2} \\
a_{13} &= x + 2\epsilon_1 - 5,\ a_{21} = x, \\
a_{22} &= \frac{x^2 + x(3\epsilon_1 - 7) + \epsilon_1^2 - 3\epsilon_1 + 4}{x + 2\epsilon_1 - 5},\ a_{23} = 0 \\
a_{31} &= a_{32} = 1,\ a_{33} = \epsilon_1 - 2
\end{aligned} \tag{34}$$

Counterexample for subcase (ii): $(-2 < x < 0)$.

$$\begin{aligned}
a_{11} &= -\frac{x^2 + x(1 - \epsilon_1) - \epsilon_1^2 + 7(\epsilon_1 - 2)}{x - 2\epsilon_1 + 5} \\
a_{12} &= -\frac{x^4 + 2x^3(2 - \epsilon_1) - x^2(\epsilon_1^2 - 7\epsilon_1 + 17) + 2x(\epsilon_1^3 - 9\epsilon_1^2 + 30\epsilon_1 - 33) + \epsilon_1^4 - 6\epsilon_1^3 + 11\epsilon_1^2 - 5\epsilon_1 + 6}{x(x - 2\epsilon_1 + 5)^2} \\
a_{13} &= -x + 2\epsilon_1 - 5,\ a_{21} = x, \\
a_{22} &= \frac{x^2 + x(3 - \epsilon_1) - \epsilon_1^2 + 3\epsilon_1 - 4}{x - 2\epsilon_1 + 5},\ a_{23} = 0 \\
a_{31} &= a_{32} = 1,\ a_{33} = -x + \epsilon_1 - 2
\end{aligned} \tag{35}$$

Case 2.3. $D = D_{10}$. For this case, we only need to consider $x \in (-2, 0)$ because (i) we must have $|x| < 2$ (Lemma 2,) and (ii) D_{10} and $-D_{10}$ have the same convex direction property.

Counterexample for D_{10}:

$$\begin{aligned}
a_{11} &= -\frac{x}{2},\ a_{12} = \frac{(\epsilon_1 x + 2\epsilon_3 + x)}{(\epsilon_3 - 3x)} \\
a_{13} &= \frac{(\epsilon_3 - x)(\epsilon_1 x + 2\epsilon_3 + x)}{(x(2x - \epsilon_3))},\quad a_{21} = 1, \\
a_{22} &= -1,\quad a_{23} = \frac{2\epsilon_3 - 3x}{x} \\
a_{31} &= a_{32} = a_{33} = 1
\end{aligned} \tag{36}$$

Case 2.4. $D = D_{11}$. Using the same argument as in Case 2.3, we only need to consider $x \in (-2, 0)$.

Counterexample for D_{11}:

$$\begin{aligned}
a_{11} &= -\frac{(2\epsilon_1 - \epsilon_2 + \epsilon_3 + 2)}{2},\quad a_{12} = -\epsilon_1 - \epsilon_3 - x \\
a_{13} &= 0.25(4\epsilon_1^2 - 4\epsilon_1(\epsilon_2 - \epsilon_3 + x - 2) + \epsilon_2 \\
&\quad - 2\epsilon_2(\epsilon_3 - x + 2) + \epsilon_3^2 - 2\epsilon_3 x - 8x) \\
a_{21} &= a_{22} = 1,\quad a_{23} = -\frac{2\epsilon_1 - \epsilon_2 + \epsilon_3 - 2x}{2} \\
a_{31} &= 0,\quad a_{32} = a_{33} = 1
\end{aligned} \tag{37}$$

Case 3. $D \in \mathcal{R}^{4\times 4}$. The discrete-time stability constraints for fourth order polynomials are the following:

$$\begin{aligned}
&1 + e_3(\mu) + e_2(\mu) + e_1(\mu) + e_0(\mu) > 0 \\
&1 - e_3(\mu) + e_2(\mu) - e_1(\mu) + e_0(\mu) > 0 \\
&e_1(\mu) - e_3(\mu) < 2(1 - e_0(\mu)) \\
&e_1(\mu) - e_3(\mu) > 2(e_0(\mu) - 1) \\
&e_0^3(\mu) + 2e_0(\mu)e_2(\mu) + e_1(\mu)e_3(\mu) - e_0(\mu) - e_2(\mu) \\
&\quad - e_0(\mu)e_3^2(\mu) - e_0^2 - e_0^2(\mu)e_2(\mu) - e_1^2(\mu) + 1 \\
&\quad + e_0(\mu)e_1(\mu)e_3(\mu) > 0
\end{aligned} \tag{38}$$

From the analysis of cases 1 and 2 and Lemma 2, we know that any Jordan forms involving complex eigenvalues are not convex directions. Further, out of the following possible Jordan matrices with real eigenvalues

$$\text{(i)} \begin{bmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & x \end{bmatrix}, \text{(ii)} \begin{bmatrix} y & 1 & 0 & 0 \\ 0 & y & 1 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & x \end{bmatrix},$$

$$\text{(iii)} \begin{bmatrix} z & 1 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \text{(iv)} \begin{bmatrix} y & 1 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & x \end{bmatrix} \tag{39}$$

(i) and (ii) above can be eliminated. In (iv) we should have $x = y = 0$. In (iii) we must have $xy = 0, z = 0$. However, if either x or y is not zero , then it will violate Case 2.2. Therefore, it cannot be a convex direction. So now the only two candidates for discrete-time convex direction are:

$$D_{12} = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad D_{13} = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} \tag{40}$$

For D_{12} , a counterexample is given by:

$$A_{12} = \begin{bmatrix} -0.0406597 & 0.262038 & 0 & -0.0118078 \\ 0.629544 & 0.278577 & 0 & 0 \\ 1 & 0 & 0 & -0.974787 \\ 0 & 0 & 1 & 1.78088 \end{bmatrix} \tag{41}$$

For D_{13}, a counterexample is as follows:

$$A_{13} = \begin{bmatrix} 0 & -3 & 0 & -5 \\ -1 & 0 & 0 & 0 \\ 0 & \epsilon_1/5 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{42}$$

The eigenvalues of $A_{13} + \mu D_{13}$ are , respectively,

$$\begin{aligned} &[\pm(\epsilon_1 - 1)^{\frac{1}{4}},\ \pm i(\epsilon_1 - 1)^{\frac{1}{4}}] && at\, \mu = 0 \\ &[\pm(\epsilon_1 - 1)^{\frac{1}{4}},\ \pm i(\epsilon_1 - 1)^{\frac{1}{4}}] && at\, \mu = 1 \\ &[\pm\tfrac{\sqrt{2}}{2}(4\epsilon_1 - 5)^{\frac{1}{4}},\ \pm i\tfrac{\sqrt{2}}{2}(4\epsilon_1 - 5)^{\frac{1}{4}}] && at\, \mu = 1/2 \end{aligned} \tag{43}$$

Clearly, for some sufficiently small $\epsilon_1 > 0$, the eigenvalues at $\mu = 0$ and $\mu = 1$ are inside the unit circle but those at $\mu = 1/2$ will go unstable.

Case 4. $D \in \mathcal{R}^{n\times n}, n > 4$. Applying Lemma 2 and the analysis for case 3, the only candidate for discrete-time matrix convex direction is αI_n which is already known to be a convex direction.

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Appendix

In this Appendix, we verify all the counterexamples used in Section 4 (Tables A.1–A.7). For each counterexample, we simply tabulate all the stability constraints at $\mu = 0$ and 1 and some $\mu^* \in (0, 1)$. All these tables can be readily verified by a mathematica program called DCD_checking.m which is available by anonymous ftp from ee.newcastle.edu.au under the directory /pub/LinXIE. As mentioned before, ϵ_1, ϵ_2 and ϵ_3 are sufficiently small positive numbers.

Table 1. Counterexample for D_5 with $x_1x_2 > 0$.

μ	Constraints	When $0 < x_1 + x_2 < 2$	When $2 < x_1 + x_2 < 4$
	$e_0 - e_1 + 1 > 0$	$-\frac{\epsilon_1+2(x_1+x_2-2)}{2} > 0$	$\epsilon_1 > 0$
0	$e_0 + e_1 + 1 > 0$	$\epsilon_1 > 0$	$\frac{2\epsilon_2-4\epsilon_3x_1x_2-x_1x_2+2(x_1+x_2)+4}{2} > 0$
	$\epsilon_0 < 1$	$\frac{\epsilon_1}{4} - \frac{(x_1+x_2)}{2} < 1$	$\frac{2(\epsilon_1+\epsilon_2)-4\epsilon_3x_1x_2-x_1x_2+2(x_1+x_2)}{4} < 1$
	$e_0 - e_1 + 1 > 0$	$(x_1 + x_2 + 2) - \epsilon_1/2 > 0$	$\frac{2\epsilon_1+4\epsilon_3x_1x_2+x_1x_2+2(x_1+x_2)-4}{2} > 0$
1	$e_0 + e_1 + 1 > 0$	$\epsilon_1 > 0$	$\epsilon_2 > 0$
	$\epsilon_0 < 1$	$\epsilon_1/4 + \frac{(x_1+x_2)}{2} < 1$	$\frac{2(\epsilon_1+\epsilon_2)+4\epsilon_3x_1x_2+x_1x_2+2x_1+2x_2-8}{4} < 1$
	$e_0 - e_1 + 1 > 0$	*	$\epsilon_1 - \frac{\mu^*[-4\epsilon_3x_1x_2-2\mu^*x_1x_2+(x_1-2)(x_2-2)]}{2} < 0$
μ^*	$e_0 + e_1 + 1 > 0$	$-\frac{1}{4}x_1x_2 + \epsilon_1 < 0\, at\, \mu^* = 1/2$	*
	$e_0 < 1$	*	*

Table 2. Counterexample for D_5 with $x_1x_2 < 0$ and $x_1 < 0 < x_2$.

μ	Constraints	When $0 < x_1 + x_2$	When $x_1 + x_2 < 0$
	$e_0 + e_1 + 1 > 0$	$\frac{2-2\epsilon_1+x_1+x_2}{2} > 0$	$\epsilon_1 > 0$
0	$e_0 - e_1 + 1 > 0$	$\frac{6-2\epsilon_1-x_1-x_2}{2} > 0$	$-\epsilon_1 - 2\epsilon_2 - 4\epsilon_3 x_1 x_2 + 4 > 0$
	$\epsilon_0 < 1$	$1 - \epsilon_1 < 1$	$-\epsilon_2 - 2\epsilon_3 x_1 x_2 + 1 < 1$
	$e_0 + e_1 + 1 > 0$	$\frac{2-2\epsilon_2-x_1-x_2}{2} > 0$	$\epsilon_1 + 2\epsilon_3 x_1 x_2 - x_1 - x_2 > 0$
1	$e_0 - e_1 + 1 > 0$	$\frac{6-2\epsilon_2+x_1+x_2}{2} > 0$	$-\epsilon_1 - 2\epsilon_2 - 2\epsilon_3 x_1 x_2 + x_1 + x_2 + 4 > 0$
	$\epsilon_0 < 1$	$1 - \epsilon_2 < 1$	$1 - \epsilon_2 < 1$
	$e_0 + e_1 + 1 > 0$	*	*
1/2	$e_0 - e_1 + 1 > 0$	*	*
	$e_0 < 1$	$\frac{4-2\epsilon_1-2\epsilon_2-x_1x_2}{4} > 1$	$1 - \epsilon_2 - \frac{x_1x_2}{4} - \epsilon_3 x_1 x_2 > 1$

Table 3. Counterexample for D_6 with $x > 0$.

μ	Constraints	When $(0 < x < 1)$	When $(1 < x < 2)$
	$e_0 + e_1 + 1 > 0$	$4x + \epsilon_2 > 0$	$-2\epsilon_3 x^2 + \epsilon_1 + 4 > 0$
0	$e_0 - e_1 + 1 > 0$	$\epsilon_1 > 0$	$\epsilon_1 > 0$
	$\epsilon_0 < 1$	$\frac{4x+\epsilon_1+\epsilon_2-2}{2} < 1$	$-\epsilon_3 x^2 + \epsilon_1 + 1 < 1$
	$e_0 + e_1 + 1 > 0$	$\epsilon_2 > 0$	$\epsilon_2 > 0$
1	$e_0 - e_1 + 1 > 0$	$\epsilon_1 > 0$	$2\epsilon_3 x^2 + 4x + \epsilon_2 - 4 > 0$
	$\epsilon_0 < 1$	$\frac{(\epsilon_1+\epsilon_2-2)}{2} < 1$	$\epsilon_3 x^2 + 2x + \epsilon_2 - 3 < 1$
	$e_0 + e_1 + 1 > 0$	*	*
$\exists\mu^*$	$e_0 - e_1 + 1 > 0$	$-\frac{x^2}{4} + \epsilon_1 < 0\, at\, \mu^* = \frac{1}{2}$	$-\mu^* x^2(-2\epsilon_3 - \mu^* + 1) + 4\mu^* x + \epsilon_1(1 - \mu^*)$ $+\mu^*(\epsilon_2 - 4) \rightarrow -\mu^*(x-2)^2 < 0\, as\, \mu^* \rightarrow 0$
	$e_0 < 1$	*	*

Table 4. Counterexample for D_7 with $x > 0$.

μ	Constraints	When $(0 < x < 1)$	When $(1 < x < 2)$
	$e_0 + e_1 + 1 > 0$	$4x + \epsilon_1 > 0$	$-2\epsilon_3(x^2 + y^2) + \epsilon_1 + 4 > 0$
0	$e_0 - e_1 + 1 > 0$	$\epsilon_1 > 0$	$\epsilon_1 > 0$
	$\epsilon_0 < 1$	$2x + \epsilon_1 - 1 < 1$	$-\epsilon_3(x^2 + y^2) + \epsilon_1 + 1 < 1$
	$e_0 + e_1 + 1 > 0$	$\epsilon_2 > 0$	$\epsilon_2 > 0$
1	$e_0 - e_1 + 1 > 0$	$\epsilon_2 > 0$	$2\epsilon_3(x^2 + y^2) + \epsilon_2 + 4(x - 1) > 0$
	$\epsilon_0 < 1$	$\epsilon_2 - 1 < 1$	$\epsilon_3(x^2 + y^2) + \epsilon_2 + (2x - 3) < 1$
	$e_0 + e_1 + 1 > 0$	*	*
$\exists\mu^*$	$e_0 - e_1 + 1 > 0$	$-\frac{(x^2+y^2)}{4} + \frac{\epsilon_1+\epsilon_2}{2} < 0\, at\, \mu^* = \frac{1}{2}$	$\mu^*[\mu^*(x^2 + y^2) - ((x-2)^2 + y^2)] + O(\epsilon_1, \epsilon_2, \epsilon_3) < 0$
	$e_0 < 1$	*	*

Table 5. Counterexample for D_8 and D_9.

μ	Constraints	D_8	D_9 when $(0 < x < 2)$	D_9 when $(-2 < x < 0)$
	$e_0 + e_1 + e_2 + 1 > 0$	1.99	$\epsilon_1 > 0$	$\epsilon_1 > 0$
0	$e_0 - e_1 + e_2 - 1 < 0$	−0.009999	$-\epsilon_1 < 0$	$2x - \epsilon_1 < 0$
	$-1 < \epsilon_0 < 1$	0.001	$0 < \epsilon_1 < 1$	$0 < \epsilon_1 < 1$
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	−1.00099	$2\epsilon_1^2 + \epsilon_1 - 2 < 0$	$-x(\epsilon_1 + 1) + 2\epsilon_1^2 + \epsilon_1 - 2 < 0$
	$e_0 + e_1 + e_2 + 1 > 0$	1.97826	$\epsilon_1 > 0$	$\epsilon_1 > 0$
1	$e_0 - e_1 + e_2 - 1 < 0$	−0.0217395	$-2x - \epsilon_1 < 0$	$-\epsilon_1 < 0$
	$-1 < \epsilon_0 < 1$	−0.0107404	$0 < \epsilon_1 < 1$	$0 < \epsilon_1 < 1$
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	−0.98926	$x(\epsilon_1 + 1) + 2\,\epsilon_1^2 + \epsilon_1 - 2 < 0$	$2\epsilon_1^2 + \epsilon_1 - 2 < 0$
	$e_0 + e_1 + e_2 + 1 > 0$	16001	$e_1 - \frac{x^2}{4} < 0$	*
$\frac{1}{2}$	$e_0 - e_1 + e_2 - 1 < 0$	15999	*	$\frac{x^2}{4} - \epsilon_1 > 0$
	$-1 < \epsilon_0 < 1$	15999	*	*
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	2.55952×10^8	*	*

Table 6. Counterexample for D_{10} and D_{11}.

μ	Constraints	$D_{10}(-2 < x < 0)$	$D_{11}(-2 < x < 0)$
	$e_0 + e_1 + e_2 + 1 > 0$	$\frac{4+2\epsilon_1-2\epsilon_3+x_1}{2}$	ϵ_3
0	$e_0 - e_1 + e_2 - 1 < 0$	$\frac{-4-2\epsilon_1-2\epsilon_3+x_1}{2}$	$-\epsilon_2$
	$-1 < \epsilon_0 < 1$	$-\epsilon_3$	$0 < 1 - \epsilon_1 < 1$
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	$\frac{2\epsilon_1+2\epsilon_3^2+\epsilon_3 x_1}{2}$	$-4\epsilon_1 + 2\epsilon_1^2\epsilon_2 - 0.5\epsilon_1\epsilon_2 + 0.5\epsilon_1\epsilon_3$
	$e_0 + e_1 + e_2 + 1 > 0$	$\frac{4+2\epsilon_1+2\epsilon_3-x_1}{2}$	$\epsilon_3 - \epsilon_1 x_1 \frac{\epsilon_2 x_1}{2} - \frac{\epsilon_3 x_1}{2} + x_1^2 > 0$
1	$e_0 - e_1 + e_2 - 1 < 0$	$\frac{-4-2\epsilon_1+2\epsilon_3-x_1}{2}$	$-\epsilon_2 - 4x_1 + \epsilon_1 x_1 - \frac{\epsilon_2 x_1}{2} + \frac{\epsilon_3 x_1}{2} - x_1^2 < 0$
	$-1 < \epsilon_0 < 1$	ϵ_3	$0 < 1 - \epsilon_1 < 1$
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	$\frac{2\epsilon_1+2\epsilon_3^2+\epsilon_3 x_1}{2}$	$-4\epsilon_1 + 2\epsilon_1^2 + \epsilon_2 - 0.5\epsilon_1\epsilon_2 + 0.5\epsilon_1\epsilon_3 + 4x_1$ $-3\epsilon_1 x_1 + 0.5\epsilon_2 x_1 - 0.5\epsilon_3 x_1 + x_1^2 < 0$
	$e_0 + e_1 + e_2 + 1 > 0$	*	*
$\frac{1}{2}$	$e_0 - e_1 + e_2 - 1 < 0$	*	*
	$-1 < \epsilon_0 < 1$	*	$1 - \epsilon_1 + 0.25x_1^2 > 1$
	$e_0^2 - 1 - e_0 e_2 + e_1 < 0$	ϵ_1	*

Table 7. Counterexample for D_{12} and D_{13}.

μ	Constraints	D_{12}	$D_{13}(\epsilon_1 = \frac{1}{1000})$
	$e_0 + e_1 + e_2 + e_3 + 1 > 0$	0.122109	$\frac{1999}{1000}$
	$1 - e_3 + e_2 - e_1 + e_0 > 0$	3.97201	$\frac{1999}{1000}$
0	$e_1 - e_3 - 2(1 - e_0) < 0$	−0.237635	$-\frac{1}{500}$
	$e_1 - e_3 - 2(e_0 - 1) > 0$	4.46291	$\frac{1}{500}$
	$e_0^3 + 2e_0e_2 + e_1e_3 - e_0 - e_2 - e_0e_3^2 - e_0^2 - e_0^2e_2 - e_1^2 + 1 + e_0e_1e_3 > 0$	1.25975×10^{-6}	$\frac{1999}{1000000000}$
	$e_0 + e_1 + e_2 + e_3 + 1 > 0$	0.0000343852	$\frac{1999}{1000}$
	$1 - e_3 + e_2 - e_1 + e_0 > 0$	1.60766	$\frac{1999}{1000}$
1	$e_1 - e_3 - 2(1 - e_0) < 0$	−0.343837	$-\frac{1}{500}$
	$e_1 - e_3 - 2(e_0 - 1) > 0$	6.81139	$\frac{1}{500}$
	$e_0^3 + 2e_0e_2 + e_1e_3 - e_0 - e_2 - e_0e_3^2 - e_0^2 - e_0^2e_2 - e_1^2 + 1 + e_0e_1e_3 > 0$	2.09615×10^{-6}	$\frac{1999}{1000000000}$
	$e_0 + e_1 + e_2 + e_3 + 1 > 0$	0.0610718	$\frac{2249}{1000}$
	$1 - e_3 + e_2 - e_1 + e_0 > 0$	2.78983	$\frac{2249}{1000}$
$\frac{1}{2}$	$e_1 - e_3 - 2(1 - e_0) < 0$	−0.290736	$\frac{249}{500}$
	$e_1 - e_3 - 2(e_0 - 1) > 0$	5.63715	$-\frac{249}{500}$
	$e_0^3 + 2e_0e_2 + e_1e_3 - e_0 - e_2 - e_0e_3^2 - e_0^2 - e_0^2e_2 - e_1^2 + 1 + e_0e_1e_3 > 0$	-3.55455×10^{-3}	$\frac{139440249}{1000000000}$