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Stability Analysis of Finite-Level Quantized Discrete-Time Linear Control Systems

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This paper investigates the stability of discrete-time linear time-invariant systems subject to finite-level logarithmic quantized feedback. Both state feedback and output feedback are considered. A linear matrix inequality (LMI) approach is developed to estimate, for a given controller and a given finite-level quantizer, a set of admissible initial states and an associated attractor set in a neighborhood of the origin such that all state trajectories starting in the first set will converge to the attractor in a finite time and will never leave it. Furthermore, when two such sets are a priori specified, we develop sufficient conditions to design a finite-level logarithmic quantizer for a given stabilizing state or output feedback controller.

Keywords: Quantized feedback systems, Finite-level Quantization, Stability analysis, Basin of attraction, Logarithmic quantizers

1. Introduction

Motivated by the huge interest in network-based feedback control systems, the study of quantization errors has become an important area of research. There are many situations in which quantization errors may arise and its effects cannot be neglected at the cost of poor closed-loop performance and even the loss of stability.

Early results on quantized feedback concentrate on analyzing and mitigating the effects of quantization [4, 12, 18]. Nowadays, networked control systems are the most popular examples of systems subject to quantization. In such systems, the plant and the control elements (sensor, controller and actuator) are interconnected through a digital communication channel with a finite bandwidth. Since in networked systems the control elements share the same communication link, a natural issue for such systems is to minimize the amount of information needed to be transmitted while achieving a certain closed-loop performance. Over the past few years, a significant number of works has focused on this topic. For instance, stabilization with a limited feedback data rate was studied in [15, 16, 19, 20]; the problem of coarse quantization in a quadratic stability setting was addressed in [5, 6]; quantized feedback stabilization with dynamic quantizer was considered in [2, 13]; the issues of sampling and quantization for stabilization of a continuous-time linear system was investigated in [10]; and [14] focused on input-to-state stabilization via quantized state feedback.

Research on quantized feedback systems can be divided into two categories depending on whether static or dynamic quantizers are used. A static quantizer is a memoryless nonlinear function and the dynamic one uses memory to improve the performance at the cost of

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higher complexity. To overcome the complexity problem, several researchers have employed a static quantizer together with a dynamic scaling method in which a scaling factor is dynamically adjusted to achieve global asymptotic stability [2, 7, 13, 19].

For static quantizers, it has been demonstrated in [5] that the coarsest quantization density for quadratic stabilization of discrete-time single-input single-output (SISO) linear time-invariant (LTI) systems using quantized state feedback is achieved by using a *logarithmic quantizer*. This result was extended in [3, 6] in several directions (such as, multi-input multi-output systems, output feedback with quadratic or H_∞ performance, and systems with input and output logarithmic quantizers) using the *sector bound approach.* Notice that in the two later works the logarithmic quantizer has an infinite number of quantization levels, which is not practically implementable. To address the issue of finite-level quantization in the context of the sector bound approach, [7] has considered a dynamic scaling method for the logarithmic quantizer.

On the other hand, when dealing with static finite-level logarithmic quantizers, the stability properties hold only locally and the state trajectory converges to a small neighborhood of the origin. This problem has been recently addressed by several researchers using different approaches. For instance, stabilization of discrete-time systems with an LQR-type controller and a finite-level logarithmic quantized obtained by truncating a logarithm quantizer has been investigated in [5] using the notion of practical stability. On the other hand, randomized algorithms for semiglobal quadratic stability analysis of quantized sampled-data systems was proposed in [11], and a systematic method to determine componentwise ultimate bounds for sampled-data systems with quantization was devised in [9]. In this paper, we extend the sector bound approach in [6] to handle finite-level logarithmic quantizers without the use of dynamic scaling. The motivation for employing logarithmic quantizers is that they bring in several advantages, such as a convex characterization of quadratic stabilization and the explicit coarsest quantization density formulae. More importantly, logarithmic quantization gives high-resolution quantization when the input is small but low-resolution quantization when the input is large, resulting in a roughly constant relative error, which is naturally required in many applications. We consider SISO discrete-time linear time-invariant systems with a given finite-level logarithmically quantized feedback and for a given state or output feedback controller. For these systems, we develop an LMI approach to estimate a set of admissible initial states and an invariant set in the neighborhood of the origin for which all state trajectories starting in the first set will be attracted to in finite time and will never leave it. Furthermore, in the case where these two such sets are *a priori* specified, we provide a procedure to design a finite-level logarithmic quantizer, obtained by truncating an infinite-level logarithmic quantizer, to guarantee the aforementioned convergence property considering either a state feedback or an output feedback controller. Numerical examples demonstrate the potentials of the proposed approach and show that it can be used as a tool to design finite-level logarithmic quantized feedback controllers.

This paper is organized as follows. The problem to be addressed is stated in Section 2 and some key results on the sector bound approach are reviewed in Section 3. The main results of the paper are developed in Section 4 and numerical examples are presented in Section 5. Finally, concluding remarks are presented in Section 6.

Notation. The notation is quite standard. For a real matrix S, S' denotes its transpose and $S > 0$ ($S \geq 0$) means that S is symmetric and positive definite (nonnegative definite). For two sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{A}$, the notation $\mathcal{A} \backslash \mathcal{B}$ stands for $\mathcal{A}$ excluded $\mathcal{B}$.

2. Problem Statement

Consider the following SISO linear system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) \quad\;\;\; = Cx(k) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, $C' \in \mathbb{R}^n$, x is the state vector, u is the control signal and y is the measurement.

The above system will be controlled by either a quantized state feedback

$$u(k) = Q(r(k)), \quad r(k) = Kx(k) \tag{2}$$

or a dynamic output feedback controller of the form

$$\begin{cases} \xi(k+1) = A_c\xi(k) + B_c s(k) \\ \quad r(k) \;\; = C_c\xi(k) + D_c s(k) \end{cases} \tag{3}$$

where $K' \in \mathbb{R}^n$ is the state feedback gain, $Q(\cdot)$ is a static symmetric quantizer to be specified later, $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c}$, $C_c' \in \mathbb{R}^{n_c}$ and $D_c \in \mathbb{R}$ are the matrices of the output feedback controller, ξ is its state, and r and s are related to u and y, respectively, as specified below.

Without loss of generality, it is assumed that (A, B, C) and (A_c, B_c, C_c, D_c) are minimal state-space realizations.

Similarly as in [6], in the output feedback case we will consider two possible configurations involving the system (1), controller (3) and a quantizer $Q(\cdot)$ as follows:

- **Configuration I.** The measurement is quantized but the control signal is not. In this case, $s(k) = Q(y(k))$ and $u(k) = r(k)$.
- **Configuration II.** The control signal is quantized but the measurement is not. In this case, $u(k) = Q(r(k))$ and $s(k) = y(k)$.

It is assumed that the quantizer $Q(\cdot)$ has a logarithmic law with quantization levels given by the set $\mathcal{V}$ as below

$$\mathcal{V} = \{\pm m_i : m_i = \rho^i \mu,\ i = 0, 1, 2, \cdots, N-1\} \cup \{0\}, \quad \rho \in (0,1)$$

where N is the number of positive quantization levels and $\mu > 0$ is the largest admissible level. Note that a small ρ implies coarse quantization and a large ρ means a dense quantization. Similarly as in [6], as an abuse of terminology, ρ will be referred to as the *quantization density*.

In this paper, we investigate the closed-loop stability of system (1) with either the state-feedback law in (2) or the output feedback controller in (3) in Configurations I or II, and a logarithmic quantizer with a finite alphabet following the constructive law defined as below

$$Q(v) = \begin{cases} \mu, & \text{if} \quad v > \frac{\mu}{(1-\delta)} \\ \rho^i \mu, & \text{if} \quad \frac{\rho^i \mu}{(1+\delta)} < v \leq \frac{\rho^i \mu}{(1-\delta)}, \\ & \qquad i = 0, 1, \ldots N-1 \\ 0 & \text{if} \quad 0 \leq v \leq \frac{\rho^{N-1}\mu}{(1+\delta)} \\ -Q(-v) & \text{if} \quad v < 0 \end{cases} \tag{4}$$

where δ and ρ are related by

$$\delta = \frac{1-\rho}{1+\rho}. \tag{5}$$

3. Previous Results

This section reviews two results proposed in [6], where the quadratic stabilization of linear feedback systems with a logarithmic quantizer with an infinite number of levels is solved using the sector bound approach and H_∞ optimization. Let the logarithmic quantizer $\bar{Q}(\cdot)$ with an infinite number of levels as shown in Fig. 1 and defined by

$$\bar{Q}(v) = \begin{cases} \rho^i \mu, & \text{if} \quad \frac{\rho^i \mu}{(1+\delta)} < v \leq \frac{\rho^i \mu}{(1-\delta)}, \\ & \qquad i = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{if} \quad v = 0 \\ -\bar{Q}(-v), & \text{if} \quad v < 0 \end{cases} \tag{6}$$

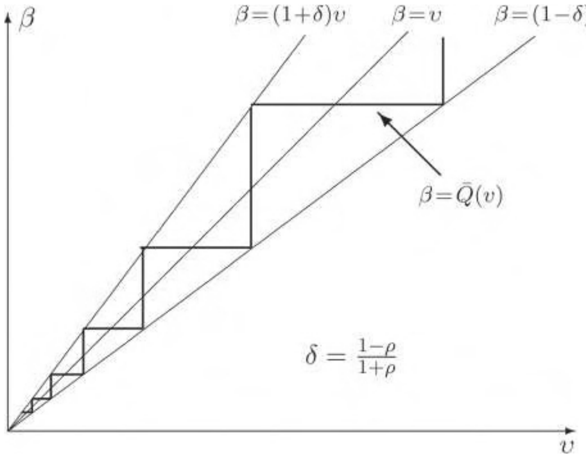

Fig. 1. Logarithmic quantizer with an infinite number of levels.

Notice from Fig.1 that the quantizer $\bar{Q}(\cdot)$ can be bounded by a sector $(1+\Delta)v$, where $\Delta \in [-\delta, \delta]$.

If we consider the system (1) with the controller of either (2) or (3) in Configurations I or II, we get from [6] and [3] the following results.

Theorem 3.1: *Consider the system (1). For a given quantization density ρ, this system is quadratically stabilizable via a quantized state feedback controller (2) with $Q(\cdot) \equiv \bar{Q}(\cdot)$, if and only if the following auxiliary system:*

$$x(k+1) = Ax(k) + B(1+\Delta)r(k), \quad |\Delta| \leq \delta \tag{7}$$

is quadratically stabilizable with $r(k) = Kx(k)$, where δ and ρ are related by (5). Moreover, the largest sector bound $\delta_{\sup}$ for quadratic stabilization, which provides the smallest quantization density $\rho_{\inf}$, is given by

$$\delta_{\sup} = \frac{1}{\inf\limits_K \|G_{sf}(z)\|_\infty} \tag{8}$$

where

$$G_{sf}(z) = K(zI - A - BK)^{-1}B. \tag{9}$$

Theorem 3.2: *Let the system (1) and a quantizer $\bar{Q}(\cdot)$ in either Configurations I or II. For a given quantization density ρ, this system is quadratically stabilizable via an output feedback controller (3) if and only if the system*

$$\begin{cases} x(k+1) = Ax(k) + Br(k) \\ \quad s(k) = (1+\Delta)Cx(k), \quad |\Delta| \leq \delta \end{cases} \tag{10}$$

in the case of Configuration I, or the system

$$\begin{cases} x(k+1) = Ax(k) + B(1+\Delta)r(k) \\ \quad s(k) = Cx(k), \quad |\Delta| \leq \delta \end{cases} \tag{11}$$

in the case of Configuration II, is quadratically stabilizable via a controller (3), where δ and ρ are related by (5). Moreover, for both configurations, the largest sector bound $\delta_{\sup}$ for quadratic stabilization, which provides the smallest quantization density $\rho_{\inf}$, is given by

$$\delta_{\sup} = \frac{1}{\inf\limits_{A_c,B_c,C_c,D_c} \|G_{of}(z)\|_\infty} \tag{12}$$

where

$$G_{of}(z) = \frac{G(z)H(z)}{1 - G(z)H(z)}, \tag{13}$$

$$\begin{aligned} G(z) &= C(zI - A)^{-1}B \\ H(z) &= C_c(zI - A_c)^{-1}B_c + D_c. \end{aligned} \tag{14}$$

Remark 3.1: *It follows from Theorems 3.1 and 3.2 that the optimal state feedback gain K and the output feedback controller (3), in the sense of achieving quadratic stability with a minimum quantization density, can be found in terms of standard H_∞ control problems. Thus, the controller parameter K (or (A_c, B_c, C_c, D_c)) in Theorem 3.1 (or Theorem 3.2) can be readily obtained via LMI algorithms [8]. The latter remark also applies to suboptimal controllers.* □

4. Stability Analysis

The results of Section 3 apply to quantized feedback systems for which the quantizer has an infinite number of quantization levels. When dealing with finite-level quantizers, in general, we cannot assure that the state trajectory will converge to the state-space origin (the equilibrium point under analysis). In the sequel we shall derive LMI conditions to ensure the convergence, in finite time, of the state trajectory to a small invariant neighborhood of the origin.

4.1. General Setup

First, we introduce an auxiliary system which encompasses the closed-loop system for the three feedback control laws with the finite-level quantizer in (4) under analysis, namely the state feedback controller (2) and the output feedback controller (3) in either Configuration I or II. To this end, we define the following system:

$$\begin{cases} \zeta(k+1) = A_i\zeta(k) + B_iQ(r(k)) \\ r(k) = C_i\zeta(k), \quad i = 1,2,3 \end{cases} \tag{15}$$

where $\zeta \in \mathbb{R}^{n_i}$, $Q(\cdot)$ is the quantizer function as defined in (4) and the index i is related to the feedback control under consideration. More specifically, $i = 1$ refers to state feedback, $i = 2$ is for output feedback in Configuration I, and $i = 3$ refers to output feedback in Configuration II. From straightforward algebraic manipulations, we obtain the following results:

$$\begin{aligned} &\zeta = x, \quad n_i = n, \quad \text{for } i = 1; \\ &\zeta = [x' \; \xi']', \quad n_i = n + n_c, \quad \text{for } i = 2,3 \end{aligned} \tag{16}$$

$$A_1 = A, \quad B_1 = B, \quad C_1 = K \tag{17}$$

$$A_2 = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad A_3 = \begin{bmatrix} A & 0 \\ B_cC & A_c \end{bmatrix},$$

$$B_2 = \begin{bmatrix} BD_c \\ B_c \end{bmatrix} \qquad B_3 = \begin{bmatrix} B' \\ 0 \end{bmatrix}', \tag{18}$$

$$C_2 = [\,C \quad 0\,], \qquad C_3 = [\,D_cC \quad C_c\,]. \tag{19}$$

Throughout the paper, we shall use the matrices A_i, B_i and C_i, and n_i in the sense as above, where the index i refers to the feedback control under consideration.

In connection with the closed-loop system (15) and the finite-level logarithmic quantizer (4), let the following sets:

$$\mathcal{B} = \{\zeta \in \mathbb{R}^{n_i} : |C_i\zeta| \le \mu/(1-\delta)\} \tag{20}$$

$$\mathcal{C} = \{\zeta \in \mathbb{R}^{n_i} : |C_i\zeta| \le \varepsilon\}, \quad \varepsilon = \rho^{N-1}\mu/(1+\delta) \tag{21}$$

for $i = 1,2$ or 3, depending on the feedback being considered, and where δ and μ are as in (4). The sets $\mathcal{B}$ and $\mathcal{C}$ are related to respectively the largest and smallest quantization levels. These sets are unbounded along the directions of the vectors of an orthogonal basis of the null space of C_i and bounded by two hyperplanes orthogonal to C_i' and symmetric with respect to origin. The distance between these hyperplanes is $2\mu(1-\delta)^{-1}/\sqrt{C_iC_i'}$ for $\mathcal{B}$ and $2\varepsilon/\sqrt{C_iC_i'}$ for $\mathcal{C}$.

Note that when the state ζ of system (15) lies in $\mathcal{C}$, then $Q(C_i\zeta) = 0$ and therefore the input signal to the latter system is zero. Thus, in general, the trajectory of ζ will not converge to the origin and hence quadratic stability will not hold. To handle this situation, and motivated by the notion of practical stability used in [5], in the sequel we will introduce the notion of stability adopted in this paper. Let the quadratic functions

$$V(\zeta) = \zeta' P\zeta, \quad V_a(\zeta) = \zeta' P_a\zeta, \quad P > 0, \quad P_a > 0 \quad (22)$$

where ζ is as in (16), and the sets

$$\mathcal{D} = \{\zeta \in \mathbb{R}^{n_i} : V(\zeta) \le 1\},$$
$$\mathcal{A} = \{\zeta \in \mathbb{R}^{n_i} : V_a(\zeta) \le 1\} \quad (23)$$

$$\mathcal{C}_p = \{\zeta \in \mathcal{C} : DV_a(\zeta) \ge 0\} \quad (24)$$

where the notation $Df(\zeta(k))$ for a real sequence $f(\cdot)$ is defined by

$$Df(\zeta(k)) := f(\zeta(k+1)) - f(\zeta(k)).$$

Definition 4.1: *Consider the closed-loop system (15) with either the state feedback (2) or the output feedback (3) in Configuration I or II. This system is* widely quadratically stable, *if there exist quadratic functions $V(\zeta)$ and $V_a(\zeta)$ as above such that the following conditions hold:*

$$\mathcal{A} \subset \mathcal{D}, \quad \mathcal{D} \subset \mathcal{B} \quad (25)$$

$$DV(\zeta) < 0, \forall\zeta \in \mathcal{D}\backslash\mathcal{C} \quad (26)$$

$$DV_a(\zeta) < 0, \forall\zeta \in \mathcal{A}\backslash\mathcal{C}_p \quad (27)$$

$$\zeta(k+1) \in \mathcal{A}, \quad \textit{whenever} \quad \zeta(k) \in \mathcal{C}_p. \quad (28)$$

Definition 4.1 implies that for any initial condition in $\mathcal{D}$, the state trajectory of system (15) will enter $\mathcal{A}$ in finite time and will remain in this set. Thus, $\mathcal{A}$ is an attractor of $\mathcal{D}$ and the latter set will be referred to as the set of admissible initial states (or conditions).

The above stability notion was inspired, and is similar, to the notion of practical stability as introduced in [5]. Note that [5] uses practical stability to construct a finite-level logarithmic quantizer employing ellipsoidals of the same shape for the set of admissible initial states $\mathcal{D}$ and its attractor estimate $\mathcal{A}$. In contrast, wide quadratic stability allows for using ellipsoidals of different shapes for $\mathcal{D}$ and $\mathcal{A}$, which is a desired feature due to the shape of $\mathcal{B}$. This fact will be illustrated in Example 2 of the next section. Observe that if we constrain the shape of $\mathcal{A}$ in the above definition to be $\mathcal{A} = \{\zeta \in \mathbb{R}^{n_i} : V(\zeta) \le \varpi^{-1}, \varpi > 1\}$, we recover the idea of practical stability as applied in [5].

4.2. Main Results

First, considering (15), condition $DV(\zeta) < 0$ is given by

$$\begin{bmatrix} \zeta \\ Q(r) \end{bmatrix}' \begin{bmatrix} A_i'PA_i - P & A_i'PB_i \\ B_i'PA_i & B_i'PB_i \end{bmatrix} \begin{bmatrix} \zeta \\ Q(r) \end{bmatrix} < 0 \quad (29)$$

where r is as defined in (15). Also, notice that for all $\zeta \in \mathcal{B}\backslash\mathcal{C}$, $Q(r)$ satisfies the following sector bound condition [6]:

$$(Q(r) - (1-\delta)r)'(Q(r) - (1+\delta)r) \le 0. \quad (30)$$

Thus, condition (26) is satisfied if and only if (29) holds subject to (30). By applying the $\mathcal{S}$-procedure [1], the latter holds if and only if

$$\eta' \begin{bmatrix} A_i'PA_i - P - \tau_1(1-\delta^2)C_i'C_i & A_i'PB_i + \tau_1 C_i' \\ B_i'PA_i + \tau_1 C_i & B_i'PB_i - \tau_1 \end{bmatrix} \eta < 0 \quad (31)$$

where $\eta = [\zeta' \quad Q(r)']'$ and $\tau_1 > 0$ is a multiplier to be found introduced by the $\mathcal{S}$-procedure.

Observe that condition (31) with P and τ_1 replaced by P_a and τ_2, respectively, ensures that $DV_a(\zeta) < 0, \forall\zeta \in \mathcal{B}\backslash\mathcal{C}$. This together with (25) and considering the definition of the set $\mathcal{C}_p$, will ensure the feasibility of (27). Further, (27) and (28) ensure that $\mathcal{C}_p$ is bounded and $\mathcal{C}_p \subset \mathcal{A}$, otherwise $\zeta(k)$ could eventually leave $\mathcal{A}$.

Theorem 4.1: *Let $Q(\cdot)$ be a finite-level quantizer as defined in (4), where μ, ρ and N are given, and consider the system (1) with either a given state feedback controller (2) or an output feedback controller (3) in Configuration I or II. The resulting closed-loop system (15) is widely quadratically stable if there exist matrices $P > 0$ and $P_a > 0$, and positive scalars $\tau_1, \cdots, \tau_4$ satisfying the following inequalities:*

$$P_a - P > 0 \quad (32)$$

$$P - (1-\delta)^2\mu^{-2}C_i'C_i > 0 \quad (32)$$

$$\begin{bmatrix} A_i'PA_i - P - \tau_1(1-\delta^2)C_i'C_i & A_i'PB_i + \tau_1 C_i' \\ B_i'PA_i + \tau_1 C_i & B_i'PB_i - \tau_1 \end{bmatrix} < 0 \quad (34)$$

$$\begin{bmatrix} A_i'P_aA_i - P_a - \tau_2(1-\delta^2)C_i'C_i & A_i'P_aB_i + \tau_2 C_i' \\ B_i'P_aA_i + \tau_2 C_i & B_i'P_aB_i - \tau_2 \end{bmatrix} < 0 \quad (35)$$

$$\tau_3 - \tau_4 \ge 0 \quad (36)$$

$$P_a - (1+\tau_3)A_i'P_aA_i + \tau_4\varepsilon^{-2}C_i'C_i \ge 0 \quad (37)$$

for $i = 1, 2$ or 3 depending on the feedback being used, where δ is related to ρ by (5) and ε is as in (21). Moreover, the set $\mathcal{D}$ of admissible initial states and its attractor $\mathcal{A}$ are given by (23).

Proof: First, in view of (20) and (23), the inequalities (32) and (33) ensure that $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{D} \subset \mathcal{B}$, respectively.

Next, (34) guarantees that (31) holds, implying that condition (26) is satisfied. Similarly, (35) together with (25) and the definition of set $\mathcal{C}_p$ ensures that condition (27) holds.

Adding (36) to (37) post-multiplied by $\phi \in \mathbb{R}^{n_i}$ and pre-multiplied by ϕ', we get

$$(1 - \phi' A_i' P_a A_i \phi) - \tau_3^{-1} \phi'(A_i' P_a A_i - P_a)\phi$$
$$-\tau_3^{-1}\tau_4(1 - \varepsilon^{-2}\phi' C_i' C_i \phi) \geq 0, \ \forall \phi \in \mathbb{R}^{n_i}.$$

By the S-procedure, the latter inequality implies that

$$\begin{aligned} &\phi' A_i' P_a A_i \phi \leq 1, \ \forall \phi \in \mathbb{R}^{n_i} : \varepsilon^{-2}\phi' C_i' C_i \phi \leq 1, \\ &\phi'(A_i' P_a A_i - P_a)\phi \geq 0. \end{aligned} \tag{38}$$

Note that the second inequality of (38) is equivalent to $\phi \in \mathcal{C}$. With $\phi = \zeta(k)$ as in (15), and considering that for $\zeta(k) \in \mathcal{C}$ the input signal $Q(r(k))$ of (15) is zero, then (38) leads to

$$\begin{aligned} &\zeta(k+1)' P_a \zeta(k+1) \leq 1, \\ &\forall \zeta(k) \in \mathcal{C} : \zeta(k+1)' P_a \zeta(k+1) - \zeta(k)' P_a \zeta(k) \geq 0 \end{aligned}$$

which ensures that condition (28) is satisfied. Hence, we conclude that system (15) is widely quadratically stable. □

Remark 4.1: *Notice that in Theorem 4.1 the controller and the quantizer $Q(\cdot)$ are considered to be known. A possible way to determine a controller (state or output feedback) is to employ the design of either Theorem 3.1 or 3.2 for logarithmic quantizers with an infinite number of quantization levels, and choose the quantization density ρ of the finite-level quantized such that $\rho \geq \rho_{\inf}$, where $\rho_{\inf}$ is the smallest quantization density given by these theorems. The* maximum quantization level μ *and the* zero-level quantization error $\varepsilon = \rho^{N-1}\mu(1+\delta)^{-1}$ *are then chosen by the designer. Observe that for a given μ, Theorem 4.1 can be used to determine the maximum admissible zero-level quantization error, which gives the smallest admissible N. This can be achieved by searching for the largest value of $\varepsilon > 0$ such that the inequalities (32)–(37) of Theorem 4.1 are feasible.* □

Remark 4.2: *Observe that (37) is not jointly convex in τ_3 and P_a. However, for a given τ_3 the inequalities (32)–(37) become LMIs. Thus, a direct approach to solve these inequalities is to search for the parameter $\tau_3 > 0$. A line search seems to be an appropriate way to optimize τ_3.* □

Remark 4.3: *It turns out that Theorem 4.1 can be readily extended to deal with linear systems subject to parameter uncertainty, where in (1) we have matrices $A(\theta)$ and $B(\theta)$ depending affinely on a convex bounded uncertain parameters vector $\theta \in \mathbb{R}^{n_p}$ that is confined to a polytope Θ with given vertices $\bar{\theta}_k \in \mathbb{R}^{n_p}, k = 1, \ldots, n_v$. In this situation, the inequalities in (34), (35) and (37) of Theorem 4.1 need to be modified as described below. Applying Schur's complement to (34), (35) and (37) with the matrices A and B replaced by $A(\theta)$ and $B(\theta)$ and using convexity arguments, it can be easily verified that these inequalities are satisfied for all $\theta \in \Theta$ if and only if the following inequalities are feasible:*

$$\begin{bmatrix} -P - \tau_1(1-\delta^2)C_i'C_i & \tau_1 C_i' & A_i(\theta)'P \\ \tau_1 C_i & -\tau_1 & B_i(\theta)'P \\ PA_i(\theta) & PB_i(\theta) & -P \end{bmatrix}$$
$$< 0, \quad \theta = \bar{\theta}_k, \quad k = 1, \ldots, n_v \tag{39}$$

$$\begin{bmatrix} -P_a - \tau_2(1-\delta^2)C_i'C_i & \tau_2 C_i' & A_i(\theta)'P_a \\ \tau_2 C_i & -\tau_2 & B_i(\theta)'P_a \\ P_a A_i(\theta) & P_a B_i(\theta) & -P_a \end{bmatrix}$$
$$< 0, \quad \theta = \bar{\theta}_k, \quad k = 1, \ldots, n_v \tag{40}$$

$$\begin{bmatrix} P_a + \tau_4 \varepsilon^{-2} C_i' C_i & (1+\tau_3)A_i(\theta)'P_a \\ (1+\tau_3)P_a A_i(\theta) & (1+\tau_3)P_a \end{bmatrix} \geq 0,$$
$$\theta = \bar{\theta}_k, \quad k = 1, \ldots, n_v \tag{41}$$

where $A_i(\theta)$ and $B_i(\theta)$ are the matrices A_i and B_i in (17)–(19) with A and B replaced by respectively $A(\theta)$ and $B(\theta)$. □

In general, it is desirable to find the set $\mathcal{D}$ of maximum size, in the sense of its volume, or the smallest $\mathcal{A}$. Since $\mathcal{D}$ is an ellipsoid, one approach to maximize its size is to minimize Trace (P). The motivation for this is that $n_i(\text{Trace}\,(P))^{-1} \leq \text{Trace}\,(P^{-1}),\ P \in \mathbb{R}^{n_i \times n_i}$, and Trace (P^{-1}) is the sum of the squared semi-axis lengths of the ellipsoid $\mathcal{D}$. Similarly, an approach to minimize the size of $\mathcal{A}$ is to maximize Trace(P_a). In the light of the latter arguments, the size of the set $\mathcal{D}$ of Theorem 4.1 can be maximized by solving the following optimization problem:

$$\begin{aligned} &\min_{\gamma_1, P, P_a, \tau_1, \cdots, \tau_4} \gamma_1, \ \text{subject to (32)-(37) and} \\ &\qquad P > 0, \quad \tau_j > 0, \ j = 1, \cdots, 4, \\ &\qquad \gamma_1 - \text{Trace}(P) \geq 0. \end{aligned} \tag{42}$$

On the other hand, we can minimize the size of $\mathcal{A}$ via the optimization problem as below

$$\max_{\gamma_2,P,P_a,\tau_1,\cdots,\tau_4} \gamma_2, \quad \text{subject to (32)-(37) and} \\ \gamma_2 > 0, \quad P > 0, \quad \tau_j > 0,\ j = 1,\cdots,4, \\ \mathrm{Trace}(P_a) \tag{43}$$

It may often be desirable to jointly optimize the size of the sets $\mathcal{D}$ and $\mathcal{A}$. This joint optimization is, in general, a difficult problem. A way to jointly achieve $\mathcal{D}$ of a large size and $\mathcal{A}$ of a small size is to minimize $\gamma := \gamma_1/\gamma_2$, where γ_1 and γ_2 are the parameters in (42) and (43). This optimization problem can be formulated as follows. First, define

$$\kappa = \gamma_2^{-1}, \quad X = \kappa P, \quad X_a = \kappa P_a, \\ \alpha_j = \kappa\tau_j, \quad j = 1,2,4, \quad \alpha_3 = \tau_3$$

where $P, P_a, \tau_1, \cdots, \tau_4$ are as in (32)–(37). Multiplying (32)–(37), (42) and (43) by κ, these inequalities become

$$\gamma - \mathrm{Trace}(X) \geq 0 \tag{44}$$

$$\mathrm{Trace}(X_a) - 1 \geq 0 \tag{45}$$

$$X_a - X > 0 \tag{46}$$

$$X - \kappa(1-\delta)^2\mu^{-2}C_i'C_i > 0 \tag{47}$$

$$\begin{bmatrix} A_i'X_aA_i - X - \alpha_1(1-\delta^2)C_i'C_i & A_i'XB_i + \alpha_2C_i' \\ B_i'XA_i + \alpha_1C_i & B_i'XB_i - \alpha_1 \end{bmatrix} < 0 \tag{48}$$

$$\begin{bmatrix} A_i'X_aA_i - X_a - \alpha_2(1-\delta^2)C_i'C_i & A_i'X_aB_i + \alpha_2C_i' \\ B_i'X_aA_i + \alpha_2C_i & B_i'X_aB_i - \alpha_2 \end{bmatrix} < 0 \tag{49}$$

$$\alpha_3\kappa - \alpha_4 \geq 0 \tag{50}$$

$$X_a - (1+\alpha_3)A_i'X_aA_i + \alpha_4\varepsilon^{-2}C_i'C_i \geq 0. \tag{51}$$

Then, the optimization problem to minimize γ is as follows:

$$\left\{ \begin{array}{l} \min\limits_{\gamma,\kappa,X,X_a,\alpha_1,\cdots,\alpha_4} \gamma, \quad \text{subject to (44)-(51) and} \\ \kappa > 0, \quad X > 0, \quad \alpha_j > 0,\ j = 1,\cdots,4 \end{array} \right\} \tag{52}$$

and we have that $P = \kappa^{-1}X$ and $P_a = \kappa^{-1}X_a$.

Note that remarks similar to those of Remarks 4.2 and 4.3 apply to the three latter optimization problems.

4.3. Finite-Level Quantizer Construction

Theorem 4.1 provides a method of deriving a set of admissible initial states $\mathcal{D}$ and its attractor $\mathcal{A}$ for a finite-level quantizer (4) with given maximum quantization level μ and zero-level error ε. However, this theorem can be also applied to design a quantizer which guarantees wide quadratic stability. Given the set $\mathcal{D}_0 = \{\zeta : \zeta' P_0 \zeta \leq 1\}$, $P_0 > 0$, of admissible initial states and an upper-bound ϑ of the volume of an attractor $\mathcal{A} = \{\zeta : \zeta' P_a \zeta \leq 1\}$[1] of $\mathcal{D}_0$, with $P_a > 0$ to be found, a suitable quadratically stabilizing controller and a finite-level quantizer $Q(\cdot)$ can be obtained by the following procedure:

Step 1: Design a quadratically stabilizing controller (either state feedback or output feedback) for a logarithmic quantizer with an infinity number of levels and an appropriate density ρ using any available method, and let $\delta = (1-\rho)(1+\rho)^{-1}$. For instance, consider the coarse quantization controller of either Theorem 3.1 or 3.2 that achieves $\delta_{\sup}$ of (8) or (12), respectively. Choose the quantization density ρ of the finite-level quantizer $Q(\cdot)$ such that

$$\rho > \frac{1-\delta_{\sup}}{1+\delta_{\sup}}.$$

Step 2: Find matrices $P > 0$ and $P_a > 0$, and positive scalars $\gamma_1, \gamma_2, \tau_1, \tau_2, \tau_3$ and $\bar{\tau}_4$ satisfying (34), (35) and

$$P_0 - P > 0, \quad P_a - P_0 > 0, \tag{53}$$

$$P - (1-\delta)^2\gamma_1 C_i'C_i > 0, \tag{54}$$

$$\tau_3\gamma_2 - \bar{\tau}_4 \geq 0, \tag{55}$$

$$P_a - (1+\tau_3)A_i'P_aA_i + \bar{\tau}_4C_i'C_i \geq 0, \tag{56}$$

$$\vartheta^{\frac{2}{n_i}}P_a - \sigma^{\frac{2}{n_i}}I \geq 0 \tag{57}$$

where n_i is as in (16). Then, the parameter μ and ε of the finite-level quantizer $Q(\cdot)$ are given by $\mu = 1/\sqrt{\gamma_1}$ and $\varepsilon = 1/\sqrt{\gamma_2}$.

Step 3: The number of positive quantization levels of $Q(\cdot)$ is given by the smallest integer N satisfying $N \geq 1 + \log_\rho \frac{\varepsilon(1+\delta)}{\mu}$.

[1] The volume of $\mathcal{A}$ is given by $\sigma\prod_{k=1}^{n_i}\lambda_k^{-\frac{1}{2}}(P_a)$, where σ is a constant and $\lambda_k(P_a)$ are the eigenvalues of P_a.

Notice that the inequalities of *Step 2* ensure that the conditions (32)–(37) of Theorem 4.1 hold for the given set $\mathcal{D}_0$ of admissible initial states and an attractor $\mathcal{A}$ with a prescribed upper-bound ϑ on its volume. The reason for this is as follows: (i) the conditions in (53) imply $\mathcal{D}_0 \subset \mathcal{D}$ and $\mathcal{A} \subset \mathcal{D}_0$, and thus (32) holds; (ii) inequality (54) is equivalent to (33) with $\gamma_1 = \mu^{-2}$; (iii) conditions (55) and (56) are equivalent to respectively (36) and (37) with $\bar{\tau}_4 = \gamma_2 \tau_4$ and $\gamma_2 = \varepsilon^{-2}$; (iv) inequality (57) ensures the upper-bound ϑ for the volume of $\mathcal{A}$. Moreover, the values of μ and ε follows directly from the definition of γ_1 and γ_2, respectively, whereas N in *Step 3* is derived from the fact that $\varepsilon = \rho^{N-1}\mu/(1+\delta)$.

Similar to Theorem 3.1, we can either minimize μ or maximize ε by solving optimization problems for minimizing γ_1 or maximizing γ_2 respectively, subject to the inequalities of *Step 2*. Moreover, we can jointly achieve a small μ and a large ε by minimizing $\gamma := \mu^2/\varepsilon^2$. This optimization problem can be readily derived by multiplying the inequalities of *Step 2* by $\kappa := \mu^2$ and setting $X = \kappa P,\ X_a = \kappa P_a,\ \alpha_j = \kappa \tau_j$, $j = 1, 2$, $\alpha_3 = \tau_3$ and $\alpha_4 = \kappa \bar{\tau}_4$, leading to

$$\min_{\gamma,\kappa,X,X_a,\alpha_1,\cdots,\alpha_4} \gamma, \quad \text{subject to:} \tag{58}$$

$$\kappa > 0, \quad X > 0, \quad \alpha_j > 0,\ j = 1, \cdots, 4 \tag{59}$$

$$\kappa P_0 - X > 0, \quad X_a - \kappa P_0 > 0, \tag{60}$$

$$X - (1-\delta)^2 C_i' C_i > 0, \tag{61}$$

$$\begin{bmatrix} A_i' X A_i - X - \alpha_1(1-\delta^2) C_i' C_i & A_i' X B_i + \alpha_1 C_i' \\ B_i' X A_i + \alpha_1 C_i & B_i' X B_i - \alpha_1 \end{bmatrix} > 0, \tag{62}$$

$$\begin{bmatrix} A_i' X_a A_i - X_a - \alpha_2(1-\delta^2) C_i' C_i & A_i' X_a B_i + \alpha_2 C_i' \\ B_i' X_a A_i + \alpha_2 C_i & B_i' X_a B_i - \alpha_2 \end{bmatrix} < 0, \tag{63}$$

$$\alpha_3 \gamma - \alpha_4 \geq 0, \tag{64}$$

$$X_a - (1+\alpha_3) A_i' X_a A_i + \alpha_4 C_i' C_i \geq 0, \tag{65}$$

$$\vartheta^{\frac{2}{n_i}} X_a - \sigma^{\frac{2}{n_i}} \kappa I \geq 0. \tag{66}$$

Moreover, we have $\mu = \sqrt{\kappa},\ \ \varepsilon = \sqrt{\kappa/\gamma},\ \ P = \kappa^{-1} X$ and $P_a = \kappa^{-1} X_a$.

Remark 4.4: *Similar to the optimization problems of Section 4.2, the latter optimization problem is non-convex. However, for a given α_3 the problem becomes convex. Thus, a way to minimize γ via convex optimization is to search for the parameter $\alpha_3 > 0$ that gives the smallest γ, and this can be readily achieved via, for instance, a line search procedure. It should be also noted that remarks along the lines of those of Remark 4.3 apply to the inequalities in (62), (63) and (65) in the case where the matrices A and B are affinely dependent on polytopic-type uncertain parameters.* □

5. Numerical Examples

5.1. Example 1

Consider the non-minimum phase open-loop unstable discrete-time system of [6, Example 3.1] given as follows:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = 2x_2(x) + u(k) \\ y(k) = -3x_1(k) + x_2(k) \end{cases} \tag{67}$$

which has the transfer function $G(z) = \frac{z-3}{z(z-2)}$.

First, state and output feedback controllers are designed considering a logarithmic quantizer with an infinite number of quantization levels. Applying Theorem 3.1 we obtain the following state feedback controller

$$K = -[0 \quad 1.99], \quad \rho_{\inf} = 1/3 \quad (\Leftrightarrow \delta_{\sup} = 1/2)$$

and by Theorem 3.2, and for configurations I and II, we get an output feedback controller with

$$A_c = -5, \quad B_c = 1, \quad C_c = -50/3,$$
$$D_c = 10/3, \quad \rho_{\inf} = 0.8182 \quad (\Leftrightarrow \delta_{\sup} = 1/10).$$

It is assumed that the finite-level quantizer has a maximum level $\mu = 2.1$ and $\rho = \rho_{\inf}$ for state feedback and output feedback, for both configurations I and II. The maximum admissible zero-level quantization error ε is then chosen such that the conditions of Theorem 4.1 are satisfied (see Remark 4.1).

For the above state feedback controller, Fig. 2 shows part of the set $\mathcal{D}$ of admissible initial states and its attractor $\mathcal{A}$, as obtained from the optimization problem in (52), along with a stable and two unstable state trajectories. The maximum admissible zero-level quantization error is $\varepsilon = 0.5$, that by (21) yields $N = 2$. Thus, the required number of bits N_b for the quantizer is $N_b = 3$.

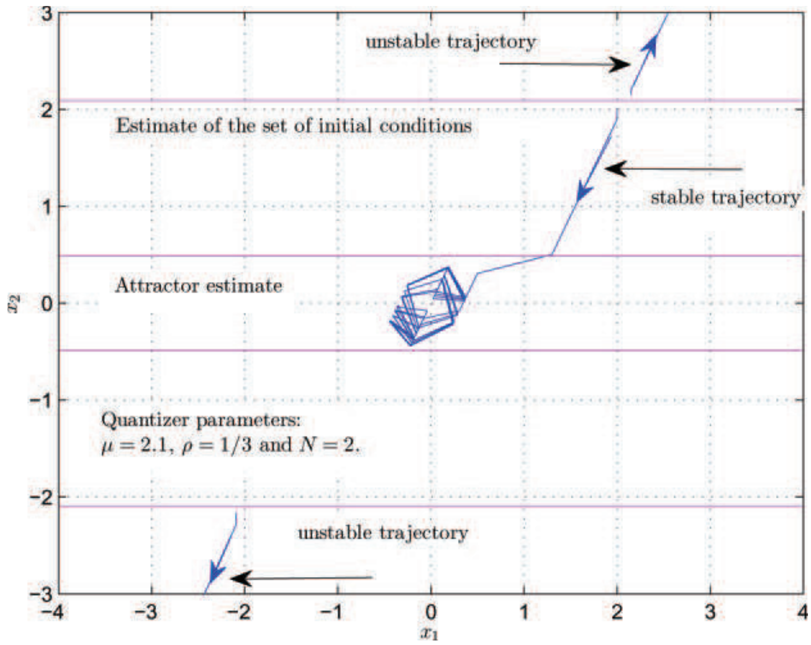

Fig. 2. Sets $\mathcal{D}$ and $\mathcal{A}$ for the state feedback controller, and a stable and two unstable state trajectories.

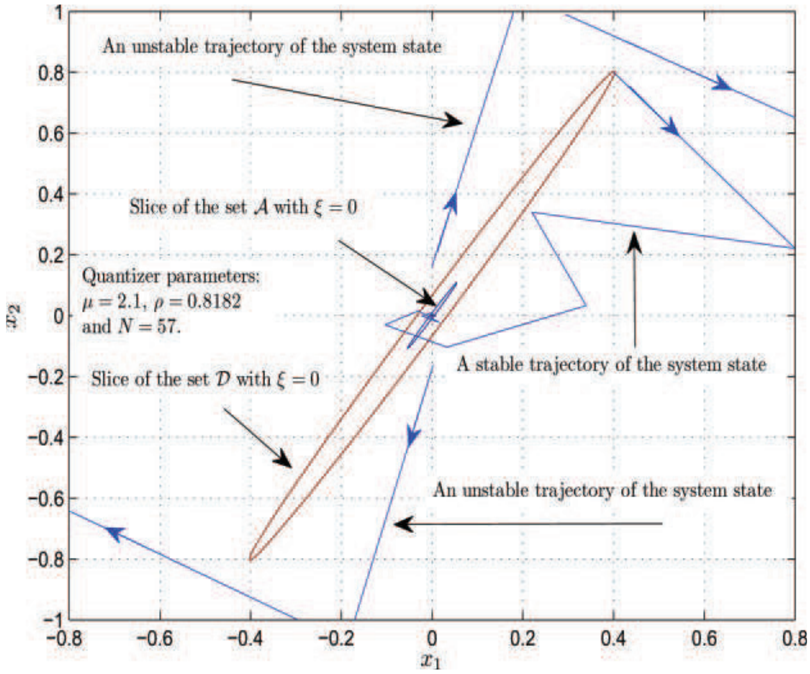

Fig. 3. A slice of the sets $\mathcal{D}$ and $\mathcal{A}$ (with $\xi = 0$) for output feedback in Configuration I, and a stable and two unstable state trajectories.

Considering the above output feedback controller with a quantized measurement, i.e. in Configuration I, we obtain the results in Fig. 3, which displays a slice of $\mathcal{D}$ and $\mathcal{A}$ with $\xi = 0$, as well as a stable and two diverging trajectories of the system state. Note that the maximum ε for the LMIs of Theorem 4.1 to be feasible is 10^{-4}, yielding $N = 57$, which requires a quantizer with $N_b = 7$. On the other hand, applying the output feedback controller with a quantized control signal, i.e. in Configuration II, leads to the results in Fig. 4, which shows a slice of $\mathcal{D}$ and $\mathcal{A}$ with $\xi = 0$, together with a stable and two unstable trajectory of the system state. In this case, the maximum admissible ε is 10^{-3}, resulting in $N = 44$ and $N_b = 7$.

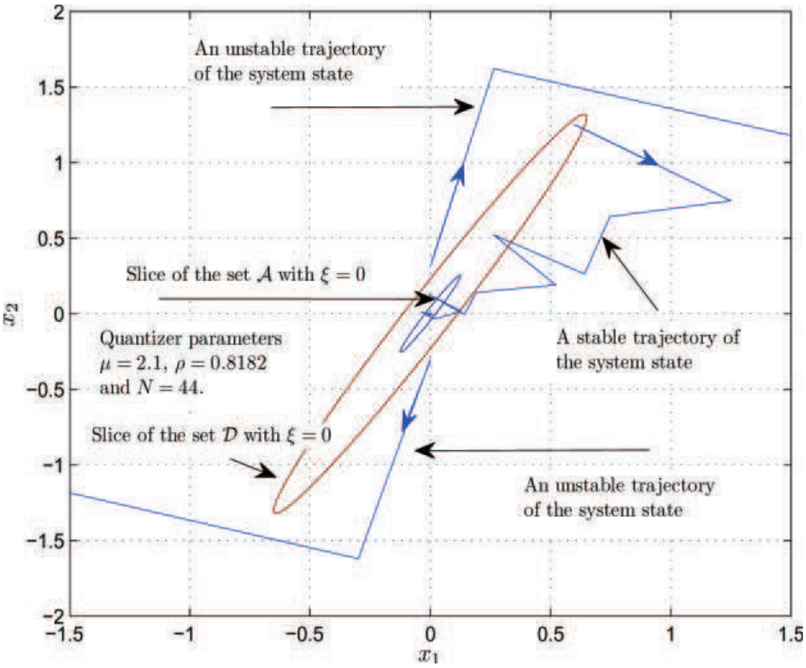

Fig. 4. A slice of the sets $\mathcal{D}$ and $\mathcal{A}$ (with $\xi = 0$) for output feedback in Configuration II, and a stable and two unstable state trajectories.

The simulation results in Figs. 2–4 demonstrate that the estimates of the set of admissible of initial conditions and of the attractor are very tight in the sense the starting points of unstable trajectories are close to the boundary of $\mathcal{D}$ and the stable trajectories lie after a finite time close to the boundary of $\mathcal{A}$.

5.2. Example 2

Consider the magnetic ball levitation system studied in [11], in which a steel ball of mass M is levitated by an electromagnet. The linearized system dynamics around an equilibrium point is given by the following state space representation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2Rg}{v_0}\sqrt{\frac{Mg}{k_1}} & 0 & -\frac{2Rg}{v_0} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t),$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{68}$$

where $v_0 = 10$ volts, $M = 0.068$ kg, $R = 10\Omega$, $L = 0.41$ H, $K_1 = 3.3 \times 10^{-5}\text{Nm}^2/\text{A}^2$ and $g = 9.8\text{m/s}^2$.

Following [11], a quantized discrete-time state feedback control is used. To this end, the state variables of the above system are uniformly sampled with a sampling period $T_s = 4.605$ms and the state feedback gain as follows is applied

$$K = [\,10315.67 \quad 195.02 \quad -49.47\,]. \tag{69}$$

The discrete-time control signal at the instant kT_s is then quantized and held by a zero-order holder at times $t \in \{t : kT_s \le t < (k+1)T_s,\ k = 0, 1, 2, \ldots\}$. The quantizer $\hat{Q}(\cdot)$ considered in [11] is a logarithmic quantizer that is truncated only towards the origin, but its constructive law differs from the one defined in (4). Nevertheless, as proposed in [9], by a scaling procedure we can write $\hat{Q}(\cdot) = \varsigma Q(\cdot)$, where $\varsigma = 1.1289$ and $Q(\cdot)$ is the quantizer as give in (4) with

$$\delta = 0.2806, \quad \rho = 0.5618, \quad \rho^{N-1}\mu = 0.5775$$
$$(\Rightarrow \varepsilon = 0.4510)$$

where N and μ are to be defined in the sequel. Notice that due to ς, the output of the quantizer $Q(\cdot)$ needs to be scaled by the factor ς as above, which corresponds to multiply the system input matrix B by ς.

In light of the above, we obtain the discrete-time closed-loop system representation in (15) with

$$A_1 = \begin{bmatrix} 1.030000 & 0.004651 & -0.000201 \\ 13.01000 & 1.030000 & -0.086240 \\ 0.000000 & 0.000000 & 0.893800 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -8.568 \times 10^{-7} \\ -0.000554 \\ 0.011990 \end{bmatrix}, \quad C_1 = K$$

where the matrix B_1 already includes the scaling factor ς.

In [11] the quantizer $\hat{Q}(\cdot)$ was designed considering a set of admissible initial states given by the ball $B_0 = \{x \in \mathbb{R}^3 : \|x\| \leq 10\}$. Note that since the linearized model describes the system dynamics for small deviations from the equilibrium point, the size of B_0 is somehow unrealistic. Nevertheless, for comparison purposes, we will use a set $\mathcal{D}$ of admissible initial state satisfying the condition $\mathcal{D} \supseteq B_0$, which is ensured by the inequality $P \leq P_0$, with $P_0 = 0.01 I_3$. On the other hand, since we must have $\mathcal{D} \subset \mathcal{B}$ and considering that $|C_1 x(0)| \leq 1.0318 \times 10^5$ for $x(0) \in B_0$, we choose $N = 22$, which implies $\mu = 1.044 \times 10^5$. Moreover, as the attractor estimate in [11] consists of a closed ball centered at the origin of $\mathbb{R}^n$, we constrain the matrix P_a to satisfy $P_a - \lambda I_3 \geq 0$ with $\lambda > 0$ to be found, which implies $\mathcal{A} \subseteq \{x \in \mathbb{R}^3 : \|x\| \leq 1/\sqrt{\lambda}\}$. In light of the latter facts and considering Theorem 4.1, we have determined $\mathcal{A}$ by solving the following optimization problem:

$$\min_{P, P_a, \tau_1, \ldots, \tau_4, \lambda} \lambda, \quad \text{subject to (32)-(37) and} \lambda > 0,$$

$$P > 0, \; P_0 - P \geq 0, \quad P_a - \lambda I_3 \geq 0,$$
$$\tau_j > 0, j = 1, \ldots, 4.$$

Considering Remark 4.2, we solve the above problem by performing a line search on τ_3 and using standard LMI solver package, yielding

$$\lambda = 972.6545,$$
$$P_a = 10^7 \times \begin{bmatrix} 9.5311 & 0.1772 & -0.0393 \\ 0.1772 & 0.0050 & -0.0007 \\ -0.0393 & -0.0007 & 0.0003 \end{bmatrix}.$$

In view of the above, we obtain $\mathcal{A} \subset \{x \in \mathbb{R}^3 : \|x\| \leq 0.0321\}$, that is significantly smaller than the estimates obtained in [11] and [9] which are given by respectively $\{x \in \mathbb{R}^3 : \|x\| \leq 0.053\}$ and $\{x \in \mathbb{R}^3 : \|x\| \leq 0.0936\}$. Notice that the actual attractor estimate derived in [9] is given in terms of upper bounds on the magnitude of the state vector components. To demonstrate that our approach also gives a less conservative componentwise estimate of the attractor $\mathcal{A}$, we have computed the minimum outerbounding box $\mathcal{A}_{\text{box}}$ for the ellipsoid $\mathcal{A}$, i.e. the minimum box that contains $\mathcal{A}$ and with edges orthogonal to the standard axes of $\mathbb{R}^3$. The obtained $\mathcal{A}_{\text{box}}$ is given by $\mathcal{A}_{\text{box}} = \{x \in \mathbb{R}^3 : |x_1| \leq 2.298 \times 10^{-4}, \quad |x_2| \leq 76.45 \times 10^{-4}, \quad |x_3| \leq 320.53 \times 10^{-4}\}$, which is significantly smaller than the result in [9], namely $\{x \in \mathbb{R}^3 : |x_1| \leq 9.2 \times 10^{-4}, \quad |x_2| \leq 363.5 \times 10^{-4}, \quad |x_3| \leq 862.2 \times 10^{-4}\}$.

Finally, to substantiate the importance of the feature of wide quadratic stability that allows for using sets $\mathcal{D}$ and $\mathcal{A}$ of different shapes, we have constrained $\mathcal{D}$ and $\mathcal{A}$ to have the same shape (by setting $P = P_a/\varpi, \varpi > 1$ to be found). In this case, we have obtained $\mathcal{A} \subset \{x \in \mathbb{R}^3 : \|x\| \leq 0.0942\}$, which is significantly more conservative than the one as above where $\mathcal{D}$ and $\mathcal{A}$ are not constrained to have the same shapes. Notice that the difference in "size" of the obtained $\mathcal{A}$ is even more accentuated when comparing componentwise bounds for $\mathcal{A}$. It turns out that the minimum outerbounding box for $\mathcal{A}$ in this case is given by $\mathcal{A}_{\text{box}} = \{x \in \mathbb{R}^3 : |x_1| \leq 18.264 \times 10^{-4}, \quad |x_2| \leq 932.43 \times 10^{-4}, \quad |x_3| \leq 942.04 \times 10^{-4}\}$, which is much large than in the case where we allow for $\mathcal{D}$ and $\mathcal{A}$ of different shapes. In particular, note the huge difference in the bounds for x_1 and x_2.

5.3. Example 3

Consider the inverted pendulum system attached to a cart taken from [17], where the system dynamics is modeled by the following linearized equations w.r.t. the desired equilibrium point:

$$\dot{x}(t) = \begin{bmatrix} 0 & 6.261 \\ 6.261 & 6.261\theta \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 6.261 \end{bmatrix} u(t),$$
$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{70}$$

where $x_1(t)$ is the pendulum angle, $x_2(t) = 0.161\dot{x}_1(t)$, $u(t)$ is the control input, and $\theta \in [0, 1/50]$ is an uncertain parameter representing the friction acting on the pendulum.

For the above system, we are interested in devising a logarithmic quantizer with a coarse quantization considering a stabilizing digital state feedback controller. To this end, we assume a constant sampling period $T_s = 0.0057$s and consider the following approximate discrete-time model for (70) (obtained with Euler's discretization):

$$x(k+1) = \begin{bmatrix} 1.000 & 0.036 \\ 0.036 & 1 - 0.036\theta \end{bmatrix} x(k)$$
$$+ \begin{bmatrix} 0 \\ 0.036 \end{bmatrix} u(k), \quad x(k) = \begin{bmatrix} x_1(kT_s) \\ x_2(kT_s) \end{bmatrix}. \tag{71}$$

We define the set of admissible initial conditions as follows:

$$\mathcal{D}_0 = \left\{ x \in \mathbb{R}^2 : x' \begin{bmatrix} 0.2 & 0 \\ 0 & 14 \end{bmatrix} x \leq 1 \right\}$$

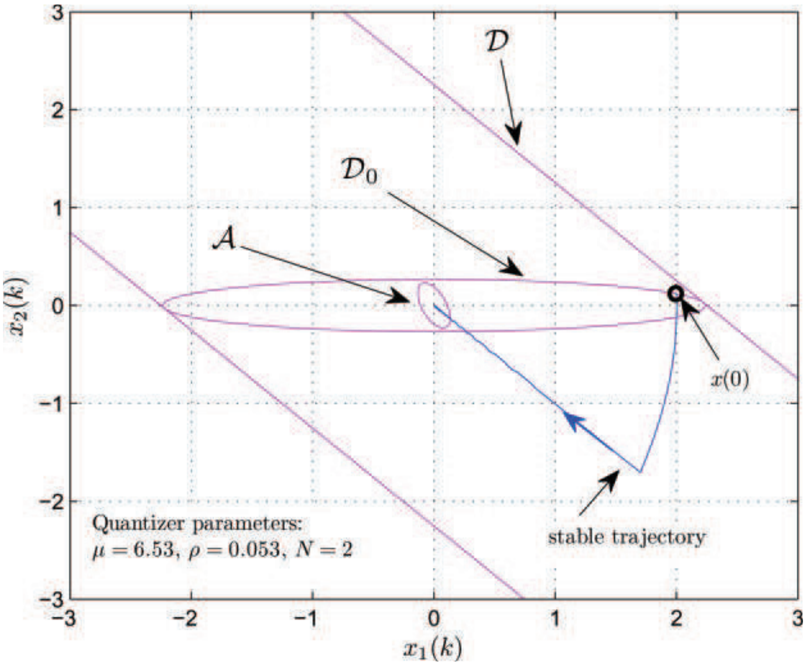

Fig. 5. The set $\mathcal{D}_0$, the estimates of $\mathcal{D}$ and $\mathcal{A}$, and a stable state trajectory of the closed-loop system.

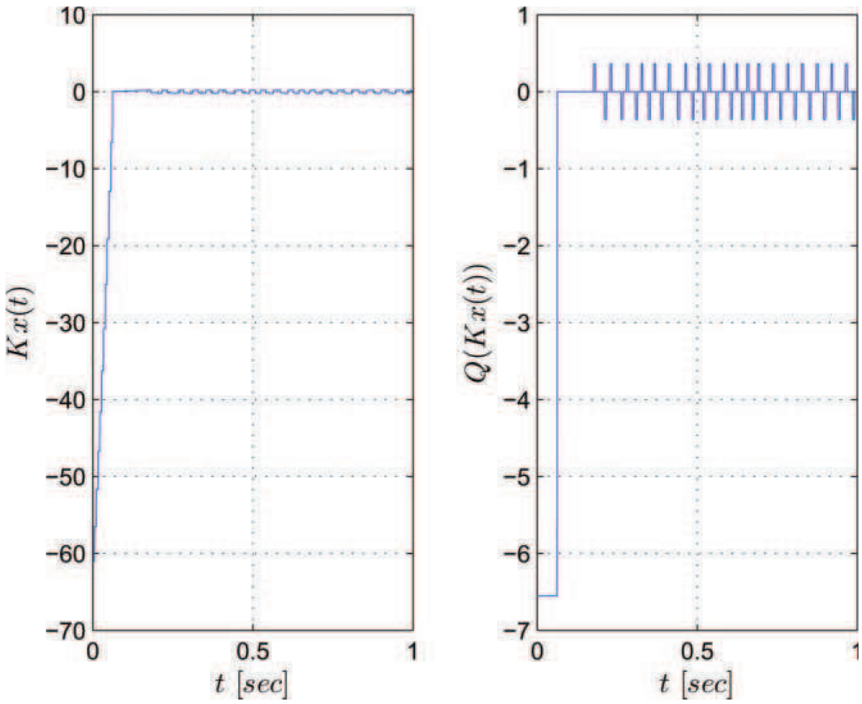

Fig. 6. Input and output quantizer signals for system (70) in closed-loop.

which is an outer approximation of the polytope of initial conditions considered in [17].

To design the coarse quantizer, we apply the procedure given in Section 4.3 and considering (58)–(66) to minimize the number of quantization levels. First, we design a state feedback by means of Theorem 3.1 (taking Remark 4.4 into account) which leads to the following result:

$$K = -[28.81428.769], \quad \delta_{\text{sup}} = 0.9652$$

Second, we take $\rho = 0.053$ (corresponding to $\delta = 0.9$) and assume that the volume of $\mathcal{A}$ is 10% of the volume of $\mathcal{D}_0$ (i.e., $\vartheta = 0.188$). Then, we solve the optimization problem (58)–(66) (taking Remark 4.4 into account) by means of standard LMI solver package and performing a line search on α_3, yielding the following results:

$$P = \begin{bmatrix} 0.1972 & 0.1969 \\ 0.1969 & 0.1970 \end{bmatrix},$$

$$P_a = \begin{bmatrix} 81.8847 & 26.9052 \\ 26.9052 & 27.8176 \end{bmatrix}, \quad \alpha_3 = 0.07,$$

$$\mu = 6.532, \quad \varepsilon = 2.532, \quad N = 2.$$

Fig. 5 shows the given set of admissible initial conditions $\mathcal{D}_0$, the estimates $\mathcal{D} = \{x \in \mathbb{R}^2 : x'Px \leq 1\}$ and $\mathcal{A} = \{x \in \mathbb{R}^2 : x'P_a x \leq 1\}$, and a state trajectory of the closed-loop system starting from $x(0) = [2 \ \ 0.12]'$, which is at the boundary of $\mathcal{D}_0$. The state trajectory simulation was carried out considering the exact discretization of system (70), the quantized control $u(t) = Q(Kx(t))$ for $t = kT_s$ that is held by a zero-order holder at times $t \in \{t : kT_s \leq t < (k+1)T_s\}$, $k = 0, 1, 2, \ldots$, and $\theta = 0$ which corresponds to the worst case of the parameter θ in the sense of the closed-loop system damping factor. For this simulation, we also show in Fig. 6 the input and output quantizer signals, demonstrating that system (70) is practically stabilized with only 5 levels of the control signal, namely $u(t) \in \{\pm 6.53, \pm 0.20, 0\}$.

6. Conclusion

This paper has addressed the stability of SISO discrete-time linear time-invariant systems with a finite-level logarithmically quantized feedback controller. Both state and output feedback controllers have been considered. Based on a relaxed stability notion, referred to as *wide quadratic stability*, we have developed an LMI based approach to estimate a set of admissible initial states and an associated invariant attractor set in a neighborhood of the origin, such that all state trajectories starting in the first set will converge to the attractor in finite time. In addition, when these two sets are *a priori* specified, we have proposed a method to design a finite-level logarithmic quantizer for either state feedback or output feedback stabilizing controllers such that wide quadratic stability is ensured. Numerical examples have shown that: (*i*) for state feedback, wide quadratic stability can be guaranteed with a relatively small number of bits, contrasting with output feedback which requires a significantly larger number of bits; (*ii*) the size of the set of admissible initial states in the output feedback setting is much smaller when compared with the state feedback case; and (*iii*) the proposed approach gives less conservative estimates of the system attractor when compared to existing methods in the literature of quantized feedback systems and provides a powerful tool for constructing logarithmic quantizers.

Acknowledgments

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