Comments on "A Procedure for the Positive Definiteness of Forms of Even Order"

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***Abstract*—The purpose of this paper is to point out that the main result in the above-mentioned paper[1] is erroneous.**

The above-mentioned paper[1] gives a necessary and sufficient condition for the positive definiteness of the following quadratic programming problem:

$$\text{minimize}\{f(x) = x^T Ax\colon h_i(x) = x^T B_i x = 0, \text{for } i = 1, \cdots, m\} \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in \boldsymbol{R}^n$, and A and B_i are $n \times n$ symmetric matrices. This result is also generalized to quartic and sixth-order forms. We point out that these results are all sufficient only.

The result in question is Theorem 1 in the cited paper (called "Theorem 1" in this paper), which is stated as follows: $f(x) \geq 0$ for all $x \in \boldsymbol{R}^n$ satisfying $h_i(x) = 0$ if and only if there exist $\lambda_i \in \boldsymbol{R}$ $(i = 1, \cdots, m)$ such that $A - \Sigma_{i=1}^m \lambda_i B_i \geq 0$ (positive semidefinite).

Our counterexample for the necessity part of "Theorem 1" is as follows.

Counterexample: Take

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 8 \end{bmatrix} = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ \hline 0 & 0 & 0 & 8 \end{array}\right] \tag{2}$$

and

$$h_i(x) = x_i^2 - x_4^2, \qquad i = 1, 2, 3. \tag{3}$$

The matrices $B_i, i = 1, 2, 3$ are naturally defined.

First, we claim that $f(x) \geq 0$ for all x satisfying $h_i(x) = 0, i = 1, 2, 3$. Obviously, $f(x) = 0$ when $x_4 = 0$ because in this case $x = 0$. When $x_4 \neq 0$, we have $x_i = r_i x_4, r_i \in \{-1, 1\}, i = 1, 2, 3$. Subsequently, $f(x) = x_4^2(r^T A_1 r + 8)$, where $r = (r_1, r_2, r_3)^T$. It is verified that

$$\min\{r^T A_1 r + 8\colon r_i \in \{-1, 1\}, \quad i = 1, 2, 3\} = 0.$$

Therefore, our claim holds.

Second, we argue that there exists no $\lambda_i, i = 1, 2, 3$ such that $A - \Sigma_{i=1}^3 \lambda_i B_i \geq 0$. To see this, we write

$$A - \sum_{i=1}^3 \lambda_i B_i = \begin{bmatrix} -\lambda_1 - 2 & 1 & 1 & 0 \\ 1 & -\lambda_2 - 2 & 1 & 0 \\ 1 & 1 & -\lambda_3 - 2 & 0 \\ 0 & 0 & 0 & 8 + \sum_{i=1}^3 \lambda_i \end{bmatrix}. \tag{4}$$

Define $\mu_i = -\lambda_i - 2, i = 1, 2, 3$, and we have $A - \Sigma_{i=1}^3 \lambda_i B_i \geq 0$ if and only if

$$\sum_{i=1}^3 \mu_i \leq 2 \quad \text{and} \quad \overline{A}_1 = \begin{bmatrix} \mu_1 & 1 & 1 \\ 1 & \mu_2 & 1 \\ 1 & 1 & \mu_3 \end{bmatrix} \geq 0. \tag{5}$$

The second inequality above implies that

$$\mu_i \geq 0, \qquad i = 1, 2, 3, \qquad \mu_1\mu_2 \geq 1, \qquad \mu_2\mu_3 \geq 1. \tag{6}$$

The last two inequalities above come from the first and last 2×2 principal minor of $\overline{A}_1$, respectively. The first two inequalities in (6) together with the first inequality in (5) give $\mu_1 = \mu_2 = 1, \mu_3 = 0$. However, the last inequality in (6) cannot be satisfied. Subsequently, no μ_i (hence no λ_i) exists to make $A - \Sigma_{i=1}^3 \lambda_i B_i \geq 0$. That is, the necessity part of Theorem 1 is incorrect.

Since the authors' results on higher order forms all rely on "Theorem 1," the error carries through.

To point out the technical error in "Theorem 1," we note that the authors misused the Lagrange multiplier theorem [1]. To be more precise, the authors claim that a necessary condition for a minimizer x_0 of (1) is that the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i h_i(x)$$

satisfies the following conditions.

1) There exists $\lambda \in \boldsymbol{R}^m$ such that $\nabla_x \mathcal{L}(x_0, \lambda) = 0$.
2) The Hessian $\nabla_x^2 \mathcal{L}(x_0, \lambda) \geq 0$.

However, these conditions are not necessary in general unless the rank of the matrix $(\partial h_i(x_0)/\partial x_0^j)$ is equal to m; see [1, p. 114, Th. 2.2]. In proving the necessity part of "Theorem 1," the authors failed to check the rank condition.

We point out that the sufficient part of "Theorem 1" is still valid. This result, however, is known as the $\mathcal{S}$-procedure [2]. Finally, we note that the minimum in (1) does not exist in general.

References

[1] M. R. Hestenes, *Optimization Theory, The Finite Dimensional Case*. New York: Robert E. Krieger, 1981.

[2] V. A. Yakubovich, "$\mathcal{S}$-procedure in nonlinear control theory," *Vestnik Leninggradskogo Universiteta, Ser. Matematika*, 1971, pp. 62–77.

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[1] M. A. Hasan and A. A. Hasan, *IEEE Trans. Automat. Contr.*, vol. 41, pp. 615–617, Apr. 1996.