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Robust Nonlinear Control: Beyond Backstepping and Nonlinear Forwarding

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Abstract. This paper considers the problem of robust global stabilization of a large class of nonlinear systems. We introduce a recursive design approach based on a powerful technique of Wei for quadratic stabilization of linear systems. This approach allows us to globally stabilize an uncertain nonlinear system which is augmented from a robustly globally stabilizable system via either down-augmentation or up-augmentation. These augmentations are similar to the so-called backstepping and integrator forwarding. However, the advantage of the proposed augmentation approach is that robust stabilizability is achieved for nonlinear systems involving large uncertain parameters.

1 Introduction

In Wei [8], a remarkable result was given for quadratic stabilization of uncertain linear systems. Wei's result gives a structure of uncertain systems, called *antisymmetric stepwise configuration*, which guarantees quadratic stabilizability via state feedback. This structure is constructed using a chain of the so-called *down-augmentations* and *up-augmentations*. The former augments a quadratically stabilizable system by cascading dynamics at the control input, while the latter adds state variables which integrate the existing ones. A remarkable feature of Wei's result is that time-varying uncertain parameters of large size are permitted. Also, the antisymmetric stepwise configuration is proved to be the only structure which can be quadratically stabilized via state feedback, under certain structural assumptions on uncertain parameters; see [8] for details.

For nonlinear systems, the down-augmentation technique corresponds the well-known back-stepping approach; see, e.g., [2], which is used for global stabilization of nonlinear systems with the so-called lower-triangular form (or strict feedback form). The up-augmentation technique, on the other hand, corresponds to a number of recent techniques for dealing with nonlinear systems with the so-called upper-triangular form (or strict feedforward form); see [7, 3, 1]. Indeed, the lower-triangular and upper-triangular forms are generated using down-augmentations and up-augmentations recursively.

The *up-augmented structure* starts with a *base system* of the following form:

$$\dot{x}(t) = f(x(t), q) + b(q)u(t) \tag{1}$$

where t represents time, $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}$ is the control, $q \in \mathbf{R}^l$ is an uncertain parameter vector contained in a compact set Ω, $b(q)$ is continuous in q, $f(x,q)$ is continuous in q and smooth in x (in this paper, if no specification, all smooth functions means globally smooth functions) with $f(0,q) = 0$.

An *up-augmented system* is given by

$$\begin{aligned} \dot{x}_0 &= f_0(x, q) \\ \dot{x} &= f(x,q) + b(q)[u + d(x, x_0, \eta, q)] \end{aligned} \tag{2}$$

where $x_0 \in \mathbf{R}$ is a new state variable, $f_0(x,q)$ is continuous in q and smooth in x with $f_0(0,q) = 0$, $d(x, x_0, \eta, q)$ is continuous in q and smooth in (x, x_0, η) with $d(0,0,0,q) = 0$. The parameter η represents state variables generated by up-augmentation other than x_0 and x if (2) is a subsystem of a larger one, or void otherwise.

On the other hand, the back-stepping technique deals with the so-called *down-augmented structure*. For the same base system (1), a down-augmented system is of the form:

$$\begin{aligned} \dot{x} &= f(x,q) + b(q)x_{n+1} \\ \dot{x}_{n+1} &= \theta_{n+1}(q)[u + d(x, x_{n+1}, \eta, q)] \end{aligned} \tag{3}$$

where $x_{n+1} \in \mathbf{R}$ is a new state variable, $\theta_{n+1}(q)$ is a continuous function bounded away from zero, and $d(\cdot)$ is the same as before.

In this paper, we intend to generalize Wei's augmentation approach to a class of uncertain nonlinear systems which often involve uncertain functions of a large size.

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The uncertain functions are also allowed to be time-varying. The robust stabilizing controller is designed recursively. When reducing to linear systems, the robust controller becomes linear, and our result recovers Wei's result on quadratic stabilization [8].

2 Robust Nonlinear Forwarding

In this section we revisit the robust forwarding technique in [5]. This technique does not yield a global smooth controller. But the controller can be made smooth as shown in [6]. The control law is designed in two steps. In the first step, a nonlinear controller is applied to the base system so that its state x converges to a "small" bounded set Ω while x_0 is not regulated. In the second step, a nonlinear controller is designed to maintain x within Ω while driving the augmented state x^+ to zero. Overall, this two-step control law achieves robust global asymptotic stabilization (RGAS) and robust local quadratic stabilization (RLQS) (see Definition 2.1 and 2.2).

Definition 2.1 An n-order system

$$\dot{x} = f(x,q) \tag{4}$$

is robustly globally asymptotically stable (RGAS) if,

$$\lim_{t\to\infty} x(t,q) = 0, \ \ \forall q \in Q, \ \ \forall x(0) \in \mathbf{R}^n.$$

Definition 2.2 An n-order system (4) is robustly locally quadratically stable (RLQS) if there are a local quadratic Lyapunov function $V(x) = x^T Px$, $P = P^T > 0$, a decay rate $\varepsilon > 0$ and a local region Ω such that,

$$\dot{V}(x) \le -\varepsilon\|x\|^2, \ \ \forall x \in \Omega.$$

ASSUMPTIONS

Assumption 2.1 (*Local Quadratic Stabilizability*): There exists a local smooth controller $u_n(x)$ for the base system (1) such that, with

$$u(t) = u_n(x(t)), \tag{5}$$

the state of the system (1) is RLQS with a local quadratic Lyapunov function $V(x) = x^T Px$, a local region Ω and a decay rate ε. □

Assumption 2.2 (*Local Smoothness Properties*): For the same local region Ω and local controller $u_n(x)$ as above, any $x \in \Omega$ and $q \in Q$, there holds

$$\begin{aligned} f(x,q) + b(q)u_n(x) &= A(x,q)x \\ b(q) = \theta_n(q)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \theta_n(q)b \end{aligned} \tag{6}$$

where

$$A(x,q) = \begin{bmatrix} 0 & A^-(x,q) \\ d_{n1}(x,q) & \star \end{bmatrix} \tag{7}$$

with $*$ representing an arbitrary term, $1 \ge \theta_n(x,q) > \underline{\theta}_n > 0$, where $\underline{\theta}_n$ is constant. □

Remark 2.1 With the local smoothness assumption, the RLQS property in Assumption 2.1 can be modified to the following: there exists $\mu > 0, \varepsilon > 0$ and matrix $P = P^T > 0$ such that

$$PA(x,q) + A^T(x,q)P \le -\varepsilon I, \ \ \forall x \in \Omega, q \in Q \tag{8}$$

where

$$\Omega = \{x : \ x^T Px < \mu\}. \tag{9}$$

Assumption 2.3 (*Global Properties*): Consider the following system derived from (2):

$$\dot{x} = f(x,q) + b(q)[u + d(x^+, \eta, q)]. \tag{10}$$

Given any smooth function $\eta(t)$ and $0 < \rho < 1$, there exists a locally (or globally) smooth controller $u_d(x^+, \eta)$ such that, with

$$u(t) = u_d(x^+(t), \eta(t)), \tag{11}$$

the state of the system (2) will be driven into $\rho\Omega$ in a finite time T, where Ω is given in (9), and $\rho\Omega = \{\rho x : \ x \in \Omega\}$. □

Assumption 2.4 (*Local Smoothness Properties*): For the local region Ω and any $x \in \Omega$ and $q \in Q$, there holds

$$f_0(x,q) = a(x,q)x \tag{12}$$

where

$$a(x,q) = [\theta_0(x,q) \ \ \star] \tag{13}$$

with $1 \ge \theta_0(x,q) \ge \underline{\theta}_0 > 0$ where $\underline{\theta}_0$ is constant. □

LYAPUNOV FUNCTION AND CONTROLLER DESIGN

Now we pay attention to controller design for (2). First, we utilize Assumption 2.3 and apply (11) to drive $x(t)$ into $\rho\Omega$ in a finite time T. In this step, $x_0(t)$ is not regulated. Once $x(t) \in \rho\Omega$, we switch to a local mode where a different controller $u^+(x^+, \eta)$ will be applied. This controller will maintain $x(t)$ in Ω while driving $x^+(t)$ to zero. The design of $u^+(x^+, \eta)$ relies on a local Lyapunov function for (2)

$$\begin{aligned} V^+(x^+) &= (x_0 - (\gamma \ \ 0)Px)^2 \\ &\quad + \int_0^{V(x)} s(w)dw > 0, \ \forall x \in \Omega \end{aligned} \tag{14}$$

where $\gamma < 0$ is a constant to be specified and $s(\cdot)$ is a locally smooth function satisfying: $s(w) > 0\ \forall w \in [0, \mu)$;

$$\int_0^v s(w)dw < \infty,\ \forall v \in [0, \mu); \qquad (15)$$

and

$$\lim_{v \to \mu} \int_0^v s(w)dw \to \infty. \qquad (16)$$

Remark 2.2 Note that (14) includes a quite large set of Lyapunov functions. A particular choice of $s(\cdot)$ is given by [5]:

$$s(w) = \frac{1}{\mu - w}.$$

For linear systems we can take $\mu = \infty$ and $s(w)$ a constant, which yields a quadratic Lyapunov function. It can be verified that this is the same Lyapunov function used in Wei [8]. In generally, these Lyapunov functions are non-quadratic. However, as $x \to 0$, $V^+(x^+)$ becomes quadratic in x^+ because $s(0) > 0$. We also note that the function $\int_0^{V(x)} s(w)dw$ resembles a "potential barrier" and the Lyapunov function (14) is valid only for $x \in \Omega$, i.e.,

$$V^+(x^+) \to \infty \quad as \quad x^T P x \to \mu. \qquad (17)$$

This implies that future $x \in \Omega$ as long as that $V^+(x^+)$ remains bounded. □

For notational simplicity, we will denote $s(V(x))$ by $s(x)$. Defining

$$P^+ = \begin{bmatrix} 1 & -(\gamma\ \ 0)P \\ -P\begin{pmatrix}\gamma \\ 0\end{pmatrix} & s(x)P + P\begin{pmatrix}\gamma \\ 0\end{pmatrix}(\gamma\ \ 0)P \end{bmatrix} \qquad (18)$$

which is positive definite for all $x \in \Omega$. The inverse of P^+ is given by

$$S^+ = s^{-1}(x)\begin{bmatrix} s(x) + (\gamma\ 0)P\begin{pmatrix}\gamma \\ 0\end{pmatrix} & (\gamma\ 0) \\ \begin{pmatrix}\gamma \\ 0\end{pmatrix} & P^{-1} \end{bmatrix}. \qquad (19)$$

Also define a (nonlinear) state transformation

$$z^+ = (z_0,\ z^T)^T = P^+ x^+. \qquad (20)$$

To simplify the analysis, we also assume in this section that η is void, i.e., $d(x^+, \eta, q) = d_1(x^+, q)$. The case $\eta \neq 0$ is a little more involved; see [5, 6]. Since $d_1(x^+, q)$ is smooth in x^+ and $d_1(0, q) = 0$, we can rewrite

$$d_1(x^+, q) = D^+(x^+, q)x^+ = D^+(x^+, q)S^+ z^+ \qquad (21)$$

for some $D^+(x^+, q)$ smooth in x^+ and continuous in q.

Differentiating $V^+(x^+(t))$ along the trajectory of (2), we have

$$\begin{aligned} \dot{V}^+ &= 2[x_0 - (\gamma\ \ 0)Px][\dot{x}_0 - (\gamma\ \ 0)P\dot{x}] \\ &\quad + 2s(V(x))x^T P\dot{x} \\ &= 2(x^+)^T P^+ \dot{x}^+. \end{aligned} \qquad (22)$$

Theorem 2.1 For the up-augmented system (2) satisfying Assumptions 2.1-2.3 and $d(x^+, \eta, q) = d_1(x^+, q)$, there exist $\gamma < 0$ and $\alpha(x^+) > 0$ such that the nonlinear controller

$$u(t) = u^+(x^+) = u_n(x) - \alpha(x^+)z^T b \qquad (23)$$

will render

$$\dot{V}^+(x^+) \le -s^{-1}(x)\varepsilon^+(x)V^+(x^+), \quad \forall x^+ \in \mathbf{R} \times \Omega \qquad (24)$$

for some continuous $\varepsilon^+(x) > 0$, $x \in \Omega$.

Moreover, the following choice of γ, $\alpha^+(x^+)$ and $\varepsilon^+(x)$ will suffice :

$$0 < \bar{\varepsilon} < \bar{\varepsilon}_{max} = \min_{q\in Q; x\in\Omega} \lambda_{min}\left[-P^{-1}\left(A^T(x,q)P + PA(x,q)\right)P^{-1}\right] \qquad (25)$$

$$\gamma < \gamma_{max} = \min_{q\in Q;\ x\in\Omega} \frac{1}{2\theta_0(q)}\left[a(x,q)\left(A^T(x,q)P + PA(x,q) + \bar{\varepsilon}P^2\right)^{-1} a^T(x,q) - \bar{\varepsilon}\right] \qquad (26)$$

$$\varepsilon^+(x) = \frac{\bar{\varepsilon}}{2}\lambda_{min}\left(P^+(x)\right) > 0 \qquad (27)$$

$$\alpha(x^+) = s^{-1}(x)\frac{\delta^2(x^+)}{\bar{\varepsilon}} \qquad (28)$$

where $\delta(x^+)$ is any smooth function satisfying

$$\delta(x^+) \ge \max_{q\in Q}\left\| s(x)D(x^+, q)S^+(s) + \left[\gamma\frac{d_{n1}(x,q)}{\theta_n(q)}\ \ 0\right]\right\|. \qquad (29)$$

Proof : Similar to the proof of Theorem 3.1 in [5]. ∇∇∇

Since the controller $u^+(x^+)$ in (23) is locally smooth and the system (2) satisfies Assumption 2.2, we have the following theorem:

Theorem 2.2 The closed-loop system (2) with the controller (2) is RLQS and has a local quadratic Lyapunov function $V_0^+ = (x^+)^T P_0^+ x^+$ when $x^+ \in \Omega^+ = \left\{x^+,\ (x^+)^T P_0^+ x^+ \le \mu^+\right\}$ where $\mu^+ > 0$,

$$P_0^+ = \begin{bmatrix} 1 & -(\gamma\ \ 0)P \\ -P\begin{pmatrix}\gamma \\ 0\end{pmatrix} & s_0 P + P\begin{pmatrix}\gamma \\ 0\end{pmatrix}(\gamma\ \ 0)P \end{bmatrix} \qquad (30)$$

and s_0 is a positive constant.

Proof: Similar to the proof of Theorem 4.1 in [5]. ∇∇∇

3 Robust Backstepping

In this section we consider the robust stabilization problem for nonlinear systems down-augmented from the base system (1). In fact, when a global Laypunov function of the base system is known, the stabilizaion of the down-augmented systems can be achieved by the well-known backstepping method [2]. However, the results in Sections 2 provide only a local Laypunov function when the base system is an up-augmented system. In this section, a modified backstepping method will be discussed for this case.

We first rewrite the base system as follows:

$$\dot{x} = f(x,q) + b(q)x_{n+1} \tag{31}$$

where x_{n+1} is the input of this system.

To simplify the analysis, we assume that vector η in (3) is void, i.e., $d(x, x_{n+1}, \eta, q) = f_{n+1}(x, x_{n+1}, q)$. Also, for notational simplicity, we assume $\theta_{n+1}(q) \equiv 1$ in (3).

Hence, the down-augmented system (3) is rewritten as

$$\begin{aligned} \dot{x} &= f(x,q) + b(q)x_{n+1} \\ \dot{x}_{n+1} &= f_{n+1}(x, x_{n+1}, q) + u. \end{aligned} \tag{32}$$

Suppose there is a globally smooth controller $u_0(x)$, $u_0(0) = 0$, such that the base system (31) with this controller satisfies the following Assumptions.

Assumption 3.1 The base system (1) with the controller $u(t) = u_0(x(t))$ is RLQS with a local quadratic Lyapunov function $V(x) = x^T P x$, a local region $x \in \Omega = \{x,\ V(x) < \mu\}$ and decay rate $\varepsilon > 0$. □

Assumption 3.2 (*Local Smoothness Properties*) For the system (31), there is a matrix function $A(x,q)$ which is smooth in x and continuous in q such that,

$$f(x,q) + b(q)u_0(x) = A(x,q)x. \tag{33}$$

for all $x \in \Omega$, $q \in Q$. □

Assumption 3.3 (*Global Properties*) Consider the given controller $u_0(x)$ and the following system

$$\dot{x} = f(x,q) + b(q)(u_0(x) + \omega) \tag{34}$$

where ω is a disturbance input with $\bar{\omega} = \sup_{t \geq 0} \|\omega(t)\| < \infty$. Then, for any $x(0)$ and sufficiently small $\bar{\omega}$, there exists $T > 0$, $s.t.\ x(t) \in \Omega$, $t > T$. □

With Assumption 3.3, we introduce a transformation

$$\begin{aligned} x &= x \\ z_{n+1} &= x_{n+1} - u_0(x). \end{aligned} \tag{35}$$

Under this transformation, the system (32) can be rewritten as

$$\dot{x} = f(x,q) + b(q)(u_0(x) + z_{n+1}) \tag{36}$$
$$\dot{z}_{n+1} = \bar{f}_{n+1}(z^+, q) + u \tag{37}$$

where $z^+ = (x^T, z_{n+1})^T$ and

$$\begin{aligned} &\bar{f}_{n+1}(z^+, q) \\ &= f_{n+1}(x, x_{n+1}, q) - \frac{\partial u_0}{\partial x}[f(x,q) + b(q)x_{n+1}]. \end{aligned} \tag{38}$$

Since $f(x,q)$, $f_{n+1}(x, x_{n+1}, q)$ and $u_0(x)$ are globally smooth functions, there is a matrix function $F_{n+1}(z^+, q)$ such that $\bar{f}_{n+1}(z^+, q) = F_{n+1}(z^+, q)z^+$.

Then we choose

$$V^+(z^+) = V(x) + z_{n+1}^2 \tag{39}$$

as a Lypaunov function candidate for (32).

Theorem 3.1 Under Assumptions 3.1-3.3, let

$$u^+(z^+) = -\frac{2}{\varepsilon}(\|z^+\|^2 + 1)z_{n+1}\delta_{n+1}^2(z^+) - \frac{\alpha}{2}z_{n+1} \tag{40}$$

where $\delta_{n+1}(z^+)$ is a smooth function satisfying

$$\max_{q \in Q} \|F_{n+1}(z^+, q)\| < \delta_{n+1}(z+); \tag{41}$$

and

$$\alpha = \max\left\{\frac{\varepsilon}{4\bar{\omega}^2},\ c + \frac{\varepsilon}{2}\right\}. \tag{42}$$

Then, $u^+(z^+)$ is a robust controller for system (32) yielding RGAS and RLQS.

Proof: Let $V_{n+1} = z_{n+1}^2$. The derivative of this function along the trajectory of (37) is

$$\begin{aligned} \dot{V}_{n+1} &= 2z_{n+1}[\bar{f}_{n+1}(z^+, q) + u] \\ &\leq \frac{4}{\varepsilon}(\|z^+\|^2 + 1)z_{n+1}^2\delta_{n+1}^2(z^+) + \frac{\varepsilon\|z^+\|^2}{4(\|z^+\|^2 + 1)} \\ &\quad + 2z_{n+1}u. \end{aligned} \tag{43}$$

The controller (40) leads

$$\begin{aligned} \dot{V}_{n+1} &\leq -\alpha z_{n+1}^2 + \frac{\varepsilon\|z^+\|^2}{4(\|z^+\|^2 + 1)} \\ &< -\alpha z_{n+1}^2 + \frac{\varepsilon}{4}. \end{aligned} \tag{44}$$

After some $T_0 > 0$, $\|z_{n+1}\| \leq \bar{\omega}$. Using Assumption 3.3, $\exists\, T > 0$, $x(t) \in \Omega$, $\forall t > T$. Now using Assumption 3.1,

$$\begin{aligned} \dot{V} &\leq -\varepsilon\|x\|^2 + 2x^T P b(q)\omega \\ &\leq -\varepsilon\|x\|^2 + \frac{\varepsilon}{2}\|x\|^2 + \frac{2}{\varepsilon}b^T(q)P^2 b(q)\omega^2 \\ &\leq -\frac{1}{2}\varepsilon\|x\|^2 + c\omega^2. \end{aligned} \tag{45}$$

Note that $V^+ = V + V_{n+1}$. For $t > T$,

$$\begin{aligned}\dot{V}^+ &= \dot{V} + \dot{V}_{n+1} \\ &\leq -\frac{1}{2}\varepsilon\|x\|^2 + \frac{\varepsilon\|z^+\|^2}{4(\|x^+\|^2+1)} - (\alpha - c)z_{n+1}^2 \\ &\leq -\frac{\varepsilon}{4}\|z^+\|^2. \end{aligned} \quad (46)$$

Therefore, this theorem holds. ∇∇∇

When combining backstepping with forwarding, we need a local quadratic Lyapunov function for (32). $V^+(z^+)$ in (39) would be an ideal candidate because it is a quadratic Lyapunov function in the coordinate z^+. Unfortunately, a careful analysis suggests that the transformation (35) can complicate the system and prevent further use of up-augmentations.

Due to this problem, we have to find a local quadratic Lyapunov function in the x coordinate.

Theorem 3.2 Suppose the system (32) satisfies the Assumptions 3.1-3.3 and the controller $u^+(x^+)$ satisfies (40). Then the closed-loop system has a local Lyapunov function

$$V_0^+(x^+) = V(x) + \left[x_{n+1} - \frac{\partial u_0(0)}{\partial x}x\right]^2, \quad x^+ \in \Omega^+$$

where

$$\Omega^+ = \{x^+ : V_0^+(x^+) < \mu^+\}, \quad \mu^+ > 0. \quad (47)$$

Proof: Omited for this conference version. ∇∇∇

In generally, the function $d(x, x_{n+1}, \eta, q)$ in (3) can be rewritten as

$$d(x, x_{n+1}, \eta, q) = f_{n+1}(x^+, q) + d_{n+1}(x^+, \eta, q) \quad (48)$$

where $d_{n+1}(0, 0, \eta, q) = 0$ for all $x^+ = (x^T \; x_{n+1})^T$ and $q \in Q$.

Also, note that $u_0(0) = 0$. Hence there is a vector function $k(x)$ such that $u_0(x) = k(x)x$.

Theorem 3.3 Let

$$\begin{aligned}&\bar{u}^+(x^+, \eta) \\ &= u^+(x^+) - \frac{1}{\varepsilon^+}(\|\eta\|^2 + 1)z_{n+1}^2\bar{\delta}_{n+1}(x^+, \eta)\end{aligned} \quad (49)$$

where

$$\max_{q\in Q} |d_{n+1}(x, z_{n+1}, \eta, q)| < \bar{\delta}_{n+1}(x^+, \eta); \quad (50)$$

$$\alpha = \max\left\{\left(\frac{\varepsilon}{4} + \frac{\varepsilon^+}{2}\right)\frac{1}{\delta^2}, c + \frac{\varepsilon}{4}\right\} \quad (51)$$

$$\varepsilon^+ = \frac{\varepsilon}{2}\min_{x^+\in\Omega^+}\frac{\lambda_{min}(\bar{P}^+(x))}{\lambda_{max}(P_0^+)}\mu^+ \quad (52)$$

$$\bar{P}^+(x) = \begin{bmatrix} P + k^T(x)k(x) & -k(x) \\ -k^T(x) & 1 \end{bmatrix} \quad (53)$$

and

$$P_0^+ = \begin{bmatrix} P + \left(\frac{\partial u_0(0)}{\partial x}\right)^T \frac{\partial u_0(0)}{\partial x} & -\frac{\partial u_0(0)}{\partial x} \\ -\left(\frac{\partial u_0(0)}{\partial x}\right)^T & 1 \end{bmatrix}. \quad (54)$$

Then, the controller $\bar{u}^+$ drives the state x^+ of the system (3) into Ω^+ in (47) in a finite time.

Proof: Omited for this conference version.

∇∇∇

Remark 3.1 Theorems 3.1-3.2 and the smoothness property of the controller (40) imply that Assumptions 3.1-3.2 can be preserved in the down-augmentation process. Theorem 3.3 assures Assumption 2.3 for the system (3). Although it is shown here, Assumption 3.3 can be guaranteed for the system (3) by slightly modifying the controller (49). □

4 Antisymmetric Stepwise Configuration

In this section, we combine the results on up-augmentations with down-augmentations to form a class of uncertain nonlinear systems which can be robustly stabilized. This class is characterized by the antisymmetric stepwise configuration (ASSC).

To explain the ASSC, we consider the following system

$$\dot{x} = f(x, q) + b(q)u \quad (55)$$

where $f(x, q)$ is smooth in x and continuous in q, $b(q)$ is continuous in q, and $q \in Q$ is an uncertain parameter vector as before. Define

$$M(x, q) = \left[\frac{\partial f(x, q)}{\partial x} \quad b(q)\right] \quad (56)$$

and adopt the following convention:

- $*$ = any scalar function of x and q with a known bound over $\Omega \times Q$;
- θ = any scalar function of x and q with $1 \geq |\theta| \geq \underline{\theta} > 0$ over $\Omega \times Q$.

Then, examples of ASSC are given as follows:

$$\begin{bmatrix} 0 & \theta & * & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 \\ * & * & * & \theta & 0 \\ * & * & * & * & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & * & * & 0 \\ 0 & 0 & \theta & 0 & 0 \\ 0 & 0 & * & \theta & 0 \\ * & * & * & * & \theta \end{bmatrix}$$

These examples are all generated via a sequence of up- and down-augmentations. For example, the first example is generated via an up-augmentation from the lower-right 2×3 structure. A precise definition of the ASSC can be found in Wei [8] with the exception that the matrix $M(x, q)$ in [8] is independent of x.

A general formula for $M(x, q)$ is given below; which is slightly genralized from [8]:

Definition 4.1 The system (55) is said to have an *antisymmetric stepwise configuration(ASSC)* if the configuration matrix $M(x, q)$ in (56) satisfies the following conditions.

1. If $p \geq k+2$ and $m_{kp}(x, q) = 0$, then $m_{uv}(x, q) \equiv 0$ for all $u \geq v$, $u \leq p-1$ and $v \leq k+1$;

2. If $m_{ij}(x, q) \neq 0$ for some $j \leq i$, then $m_{i,i+1}(x, q)$ is independent of x_{i+1} and $m_{i,i+1}(x, q) \neq 0$ for $\forall x_i \in \mathbf{R}$ and $\forall q \in Q$.

Lemma 4.1 A nonlinear system (55) with an antisymmetric stepwise configuration can be obtained via a series of up-augmentations and down-augmentations.

Proof: [8] proves the result for linear systems. Since the ASSC is a feature of the system's structure, the result obviously holds for the nonlinear case. $\nabla\nabla\nabla$

With this lemma, we will recursively apply the robust stabilization results on up-augmentations and down-augmentations to solve the stabilization problem of the ASSC system.

Theorem 4.1 Given the nonlinear system (55) with ASSC. There exists a nonlinear controller $u(x)$ such that the closed-loop system is RGAS and RLES.

Proof: Omitted for this conference version. $\nabla\nabla\nabla$

5 Conclusion

In this paper, we have proposed a new design technique for robust nonlinear forwarding and robust backstepping. This technique is inherently different from existing ones for nonlinear forwarding and backstepping in the sense that we can handle nonlinear systems with large size uncertain parameters. By recursive applications of up-augmentations and down-augmentations, we have identified a new class of uncertain nonlinear systems which can be robustly stabilized. This class of systems is characterized by the so-called *antisymmetric stepwise configuration* (ASSC) and includes the well-known lower-triangular structure and upper-triangular structure as special cases. The ASSC is generalized from a result of Wei [8] which shows that the ASSC is a complete characterization of the class of uncertain linear systems that can be robustly stabilized via state feedback, subject to some structural assumptions on the system. It is worth to note that the results of this paper reduce to Wei's results for uncertain linear systems.

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