Adaptive Stabilization of Linear Systems Via Switching Control

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Abstract—**In this paper, we develop a method for adaptive stabilization without a minimum-phase assumption and without knowledge of the sign of the high-frequency gain. In contrast to recent work by Martensson [8], we include a compactness requirement on the set of possible plants and assume that an upper bound on the order of the plant is known. Under these additional hypotheses, we generate a piecewise linear time-invariant switching control law which leads to a guarantee of Lyapunov stability and an exponential rate of convergence for the state. One of the main objectives in this paper is to eliminate the possibility of "large state deviations" associated with a search over the space of gain matrices which is required in [8].**

I. Introduction

The recent literature on adaptive stabilization includes a number of papers indicating a variety of situations where one can dispense with some of the so-called classical assumptions, e.g., see [1]–[8]. In contrast to earlier research in adaptive control, the emphasis in this new work has been on reducing the *a priori* information which is required of the system. That is, the issue of concern is to determine the extent to which one can relax the requirements that the plant's degree and relative degree are known, the plant is minimum phase, and the sign of the high-frequency gain is known.

This new line of research can be traced back to a paper by Morse [1] which raised a number of open questions involving the classical assumptions in parameter adaptive control. Subsequently, in [2], Nussbaum paved the way for adaptive control in the absence of information on the sign of the high-frequency gain. He considered the problem of finding a smooth stabilizing controller

$$\dot{z}(t)=f(y(t),\ z(t));$$
$$u(t)=g(y(t),\ z(t)) \quad (1.1)$$

for the one-dimensional system

$$\dot{x}(t)=ax(t)+qu(t);$$
$$y(t)=x(t) \quad (1.2)$$

with both $q \neq 0$ and $a > 0$ unknown. In his paper [2], Nussbaum describes a whole family of controllers of the form (1.1) which achieve the desired stabilization for system (1.2).

Following this work, a number of more general results emerged for adaptive stabilization of higher order linear time-invariant systems with unknown high-frequency gain; see, for example, the papers by Byrnes and Willems [3], Mudgett and Morse [4], Willems and Byrnes [5], and Lee and Narendra [6]. Another breakthrough is contained in a recent paper by Morse [7] where it is shown that adaptive stabilization is possible with even less *a priori* information than heretofore required. In his paper, Morse developed a "universal controller" which can adaptively stabilize any strictly proper, minimum-phase system with relative degree not exceeding two.

Another surprising result is due to Martensson [8]. For a set of minimal plants, it is established that adaptive stabilization is possible with only one rather weak assumption. Namely, it is assumed that there exists some nonnegative integer l having the property that each possible plant admits an lth-order stabilizing compensator. Subsequently, it is shown how even this assumption can be relaxed. As Martensson points out, however, his controller is severely limited from an implementation point of view. The first limitation stems from the fact that the controller may end up performing a rather exhaustive on-line search over the space of candidate gain matrices before "latching on" to an appropriate stabilizer. Consequently, Lyapunov stability *cannot* be guaranteed; it is only shown that the state is bounded and converges to zero. Hence, there is no control over large excursions in the state space even when the initial state is arbitrarily small. From a practical point of view, the consequence of this exhaustive on-line search may be excessive overshoot. This situation is illustrated in Fig. 1 for the scalar plant in (1.2). For this system, a suitable Martensson-type controller is described by

$$\dot{z}(t)=y^2(t); \qquad z(0)\geq 1;$$
$$u(t)=y(t)h(z(t))^{1/4}\ [\sin\ h(z(t))^{1/2}+1]\ \cos\ h(z(t)) \quad (1.3)$$

where

$$h(z)=\log^{1/2}\ z.$$

Notice in Fig. 1 that for the initial condition of $x(0) = 1$, $z(0) = 1$ and parameter values $a = 1$ and $q = -1$, the peak overshoot in $y(t)$ is 300 000! A second practical limitation of the Martensson controller stems from the susceptibility of the so-called Nussbaum gain to measurement noise. This limitation is also inherent in [2]–[8] where a similar Nussbaum structure is used. To illustrate, we again consider plant (1.1) with the adaptive Nussbaum-type stabilizer (see [4])

$$\dot{z}(t)=y^2(t);$$
$$u(t)=y(t)z^2(t)\ \cos\ z(t) \quad (1.4)$$

and suppose that the measured output $y(t)$ is additively corrupted by some "small" disturbance $\epsilon(t)$; say, for example, $\epsilon(t)$ is white noise and

$$y(t)=x(t)+\epsilon(t). \quad (1.5)$$

Then it is easy to see from (1.4) that $z(t)$ may tend to infinity if $\epsilon(t)$ has nonvanishing covariance. This will happen when $y^2(t)$ is nonintegrable as a consequence of variations in $\epsilon(t)$. Therefore,

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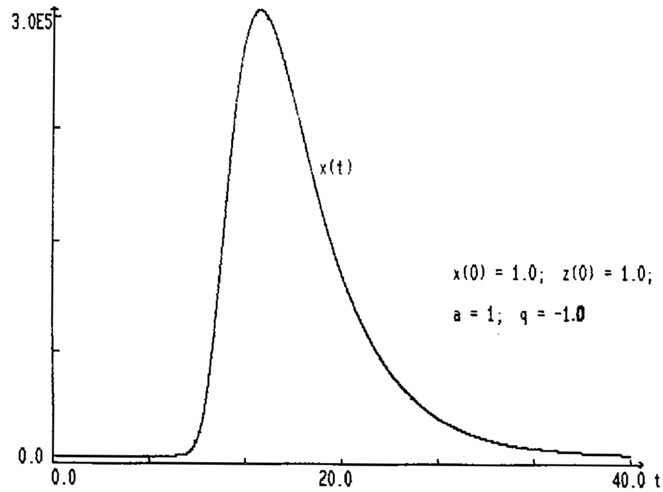

Fig. 1. Simulation using controller (1.3) for system (1.2).

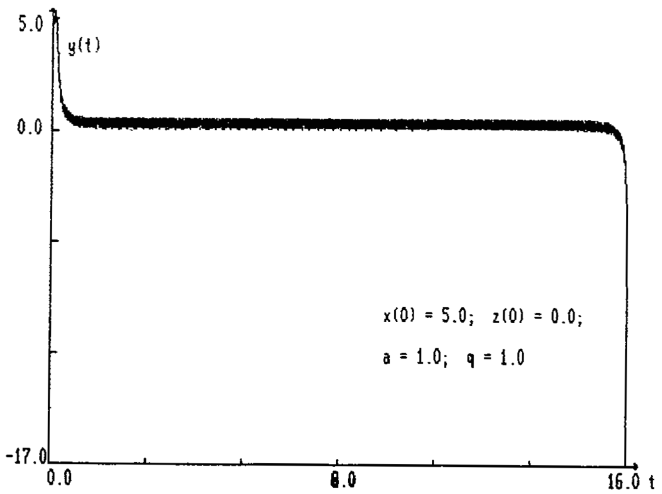

Fig. 2. Simulation using controller (1.4) for system (1.2) with additive measurement disturbance.

the control gain may not converge and we see that an arbitrarily small persistant measurement perturbation may destabilize the system. Fig. 2 demonstrates this phenomenon for the initial condition $x(0) = 5$, $z(0) = 0$, parameters $a = 1$ and $q = 1$, and measurement disturbance $\epsilon(t) = 0.25 \sin 100t$.

Given the motivation above, the objective in this paper is to develop a controller which not only stabilizes the system (as in [2]–[8]) but does so in the sense of Lyapunov. This distinction is important because with Lyapunov stability we can get a handle on the types of undesirable "overshoot" behavior described above.

The results of this paper are obtained by strengthening Martensson's hypotheses for the sake of generating a more "practical" controller. To this end, there are two more assumptions which we impose beyond those in [8]. Our first assumption is that an upper bound on the order of the plant is known. Second, we make a compactness assumption on the set of possible plants. Within this framework, we achieve the stated stability objectives using a switching control law which is a piecewise linear time-invariant feedback. It is shown that only a finite number of switches occur and then the controller remains fixed with a constant compensator gain matrix.

II. System and Assumptions

A finite *upper bound on state dimension* $n_{\max} < \infty$ is specified and each possible plant is a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t);$$

$$y(t) = Cx(t); \qquad t \in [0, \infty) \tag{2.0.1}$$

with *state* $x(t) \in R^n$ for some $n \leq n_{\max}$, *control* $u(t) \in R^m$, and *measured output* $y(t) \in R^r$. The given set of possible plants Σ consists of triples (A, B, C) and we use the notation Σ_n to denote the subset of Σ consisting of those plants having dimension n, i.e.,

$$\Sigma_n \triangleq \{(A, B, C) \in \Sigma : \dim A = n \times n\}$$

for $n = 1, 2, \cdots, n_{\max}$. Throughout this paper, it is assumed that Σ_n is compact for $n = 1, 2, \cdots, n_{\max}$ and that every possible plant $(A, B, C) \in \Sigma$ is a minimal realization.

Remarks 2.1: The assumptions above guarantee that for every possible plant $(A, B, C) \in \Sigma$, there exists an lth-order linear time-invariant dynamic compensator (l of course depends on the dimension of A)

$$\dot{z}(t) = Fz(t) + Gy(t);$$

$$u(t) = Hz(t) + Ky(t) \tag{2.1.1}$$

so that with state

$$\boldsymbol{x}(t) \triangleq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},$$

the closed-loop system

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} A + BKC & BH \\ GC & F \end{bmatrix} \boldsymbol{x}(t) \tag{2.1.2}$$

is asymptotically stable. Since the upper bound on the state dimension $n_{\max}$ is assumed to be known, the order l of this dynamic compensator can be taken to be the same for all $(A, B, C) \in \Sigma$. This follows because if $(A, B, C) \in \Sigma$ and $\dim A = n \times n$, then stability can be guaranteed using an nth-order Luenberger observer which implies that a compensator of dimension $n_{\max}$ can also be used to guarantee stability. This higher dimensional compensator is trivially obtained by augmenting the nth-order Luenberger observer with a stable subsystem of order $n_{\max} - n$ with states which are decoupled from the states of the observer. This observation will be used to our advantage in Lemma 3.1 to follow.

The compactness assumption on each Σ_n implies that the class of systems under consideration does not include singular perturbations. In other words, the model does not handle parasitics. A simple example illustrating this restriction is given by the singularly perturbed system

$$\epsilon\dot{x}(t) = x(t) + u(t); \ \epsilon \in [0, \epsilon_{\max}];$$

$$y(t) = x(t). \tag{2.1.3}$$

It is straightforward to verify that

$$\Sigma_1 = \left\{ \left(\frac{1}{\epsilon}, \frac{1}{\epsilon}, 1 \right) : \epsilon \in (0, \epsilon_{\max}] \right\} \tag{2.1.4}$$

which is not compact. It should also be noted that the compactness assumption on the Σ_n implies that some bound is available on the system parameters. This assumption is what distinguishes this work from the cited literature on adaptive stabilization.

A. Notation for the Closed-Loop System

Given any fixed triple $(A, B, C) \in \Sigma$ and a set of gain matrices (F, G, H, K) for an lth-order compensator, the closed-loop system is described by

$$\dot{x}(t)=Ax(t)+Bu(t);$$
$$y(t)=Cx(t);$$
$$u(t)=Ky(t) \tag{2.2.1}$$

where

$$A \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix};\ B \triangleq \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix};$$
$$C \triangleq \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix};\ K \triangleq \begin{bmatrix} K & H \\ G & F \end{bmatrix}$$

and

$$x(t) \triangleq [x(t)'z(t)']';\ u(t) \triangleq [u(t)'\dot{z}(t)']'.$$

To denote the dependence of the closed-loop system matrix on the chosen compensator gain matrix K, we use the notation

$$A_*(K) \triangleq A+BKC.$$

III. A Preliminary Lemma

The following technical lemma will be useful in Section IV where we construct a switching compensator leading to Lyapunov stability with an exponential rate of convergence for the state.

Lemma 3.1: Let (decay rate) $\gamma > 0$ be arbitrarily specified. Then, there exist a (compensator dimension) $l \leq n_{max}$, a constant $M_* > 0$, a finite number of compensator gain matrices $K_1, K_2, \cdots, K_f \in R^{(l+m)\times(l+r)}$, and compact sets $\Sigma_1^*, \Sigma_2^*, \cdots, \Sigma_f^*$ such that

i) $$\bigcup_{i=1}^{f} \Sigma_i^* = \Sigma; \tag{3.1.1}$$

ii) *For each $i \in \{1, 2, \cdots, f\}$ and each $(A, B, C) \in \Sigma_i^*$, we have*

$$\|e^{A_*(K_i)t}\| \leq M_* e^{-\gamma t} \tag{3.1.2}$$

for all $t \in [0, \infty)$.

Proof: Recalling the Remarks in Section II-A, it suffices to take the compensator dimension $l = n_{max}$ in the proof to follow. Note, however, that it may be possible to use a lower order compensator as far as implementation is concerned, e.g., see Example 1 in Section VII.

We first choose $\epsilon > 0$ to be any fixed number. Now, given any $\gamma > 0$ and any triple $\sigma = (A, B, C) \in \Sigma$, we can select $K_\sigma \in R^{(l+m)\times(l+r)}$ so that the closed-loop system matrix

$$A_*(K_\sigma)=A+BK_\sigma C$$

has eigenvalues all having real part less than $-(\gamma + \epsilon)$.

Let n_σ be the dimension of A and note that by continuity of the eigenvalues of $A_*(K_\sigma)$ with respect to the system matrices, we can find an open neighborhood V_σ of systems around σ (all having dimension n_σ) satisfying the following condition. For each $\tilde{\sigma} = (\tilde{A}, \tilde{B}, \tilde{C}) \in V_\sigma$, the eigenvalues of $\tilde{A} + \tilde{B}K_\sigma\tilde{C}$ also have real part less than $-(\gamma + \epsilon)$. Consequently, for each $n \in \{1, 2, \cdots, n_{max}\}$, we generate an open covering of Σ_n by taking the union of the sets V_σ as σ ranges over Σ_n. Now, using compactness of each Σ_n, we can extract a finite set of gain matrices $K_{n,1}, K_{n,2}, \cdots, K_{n,f(n)}$ and associated open neighborhoods $V_{n,1}, V_{n,2} \cdots V_{n,f(n)}$ such that for each $\sigma \in V_{n,i}$, $A_*(K_{n,i})$ has all its eigenvalues with real part less than $-(\gamma + \epsilon)$.

To complete the construction of the compensator gain matrices, we simply take the set $\{K_1, K_2, \cdots, K_f\}$ to be the union of the sets $\{K_{n,1}, K_{n,2}, \cdots, K_{n,f(n)}\}$ as n ranges from 1 to n_{max}. Now, for any fixed $i \in \{1, 2, \cdots, f\}$, define

$$\Sigma_i^* \triangleq \{(A, B, C) \in \Sigma : \text{all eigenvalues of } A_*(K_i) \text{ have real part} \leq -(\gamma+\epsilon)\}.$$

Again, using compactness of the Σ and continuity of eigenvalues of $A_*(K_i)$ with respect to the system matrices, it follows that Σ_i^* is compact. Then the definition of Σ_i^* guarantees that for each $\sigma = (A, B, C) \in \Sigma_i^*$,

$$\|e^{A_*(K_i)t}\|e^{\gamma t}\to 0$$

as $t \to \infty$. Hence, (3.1.2) is satisfied by taking

$$M_* \triangleq \max\{\max_{\sigma\in\Sigma_i^*, t\in[0,\infty)} \|e^{A_*(K_i)t}\|e^{\gamma t} : i=1, 2, \cdots, f\}.$$

■

IV. Construction of the Switching Compensator

In this section, we provide the formal construction of a switching compensator which achieves the desired Lyapunov stability with exponential decay rate. First, however, we give some heuristic motivation for the basic idea behind the construction. We begin at time zero with compensator gain matrix K_1 and use the output information to construct a "monitoring function" $V(t, \tau_1)$; see Step 4 to follow. This function, being related to the state of the system, is used to decide when to switch from K_1 to K_2. Once this switch has taken place, we then use $V(t, \tau_2)$ to decide when to switch from K_2 to K_3; this process continues with switching from K_3 to K_4, K_4 to K_5, etc. Eventually (see the proof of Theorem 5.1), the compensator gain matrix will "latch" onto some K_p $(p \leq f)$ which does indeed stabilize the system. Subsequently, no further switching occurs. The proof of stability of the compensated system is relegated to Section V where the main result of this paper is stated.

Step 1: Select any desired decay rate $\gamma > 0$ and take $K_1, K_2, \cdots, K_f \in R^{(l+m)\times(l+r)}$ and $\Sigma_1^*, \Sigma_2^*, \cdots, \Sigma_f^*$ satisfying the requirements of Lemma 3.1.

Step 2: For each $i \in \{1, 2, \cdots, f\}$ and each triple $\sigma = (A, B, C) \in \Sigma_i^*$, define the observability Gramian

$$W_i(\tau, \sigma) \triangleq \int_0^\tau e^{A_*'(K_i)\eta}C'Ce^{A_*(K_i)\eta}\, d\eta \tag{4.0.1}$$

and the scalar function

$$\rho_i(\tau, \sigma) \triangleq \lambda_{max}[W_i(\tau, \sigma)^{-1/2}e^{A_*'(K_i)\tau}W_i(\tau, \sigma)e^{A_*(K_i)\tau}W_i(\tau, \sigma)^{-1/2}]$$

where $\lambda_{max(min)}[\cdot]$ denotes the operation of taking the largest (smallest) eigenvalue.

Step 3: For each fixed $i \in \{1, 2, \cdots, f\}$ and each $\sigma = (A, B, C) \in \Sigma_i^*$, we claim that $\rho_i(\tau, \sigma) \to 0$ as $\tau \to \infty$. To this end, for fixed $\sigma = (A, B, C) \in \Sigma_i^*$, we first notice that $\|W_i(\tau, \sigma)\| = \lambda_{max}[W_i(\tau, \sigma)]$ is nondecreasing. Also, since $A_*(K_i)$ is asymptotically stable, $\|W_i(\tau, \sigma)\|$ is bounded with respect to τ. Hence, for any fixed τ_0 and $\tau \geq \tau_0$, we use norm inequalities and Lemma 3.1 to obtain

$$\begin{aligned}\rho_i(\tau, \sigma) &= \|W_i(\tau, \sigma)^{-1/2}e^{A_*'(K_i)\tau}W_i(\tau, \sigma)^{1/2}\|^2\\ &\leq \|W_i(\tau_0, \sigma)\|^{-1}\|e^{A_*(K_i)\tau}\|^2\|W_i(\infty, \sigma)\|\\ &\leq \|W_i(\tau_0, \sigma)\|^{-1}\|W_i(\infty, \sigma)\|M_*^2e^{-2\gamma\tau}.\end{aligned}$$

From this inequality, it follows that $\rho_i(\tau, \sigma) \to 0$ as $\tau \to \infty$. Now, we further bound $\rho_i(\tau, \sigma)$ independently of σ. That is,

$$\rho_i(\tau, \sigma) \leq \max_{\sigma\in\Sigma_i^*}\{\|W_i(\tau_0, \sigma)\|^{-1}\|W_i(\infty, \sigma)\|\}M_*^2e^{-2\gamma\tau}.$$

Using this bound, we conclude that for each $i \in \{1, 2, \cdots, f\}$, there exists a finite constant $\tau_i > 0$ such that

$$1 > \max_{\sigma \in \Sigma_i^*} \rho_i(\tau_i, \sigma) \triangleq \rho_i. \tag{4.0.2}$$

Step 4: The generation of the controller is accomplished by defining a *switching index* $h(t)$ and an associated sequence of *switching instants* $t_0, t_1, \cdots, t_p$. First, using the available output $y(t)$, the controller generates the signal

$$\dot{\phi}(t) \triangleq \|y(t)\|^2. \tag{4.0.3}$$

Next, we define

$$V(t, \tau) \triangleq \phi(t) - \phi(t-\tau) \tag{4.0.4}$$

for $t \in [0, \infty)$ and $\tau \in [0, t]$ and initialize the controller by taking $t_0 \triangleq 0$. Now, for $i = 1, 2, \cdots, f-1$, define

$$t_i \triangleq \sup \{t : t \geq t_{i-1} + 2\tau_i;\ V(t, \tau_i) \leq \rho_i V(t-\tau_i, \tau_i)\} \tag{4.0.5}$$

and the *switching index*

$$h(t) \triangleq i \tag{4.0.6}$$

for $t \in [t_{i-1}, t_i)$. Subsequently, the control is recursively generated using the formula

$$u(t) \triangleq K_{h(t)} y(t). \tag{4.0.7}$$

In case $t_i = \infty$ for some $i < f - 1$, the generation of t_i is terminated and the control gain matrix $K_{h(t)}$ remains constant at K_{i-1}.

Remark 4.1: In effect, the control $u(t)$ given by (4.0.7) is a piecewise linear time-invariant feedback. In Section V below, our objective is to show that the control $u(t)$ above leads to an exponential rate of convergence (hence, Lyapunov stability) for the closed-loop system.

V. Main Result

We are now prepared to state and prove the main result of this paper.

Theorem 5.1: Consider the set of possible systems Σ in (2.0.1) *with control $u(t)$ given by* (4.0.7). *Then there exist constants $M > 0$ and $\lambda > 0$ such that for all $(A, B, C) \in \Sigma$, all initial conditions $x(0) = (x(0), z(0))$ and all $t \in [0, \infty)$, it follows that*

$$\|x(t)\|^2 \leq Me^{-\lambda t}\|x(0)\|^2. \tag{5.1.1}$$

Proof: Let $\sigma = (A, B, C) \in \Sigma$ be any possible system with arbitrary initial condition $x(0)$ and note that in accordance with Lemma 3.1, $\sigma \in \Sigma_i^*$ for some $i \leq f$. Our first claim is that the switching index $h(t)$ converges to some $p \leq i$. This claim is established by noting that if $h(t) = i$, then for all $t \geq t_{i-1} + 2\tau_i$, we have

$$\begin{aligned} V(t, \tau_i) &= \phi(t) - \phi(t-\tau_i) \\ &= \int_{t-\tau_i}^{t} \|y(\eta)\|^2 \, d\eta \\ &= x'(t-\tau_i) W_i(\tau_i, \sigma) x(t-\tau_i) \\ &\leq \rho_i(\tau_i, \sigma) V(t-\tau_i, \tau_i) \\ &\leq \rho_i V(t-\tau_i, \tau_i). \end{aligned} \tag{5.1.2}$$

In view of this inequality and the definition of the switching instants, it follows that $t_i = \infty$ and $h(t) = i$ for all $t \geq t_{i-1}$. Hence, let $t_1, t_2, \cdots, t_p$ denote the finite set of switching instants which result and note that $p \leq i$ and $t_p = \infty$.

The next step of the proof involves bounding the state $x(t)$. Indeed, with $\sigma \in \Sigma_i^*$ as above and $j \leq p-1$, we consider the time interval

$$\begin{aligned} T_j &\triangleq [t_{j-1}, t_j) \\ &= [t_{j-1}, t_{j-1} + 2\tau_j) \cup [t_{j-1} + 2\tau_j, t_j) \\ &\triangleq T_{j,1} \cup T_{j,2}. \end{aligned}$$

For $t \in T_j$, we use control $u(t) = K_j y(t)$ and consider two cases whose results will be combined at the end.

Case 1: $t \in T_{j,1}$. In this case, it is apparent that

$$\|x(t)\|^2 \leq \beta_j^2 \|x(t_{j-1})\|^2 \tag{5.1.3}$$

where

$$\beta_j \triangleq \max \{\|e^{A*(K_j)\eta}\| : (A, B, C) \in \Sigma;\ \eta \in [0, \tau_j]\}. \tag{5.1.4}$$

Note that β_j is finite because $[0, \tau_j]$ and the Σ_i are compact and the matrix exponential is continuous with respect to σ and η.

Case 2: $t \in T_{j,2}$. In order to bound $x(t)$, we first bound $V(t, \tau_j)$. To this end, select the integer $\mu \geq 1$ such that

$$t_{j-1} + (\mu+1)\tau_j \leq t < t_{j-1} + (\mu+2)\tau_j$$

and let

$$\delta \triangleq t - t_{j-1} - (\mu+1)\tau_j. \tag{5.1.5}$$

By definition of μ, it follows that $\delta \in [0, \tau_j)$. Recalling expressions (4.0.1) for $W_j(\tau_j, \sigma)$ and (4.0.4) for $V(t, \tau_j)$, we obtain a bound

$$\lambda_{\min}^* \|x(t-\tau_j)\|^2 \leq V(t, \tau_j) \leq \lambda_{\max}^* \|x(t-\tau_j)\|^2 \tag{5.1.6}$$

where

$$\lambda_{\max}^* \triangleq \max_{\sigma \in \Sigma} \lambda_{\max}[W_j(\tau_j, \sigma)];\ \lambda_{\min}^* \triangleq \min_{\sigma \in \Sigma} \lambda_{\min}[W_j(\tau_j, \sigma)].$$

Note that $\lambda_{\max}^*$ and $\lambda_{\min}^*$ are positive (by invariance of observability under output feedback) and finite (by compactness of Σ and continuity of $W(\tau_j, \sigma)$ with respect to σ). Now using the state bound (5.1.4) and the bounds on $V(t, \tau_j)$ in (5.1.2) and (5.1.6), we obtain

$$\begin{aligned} \|x(t)\|^2 &\leq \beta_j \|x(t-\tau_j)\|^2 \\ &\leq \frac{\beta_j}{\lambda_{\min}^*} V(t, \tau_j) \\ &\leq \frac{\beta_j}{\lambda_{\min}^*} \rho_j^\mu V(t-\mu\tau_j, \tau_j) \\ &\leq \frac{\lambda_{\max}^*}{\lambda_{\min}^*} \beta_j \rho_j^\mu \|x(t-(\mu+1)\tau_j)\|^2 \\ &= \frac{\lambda_{\max}^*}{\lambda_{\min}^*} \beta_j \rho_j^\mu \|x(t_{j-1}+\delta)\|^2 \\ &\leq \frac{\lambda_{\max}^*}{\lambda_{\min}^*} \beta_j^2 \rho_j^\mu \|x(t_{j-1})\|^2. \end{aligned} \tag{5.1.7}$$

To complete the analysis for Case 2, we note that $\rho_i \in (0, 1)$ makes it possible to choose $\lambda_j > 0$ such that

$$\rho_j = e^{-\lambda_j \tau_j}.$$

Hence, (5.1.7) becomes

$$\|x(t)\|^2 \leq \frac{\lambda_{\max}^*}{\lambda_{\min}^*} \beta_j^2 e^{-\mu\lambda_j\tau_j} \|x(t_{j-1})\|^2. \tag{5.1.8}$$

Now using the definition of δ in (5.1.5), we can further bound the state; i.e.,

$$\|x(t)\|^2 \le \frac{\lambda^*_{\max}}{\lambda^*_{\min}} \beta_j^2 e^{\lambda_j(\tau_j-\delta)} e^{-\lambda_j(t-t_{j-1})} \|x(t_{j-1})\|^2$$

and recalling that $\delta \le \tau_j$, we finally obtain

$$\|x(t)\|^2 \le M_j e^{-\lambda_j(t-t_{j-1})} \|x(t_{j-1})\|^2 \tag{5.1.9}$$

where

$$M_j \triangleq \frac{\lambda^*_{\max}}{\lambda^*_{\min}} \beta_j^2 e^{2\lambda_j \tau_j}. \tag{5.1.10}$$

Combining Cases 1 and 2: We claim that the state bound in (5.1.9) is actually valid over all of T_j even though it was only developed for $t \in T_{j,2}$. To see this, note that $\lambda^*_{\max}/\lambda^*_{\min} > 1$ and that $t - t_{j-1} < 2\tau_j$ for $t \in T_{j,1}$. Consequently, if $t \in T_{j,1}$, we can further bound the state in (5.1.3). Namely,

$$\begin{aligned}\|x(t)\|^2 &\le \beta_j^2 \|x(t_{j-1})\|^2 \\ &\le \frac{\lambda^*_{\max}}{\lambda^*_{\min}} \beta_j^2 e^{2\lambda_j \tau_j} e^{-\lambda_j(t-t_{j-1})} \|x(t_{j-1})\|^2 \\ &= M_j e^{-\lambda_j(t-t_{j-1})} \|x(t_{j-1})\|^2. \end{aligned} \tag{5.1.11}$$

Finally, to complete the proof of the theorem, let

$$M \triangleq M_1 M_2 \cdots M_f;$$
$$\lambda \triangleq \min \{\lambda_1, \lambda_2, \cdots, \lambda_f\}.$$

Now, given any $t \in [0, \infty)$, it follows that $t \in T_j$ for some $j \le p$. By using (5.1.11), we obtain

$$\begin{aligned}\|x(t)\|^2 &\le M_j e^{-\lambda(t-t_{j-1})} \|x(t_{j-1})\|^2 \\ &\le M_j M_{j-1} e^{-\lambda(t-t_{j-2})} \|x(t_{j-2})\|^2.\end{aligned}$$

Continuing recursively in this manner and noting that each M_i exceeds unity by (5.1.10) and (5.1.4), it follows that

$$\begin{aligned}\|x(t)\|^2 &\le \left(\prod_{k=1}^{j} M_k\right) e^{-\lambda t} \|x(0)\|^2 \\ &\le M e^{-\lambda t} \|x(0)\|^2. \end{aligned}$$

■

VI. Extensions

In this section, we briefly indicate two extensions of the theory. First, the results are strengthened for the special case of full state feedback. Second, the theory is extended to deal with additive measurement noise.

A. Full State Feedback

One of the key ideas underlying the switching control (4.0.7) is the construction of the function $V(t, \tau)$ which provides information making it possible to decide when to stop switching, i.e., to decide if the controller is using the "right" gain matrix. Note, however, that the controller "waits" for a period $2\tau_i$ before deciding whether to switch from $\boldsymbol{K}_i$ to $\boldsymbol{K}_{i+1}$ and also recall that the τ_i were chosen to guarantee the decreasing property of $V(\cdot, \tau_i)$ which is essential to attainment of the main result. In view of these remarks, it is of interest to know under what conditions one can reduce the waiting period $2\tau_i$ so as to "speed up" the system response. We claim that under the strengthened hypothesis of full state feedback, the "waiting period" can in fact be made arbitrarily small. For brevity, we omit a rigorous proof and only provide a sketch of the main ideas behind this extension to the theory.

When the full state $x(t)$ is available for feedback, we use a static compensator (of dimension $l = 0$) and can therefore omit boldface notation when referring to system and compensator matrices. Since $C = I$ for all possible systems, we now use the notation (A, B) instead of (A, B, C). First, it is noted that we can extend Lemma 3.1 and generate a finite number of gain matrices $K_1, K_2, \cdots, K_f$, a finite number of compact sets $\Sigma_1^*, \Sigma_2^*, \cdots, \Sigma_f^*$, and a finite number of *Lyapunov matrices* $P_1, P_2, \cdots, P_f$ such that for each $(A, B) \in \Sigma_i^*$, the following condition holds:

$$(A+BK_i)'P_i + P_i(A+BK_i) < -I. \tag{6.1.1}$$

Hence, for each $(A, B) \in \Sigma_i^*$, the *Lyapunov function* defined by

$$V_i(x) \triangleq x'P_ix \tag{6.1.2}$$

decreases along state trajectories when control $u(t) = K_i x(t)$ is used. Next, analogous to Section IV, the function $V_i(x(t))$ can be used instead of $V(t, \tau_i)$ in the construction of the switching control. Indeed, for any arbitrarily small desired waiting period τ, define the switching instants

$$t_i \triangleq \sup \{t : t > t_{i-1} + \tau;\ V_i(x(t)) \le \rho_i V_i(x(t-\tau))\} \tag{6.1.3}$$

for $i = 1, 2, \cdots, f-1$, where

$$\rho_i \triangleq e^{-(\tau/\lambda_{\max}[P_i])}. \tag{6.1.4}$$

Then, it can be shown that with the switching index given by (4.0.6) and switching control given by (4.0.7), we obtain Lyapunov stability with exponential convergence rate as in Theorem 5.1.

B. Modification for Measurement Noise Rejection

We now provide a brief sketch indicating how the controller can be modified to handle measurement noise as discussed in Section I. In this case, the switching index $h(t)$ in (4.0.6) may never converge because $V(t, \tau_i)$ may be dominated by noise when $\|y(t)\|$ is small. Therefore, the decreasing property (5.1.2) of $V(t, \tau_i)$ may be destroyed and the switching index may keep jumping indefinitely leading to instability. To overcome this problem, we modify the switching index in such a way that:

1) the state tends to a bounded neighborhood of the origin if the measurement noise is bounded;

2) the size of the neighborhood in 1) to which the state is eventually confined vanishes as the noise amplitude vanishes.

This modification is simply accomplished by reinitializing $h(t)$ to 1 whenever $h(t)$ exceeds f. The basic idea behind this type of modification can be heuristically motivated. First, note that the measurement noise will not affect the decreasing property of $V(t, \tau_i)$ when $\|y(t)\|$ is sufficiently large. Therefore, for outputs with large norm, the modified switching rule leads to a "good" compensator gain matrix and $\|x(t)\|$ is reduced until it reaches the point that it is "comparable" to the amplitude of the measurement noise. It can be readily shown that the size of the neighborhood to which the state converges can be bounded in norm by $M'\epsilon_{\max}$ where $M' > 0$ is a constant and $\epsilon_{\max}$ is the upper bound on the norm of the measurement noise.

VII. Examples and Simulations

Two examples are provided in this section to illustrate the behavior of systems subjected to the switching control (4.0.7). In the first example, we indicate a typical construction of the controller and provide sample state trajectories for various possible plants in the given collection. In the second example, we return to system (1.2) and consider the problem of measurement noise rejection recalling the motivating instability problem described in Section I. Using the modification of the switching control as prescribed in Section VI, it is seen that the state trajectories are no longer unbounded. As a matter of fact, the state tends to a bounded neighborhood of the origin whose size is comparable to the amplitude of measurement noise.

Example 1: Consider the set of possible systems Σ described parametrically by the state equation

$$\dot{x}(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)+\begin{bmatrix}q\\ 1\end{bmatrix}u(t);\ q\in Q\triangleq[-0.5,\ 0.5];$$

$$y(t)=[1\quad 0]x(t);\ t\in[0,\ \infty). \tag{7.1.1}$$

It is straightforward to verify that for each triple $(A, B, C) \in \Sigma$, the system is controllable and observable. Also, the system order is fixed at $n = 2$ and Σ_2 is compact by inspection. Hence, Theorem 5.1 applies and we can use the recipe in Section IV to obtain a stabilizing compensator. First, we need to generate a finite number of compensator gain matrices $K_1, K_2, \cdots, K_f$ as prescribed in Lemma 3.1. To this end, we construct a reduced-order Luenberger observer (parameterized in q); we assign the poles of the state $x(t)$ at -1 and -2 and the pole of the observer at -4. It turns out that an appropriate compensator gain matrix has the form

$$K(q)=\begin{bmatrix}-31+30q & -5+6q\\ -150q^2+185q-56 & -30q^2+31q-9\end{bmatrix}.$$

Now, to satisfy the requirements of Lemma 3.1, we take $\gamma = 0.30$, and perform a lengthy but straightforward calculation and verify that the requirements of Lemma 3.1 are satisfied by taking $f = 5$ and

$$K_1=K(-0.5)=\begin{bmatrix}-46 & -8\\ -186 & -32\end{bmatrix};$$

$$K_2=K(-0.25)=\begin{bmatrix}-38.5 & -6.5\\ -111.625 & -18.625\end{bmatrix};$$

$$K_3=K(0)=\begin{bmatrix}-31 & -5\\ -56 & -9\end{bmatrix};$$

$$K_4=K(0.25)=\begin{bmatrix}-23.5 & -3.5\\ -19.125 & -3.125\end{bmatrix};$$

$$K_5=K(0.5)=\begin{bmatrix}-16 & -2\\ -1 & -1\end{bmatrix}.$$

Now, to satisfy the requirement on the ρ_i [see (4.0.2)], we increase the τ_i and find that for $\tau_1 = 2.1$, $\tau_2 = 1.8$, $\tau_3 = 1.6$, $\tau_4 = 1.2$, and $\tau_5 = 1.2$, we have $\rho_i < 1$ for $i = 1, 2, 3, 4, 5$. Hence, the parameters of the switching control in (4.0.7) are now completely specified. Figs. 3–5 are obtained by computer simulation using different values of the parameter $q \in Q$. Sample state trajectories and the switching behavior of the control are indicated.

Example 2: We consider system (1.2) for $a = 1$ and $q \in \{-1, 1\}$ with additive measurement noise. Again, the compactness of the set of possible plants and boundedness of the state dimension are trivially verified. The state feedback control derived in Section VI-A is used since the output and the state are the same. To satisfy the requirements of Lemma 3.1 for any $\gamma < 1$, we use two compensator gains $K_1 = 2$ and $K_2 = -2$. The simple Lyapunov function

$$V(x) \triangleq x^2$$

is chosen to satisfy the condition (6.1.1). The "waiting period" is taken to be $\tau_1 = \tau_2 = 0.5$.

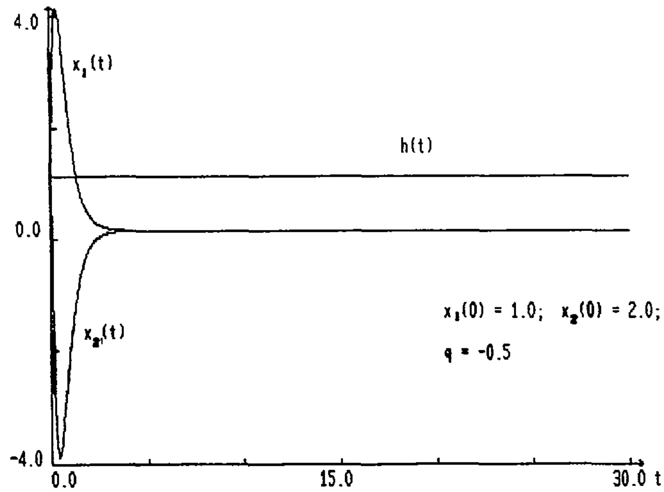

Fig. 3. Simulation for Example 1: $q = -0.5$.

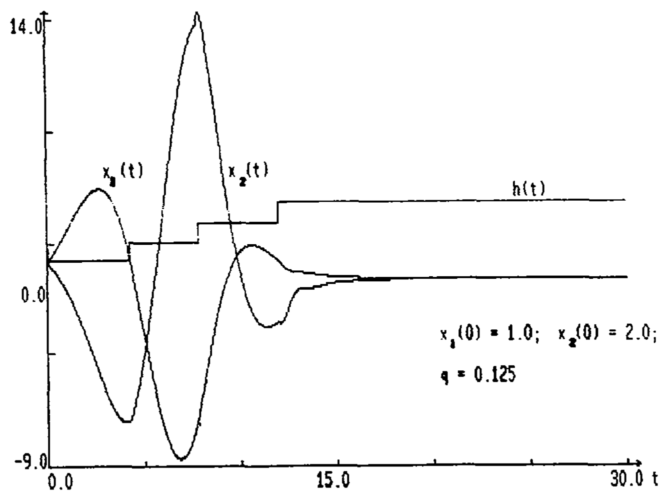

Fig. 4. Simulation for Example 1: $q = 0.125$.

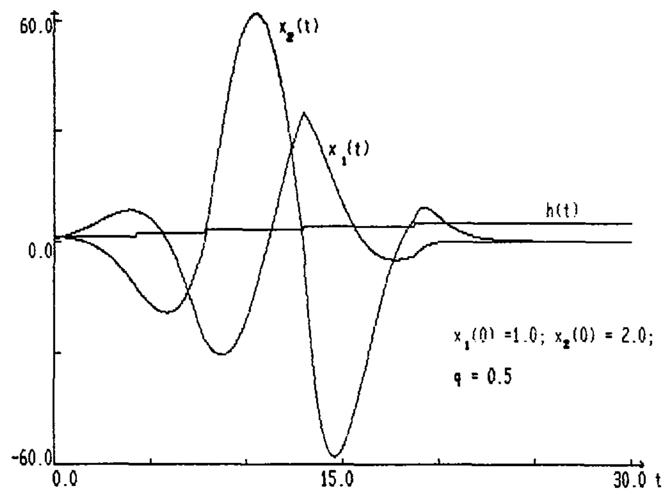

Fig. 5. Simulation for Example 1: $q = 0.5$.

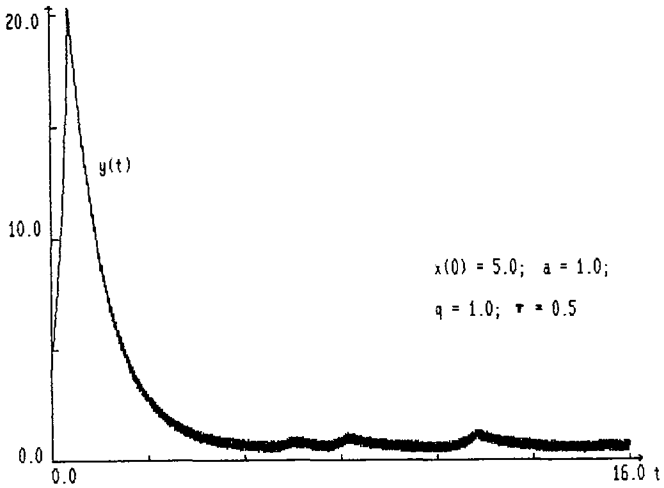

Fig. 6. Simulation for Example 2.

To illustrate the behavior of the closed-loop system, the same destabilizing disturbance $\epsilon(t) = 0.25 \sin 100t$ which we previously considered is added once again. This time, however, the system is compensated by the modified switching control described in Section VI-B. The simulation result given in Fig. 6 indicates that the state no longer "blows up." In fact, $x(t)$ settles into a small neighborhood about zero as predicted by the theory.

VIII. Conclusion

Theorem 5.1 strengthens recent results on adaptive stabilization to include a guarantee of Lyapunov stability with an exponential rate of convergence for the state. Furthermore, using the modification of the control law described in Section IV, the state remains bounded in the presence of measurement noise and the norm bound on the system state tends to zero as noise bound tends to zero. We do, however, pay a price for this "more practical" controller. That is, to obtain stronger results, we have to impose additional requirements, beyond those in [8], on the set Σ of possible plants: compactness and an *a priori* upper bound $n_{\max}$ on the order of plants in Σ.

From an implementation point of view, the switching controller in (4.0.7) has the desirable feature that it is a piecewise linear time-invariant feedback. Moreover, after a finite number of switches, the controller becomes a classical linear time-invariant feedback and remains as such thereafter. On the other hand, there is one potential "stumbling block" when performing numerical computations. Namely, the construction of the gain matrices K_1, $K_2, \cdots, K_f$ (see Lemma 3.1) may be computationally prohibitive. As indicated in the proof of the lemma, these gain matrices are obtained by extracting a finite subcovering from a specially constructed open covering of Σ. In view of this limitation, it is felt that future research should be aimed at developing alternatives to Lemma 3.1. In other words, it would be worthwhile investigating alternative procedures for construction of the controller while preserving the desirable properties obtained for the closed-loop system. The stability result established here should really be viewed as a benchmark against which to compare new control schemes.

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