On the Stepwise Hurwitz Property: A New Tool for Robust Output Feedback Stabilization

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***Abstract*—In this paper, a new theoretical concept is introduced for polynomials: the *Stepwise Hurwitz Property.* Subsequently, it is shown how this concept can be used to systematically achieve robust output feedback stabilization for large classes of uncertainty structures. A principal motivation for this paper is the fact that the state feedback controller construction methods do not readily admit modifications to handle the output feedback case. One of the fundamental technical issues addressed in this paper involves the handling of poles and zeros at the origin. For example, high-gain control results which are available to robustly stabilize an uncertain minimum phase plant $G(s,q)$ via output feedback, do not readily extend to plants of form $s^m G(s,q)$.**

1. Introduction

The main results of this paper bear on the large body of literature involving construction of robustly stabilizing controllers for systems which include an uncertain parameters or nonlinear elements with known bounds. A principal motivation for this paper is the fact that results for robust stabilization via state feedback do not readily admit modifications to handle the output feedback case; e.g., for state feedback solutions, see [10], [1], [16], [20], [21], [19] and for the case of linear systems and [3], [8], [9], [11], [13], [14] and [15] for the case of nonlinear systems. As far as the literature on robust output feedback stabilization is concerned, results for minimum phase plants are the benchmark against which the results this paper can be compared; e.g., see [2], [18] and [22] where an input-output linear system description is the starting point and [6] and [12] where lower triangular state space uncertainty structures are considered.

By way of motivation, let q denote a finite-dimensional vector of uncertain parameters with known compact bounding set Q and take $G(s,q)$ to be a plant transfer function with one-sign high-frequency gain, q entering continuously into its coefficients and having the form

$$G(s,q) = \frac{s^m N(s,q)}{D(s,q)}$$

with $N(s,q)$ and $D(s,q)$ being uncertain polynomials. Notice that if $N(s,q)$ is robustly Hurwitz and $m = 0$, this minimum phase uncertain system is readily stabilizable via high-gain output feedback. On the other hand, with $m > 0$, it turns out that a naive high-gain approach will fail.

The simple uncertainty structure above generalizes a number of uncertainty structures in the literature; see [1], [17], [19] and [21]. The structure is generated via a sequence of up and down augmentations. Special cases of this structure include the well-known lower-triangular structure and upper-triangular structure. By way of concrete illustration, the system

$$\begin{aligned} \dot{x_1} &= -q_1 x_1 + x_2 + q_2 x_4; \\ \dot{x_2} &= x_3; \\ \dot{x_3} &= q_3 x_3 + x_4; \\ \dot{x_4} &= -x_1 + u; \\ y &= x_3 \end{aligned}$$

with $q \doteq (q_1, q_2, q_3)$ does not admit a parameter-independent transformation taking it to a triangular form but has transfer function

$$G(s,q) = \frac{s(s+q_1)}{s^4 + sd(s,q) + 1}$$

with $d(s,q)$ being a second order polynomial. Now, with arbitrarily large uncertainty bounds $q_i^- \le q_i \le q_i^+$ with $q_1^- > 0$, the non-minimum phase zero at $s = 0$ is problematic as far as high-gain robust output feedback stabilization. However, for this system, since the sign of the low-frequency gain is positive, the results given in this paper lead to a systematic construction of a robust output feedback stabilizing compensator; see Section 3 where this example is revisited.

Analogous to the case above, when the transfer function has one-sign low-frequency gain and is of the form

$$G(s,q) = \frac{N(s,q)}{s^m D(s,q)}$$

with $D(s,q)$ robustly stable, while zero feedback is needed when $m = 0$, the case with $m > 0$ and non-minimum phase becomes challenging. This provides a second example from the class of systems for which a robust stabilizer can be constructed using the results in this paper.

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2. The Stepwise Hurwitz Property

In this section, we introduce two new concepts: the *Stepwise Hurwitz Property* and the notion of *Robust Hurwitz inducibility*, are introduced.

2.1 Preliminaries for Polynomials: Let $\mathcal{H}$ denote the set of Hurwitz polynomials; i.e., polynomials with all roots in the strict left half plane. Now, if $f_0(s), f_1(s), f_2(2), \ldots, f_N(s)$ are polynomials and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbf{R}^N$ is fixed, we consider the *parameterized polynomial*

$$f(s) \doteq f_0(s) + \alpha_1 f_1(s) + \alpha_2 f_2(s) + \cdots \alpha_N f_N(s).$$

For given $\alpha \in \mathbf{R}^N$, this parameterized polynomial is said to have the *Stepwise Hurwitz Property* if $f(s) \in \mathcal{H}$ and, for $k = 0, 1, \cdots, N-1$, the partial sums

$$F_k(s) \doteq f_0(s) + \alpha_1 f_1(s) + \alpha_2 f_2(s) + \cdots \alpha_k f_k(s)$$

satisfy the following condition: For each k, there exists a non-negative integer i_k such that

$$s^{-i_k} F_k(s) \in \mathcal{H}.$$

When such a selection of α exists, the polynomial sequence $\{f_k(s)\}_{k=0}^N$ is said to be *(Stepwise) Hurwitz inducible* and $f_0(s)$ is called the *Hurwitz core.*

2.2 Remarks: In a control theoretic context, if we view the α_i above as parameters which correspond to compensator coefficients, it is apparent that Hurwitz inducibility is equivalent to stabilizability. In the sequel, one of the main technical novelties associated with robust stabilization is our relabelling of the compensator coefficients α_i so that the Stepwise Hurwitz inducibility is guaranteed. That is, the obvious ordering for selection of compensator parameters corresponding to the increasing or decreasing degrees of s^k in $N_c(s)$ or $D_c(s)$ does not necessarily lead to satisfaction of the Stepwise Hurwitz Property.

2.3 Example: To illustrate the concepts above, we consider the parameterized polynomial $f(s)$ above with components $f_0(s) = s^3 + s^2$; $f_1(s) = 2s^4 - s^2 + s$; $f_2(s) = s^5 + s$ and $f_3(s) = s^2 - s + 3$. Now, for the fixed choice of parameters $\alpha_1 = \alpha_2 = 0.25$, $\alpha_3 = 0.05$ and indices $i_0 = 2$, $i_1 = i_2 = 1$ and $i_3 = 0$, a straightforward calculation leads to the s^{-i_k} shifted partial sums $s^{-2}F_0(s)$, $s^{-1}F_1(s)$, $s^{-1}F_2(s)$ and $F_3(s)$ are readily verified to be Hurwitz polynomials. Hence, for this selection of the α_k, the resulting polynomial has the Stepwise Hurwitz Property.

2.4 Notation: Given a sequence of polynomials $f_1(s), f_2(s), \ldots, f_N(s)$, we define the associated sequence of *partial polynomial vectors*

$$f^{(k)}(s) \doteq [f_0(s)\ f_1(s) \ldots\ f_k(s)];\ k = 0, 1, \ldots, N.$$

We now define I_k, the *maximum index* of $f^{(k)}(s)$, to be the maximum degree of the $f_i(s)$ comprising this partial vector; i.e.,

$$I_k \doteq \max\{\deg f_1(s),\ \deg f_2(s),\ \ldots,\ \deg f_N(s)\}$$

We also define i_k, the *minimum index* of $f^{(k)}(s)$ as follows: Let s^{j_i} be the lowest power of s appearing in $f_i(s)$ with a non-zero coefficient. Then,

$$i_k \doteq \min\{j_1, j_2, \ldots, j_k\}.$$

Note that I_k and i_k are non-decreasing and non-increasing, respectively. We conclude this section by generalizing the discussions above to uncertain polynomials.

2.5 Robustness Generalizations: For the case when the polynomials $f_k(s)$ have coefficients depending continuously on a vector q of uncertain parameters, we replace $f(s), f_k(s)$ and $F_k(s)$ by their uncertain counterparts $f(s,q), f_k(s,q)$ and $F_k(s,q)$, respectively. Then, given a compact bounding set Q for the parameters q, we say that the Stepwise Hurwitz Property holds *robustly* if $f(s,q) \in \mathcal{H}$ for all $q \in Q$ and for each partial sum $F_k(s,q)$ with $k < N$, there exists a non-negative integer i_k such that $s^{-i_k}F_k(s,q) \in \mathcal{H}$ for all $q \in Q$. Finally, it should be noted that in this case, the maximum and minimum indices of the partial polynomial vector $f^{(k)}(s,q)$, respectively denoted by $I_k(q)$ and $i_k(q)$, are functions of q.

3. Robust Hurwitz Inducibility

3.1 Theorem of Robust Hurwitz Inducibility (see Section 4 for proof): *Given the sequence of uncertain polynomials $f_k(s,q)$, $q \in Q, k = 0, 1, \ldots, N$, suppose the following conditions are satisfied:*

- (i) *The polynomial $s^{-i_0} f_0(s,q)$ is a robustly Hurwitz with a positive highest degree coefficient.*
- (ii) *The maximum and minimum indices of the partial uncertain polynomial vectors $f^{(k)}(s,q)$ are invariant, and thus denoted by I_k and i_k, respectively.*
- (iii) *As k increases from zero to N, for each transition of $k \to k+1$, I_k (respectively, i_k) can increase (respectively, decrease) by one at most.*
- (iv) *For $k = N$, the minimum index is $i_N = 0$.*

Then, the sequence $\{f_k(s,q)\}_{k=0}^N$ is robustly Hurwitz inducible.

3.2 Motivating Example Revisited: To illustrate how the Stepwise Hurwitz Theorem applies to classes of systems which are not covered by the existing literature, we revisit the motivating example given in Section 1. This example, while analyzed in somewhat of an ad hoc manner here, is addressed more formally in Section 5. Accordingly, the procedure below will be formalized as part of a step-by-step procedure. We begin with the plant transfer function

$$G(s,q) = \frac{s(s+q_1)}{s^4+(q_3-q_1)s^3+(q_2-q_1q_3)s^2-q_2q_3s+1},$$

with its uncertain parameter bounds given by $1 \le q_1 \le 2$, $-1 \le q_2 \le 1$, $-1 \le q_3 \le 1$ and note that the analysis to follow could equally well be carried out with arbitrarily large uncertainty bounds with the the proviso that $q_1 > 0$. We now specify a second order controller of the form

$$C(s) = \frac{n_2s^2+n_1s+n_0}{d_2s^2+d_1s+d_0} \doteq \frac{N_c(s)}{D_c(s)}.$$

To demonstrate that all hypotheses in the Stepwise Hurwitz Theorem are satisfied, we first consider the closed loop polynomial

$$f(s,q) = sN(s,q)N_c(s) + D(s,q)D_c(s)$$

which is rewritten as

$$f(s,q) = N(s,q)N_c(s)+d_0D(s,q)+d_1sD(s,q)+d_2s^2D(s,q).$$

We now claim that the hypotheses of the theorem are satisfied by taking $N_c(s)$ to be any positive coefficient Hurwitz polynomial and letting $f_0(s,q) = sN(s,q)N_c(s)$, $f_1(s,q) = sD(s,q)$, $f_2(s,q) = D(s,q)$ and $f_3(s,q) = s^2D(s,q)$. Now with $i_0 = 1$ and recalling that $q_1 \ge 1$, it is readily verified that $s^{-i_0}f_0(s) = (s+q_1)N_c(s)$ is robustly Hurwitz. Also, the corresponding minimum and maximum indices $i_0 = i_1 = 1$, $i_2 = i_3 = 0$, $I_0 = 4$, $I_1 = I_2 = 5$ and $I_3 = 6$ and the associated coefficients satisfy the required invariance requirements of the theorem. It follows that the parameterized closed loop polynomial $f(s,q)$ is robustly Hurwitz inducible. Now, in accordance with the previous section, it is now possible to construct a robustly stabilizing compensator. Indeed, a lengthy but straightforward computation leads to a robust stabilizer given by

$$C(s) = 20\frac{s^2+s+1}{s^2+5s+5}.$$

4. Proof of Theorem 3.1

4.1 Preliminaries: Given an n-th order polynomial

$$p(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots a_0$$

with $a_n > 0$, the associated Hurwitz matrix is denoted by H. Note that the last column of H has zero entries except its last element and that $p(s)$ is Hurwitz if and only if all the principal minors of H are positive.

4.2 Proof of Theorem: We proceed by induction. That is, assuming $\alpha_1, \cdots, \alpha_k$ are chosen such that $s^{-i_k}F_k(s,q)$ is robustly Hurwitz, we need to prove that α_{k+1} can be chosen to make $s^{-i_{k+1}}F_{k+1}(s,q)$ robustly Hurwitz. Adopting the shorthand notation $\alpha \doteq \alpha_{k+1}$, $g_0(s,q) \doteq s^{-i_{k+1}}F_k(s,q)$, $g_1(s,q) \doteq s^{-i_{k+1}}F_{k+1}(s,q)$ and $g(s,q) \doteq s^{-i_{i+1}}f_{k+1}(s,q)$ and suppressing the (s,q) arguments, we have $g_1 = g_0 + \alpha g$. We now consider four cases corresponding to various combinations of I_{k+1} and i_{k+1}.

Case 1: $I_{k+1} = I_k$ and $i_{k+1} = i_k$. In this case, g_0 is robustly Hurwitz and deg $g_0 \ge$ deg g. It follows from continuous dependence of the roots of a polynomial on its coefficients that g_1 is robustly Hurwitz for sufficiently small $\alpha > 0$.

Case 2: $I_{k+1} = I_k + 1$ and $i_{k+1} = i_k$. In this case, we observe that g_0 is robustly Hurwitz, deg g = deg $g_0 + 1$, and the highest degree coefficient of g_1 is positively invariant. Hence, for this special case, it follows from existing results in the literature (see [2] or [18]) that g_1 is robustly Hurwitz for suitably small $\alpha > 0$.

Case 3: $I_{k+1} = I_k$ and $i_{k+1} = i_k - 1$. In this case, we define $\bar{g}_0 = s^{-i_k}F_k$ and observe that $g_0 = s\bar{g}_0$ and $\bar{g}_0$ are robustly Hurwitz. Furthermore, deg $g \le$ deg g_0, and the zeroth degree coefficient of g is positively invariant. We now reduce this situation to Case 2 as follows: Forming the reversed-order polynomial $\hat{g}_1$ by reversing the ordering of the coefficients of g_1, it is straightforward to see that g_1 is robustly Hurwitz if and only if $\hat{g}_1$ is robustly Hurwitz. Now for $\hat{g}_1$, the problem of selecting α reduces to that in Case 2.

Case 4: $I_{k+1} = I_k + 1$ and $i_{k+1} = i_k - 1$. In this case, $g_0 = s\bar{g}_0$, where $\bar{g}_0 = s^{-i_k}F_k$ is robustly Hurwitz, deg g = deg $g_0 + 1$, and both the highest and the zeroth degree coefficients of g are positively invariant. Letting n = deg $\bar{g}_0$, it follows that deg g_1 = deg $g = n+2$.. Furthermore, expressing the highest and lowest coefficients of g_1 as αg_{n+2} and αg_0 respectively, it follows that these quantities are positively invariant when $\alpha > 0$. It is now straightforward to verify that that the Hurwitz matrix for g_1 is given by

$$H_1(\alpha,q) = \begin{bmatrix} H_{1,0}(q)+\alpha H_{1,1}(q) & 0 \\ H_{1,2}(q,\alpha) & \alpha g_0(q) \end{bmatrix}$$

where $H_{1,0}(q)$ is the Hurwitz matrix of $\bar{g}_0$ when viewed as an $(n+1)$-th order polynomial; i.e., with $a_{n+1} = 0$, forming the Hurwitz matrix for $a_{n+1}s^{n+1} + \bar{g}_0$. Also, in the expression above, $H_{1,1}(q)$ is the part of the Hurwitz matrix for g with the last row and column deleted. Further examination shows that $H_{1,0}(q)$ has the structure

$$H_{1,0}(q) = \begin{bmatrix} a_n(q) & [a_{n-2}(q)\ \cdots] \\ 0 & H_0(q) \end{bmatrix}$$

where $a_0(q), a_1(q), \cdots, a_n(q)$ are the coefficients of $\bar{g}_0$ and $H_0(q)$ is its Hurwitz matrix.

In view of the structural properties above, we claim that all the leading principal minors of $H_1(q,\alpha)$ are positively invariant for sufficiently small $\alpha > 0$. To prove the claim, we consider the highest order minor $\det H_1(q,\alpha)$, noting that a similar proof applies to the other lower order minors as well. Indeed, we write

$$\det H_1(q,\alpha) = \alpha g_0(q)(a_n(q)\det H_0(q) + o(q,\alpha))$$

where the term $o(q,\alpha)$ vanishes uniformly in q as $\alpha \to 0$. That is, given any $\varepsilon > 0$, there exists a suitably small $\alpha > 0$ such that $|o(q,\alpha)| \leq \varepsilon$ for all $q \in Q$. Now using the properties of $\bar{g}_0$, we know that $\det H_0(q)$ and $a_n(q)$ are both positively invariant. Therefore, for suitably small $\alpha > 0$, $\det H_1(q,\alpha)$ is positively invariant. In view of this claim, we now conclude that g_1 is robustly Hurwitz for suitably small $\alpha > 0$. It follows by induction that $\alpha_1, \alpha_2, \cdots, \alpha_N > 0$ can be selected recursively to make $s^{-i_1}F_1(s,q)$, $s^{-i_2}F_2(s,q)$, $\cdots, s^{-i_N}F_N(s,q)$ robustly Hurwitz with $i_N = 0$.

5. Stabilizable Transfer Function Structures

In this section, we provide robust stabilization results for the two transfer function structures discussed in Section 1. As previously mentioned, poles or zeros at the origin preclude the use of simple high-gain or low-gain results.

5.1 Pseudo-Minimum Phase Uncertain Plants: Recalling the discussion in Section 1, we consider a proper transfer function of the form

$$G(s,q) = \frac{s^m N(s,q)}{D(s,q)}$$

where $m \geq 0$, $N(s,q)$ is an v-th order robustly Hurwitz polynomial with a positively invariant zeroth degree coefficient and $D(s,q)$ is an n-th order uncertain polynomial with a positively invariant highest degree coefficient. When $m > 0$, it is further assumed that the zeroth order coefficient, $d_0(q)$, of $D(s,q)$, is sign-invariant so that there is no unstable zero-pole cancellation. Since the numerator of the plant has its zeros at the origin and in the open left half plane, we refer to the plant as *pseudo-minimum phase*.

Now for the non-trivial case when $D(s,q)$ is non-Hurwitz, we apply a proper compensator $C(s) = N_c(s)/D_c(s)$ and the objective is to select the coefficients of $N_c(s)$ and $D_c(s)$ to assure that the resulting closed loop polynomial

$$f(s,q) = s^m N(s,q)N_c(s) + D(s,q)D_c(s)$$

is robustly Hurwitz. When such a compensator exists, the system is *robustly stabilizable via output feedback*.

5.2 Theorem: *The pseudo-minimum phase uncertain plant $G(s,q)$ is robustly stabilizable via output feedback. Furthermore, a robustly stabilizing proper controller $C(s) = N_c(s)/D_c(s)$ can be chosen to be minimum phase and satisfying the following conditions:*

(i) *When $m = 0$,*

$$\deg N_c(s) = \deg D_c(s) = r - 1;$$

(ii) *When $m > 0$ and $d_0(q) > 0$,*

$$\deg N_c(s) = m + r - 2;$$
$$\deg D_c(s) = \max\{m-1,\ m+r-2\};$$

(iii) *When $m > 0$ and $d_0(q) < 0$,*

$$\deg N_c(s) = \deg D_c(s) = m + r - 1.$$

Furthermore, the controller $C(s)$ can be designed using the following procedure:

Step 1 *: Choose $N_c(s)$ to be any Hurwitz polynomial with the degree as given above and take the Hurwitz core to be*

$$f_0(s,q) = s^m N(s,q)N_c(s).$$

Step 2 *: If $m = 0$, for $k = 1, 2, \ldots, r$, let $f_k(s,q) = s^{k-1}D(s,q), k = 1, 2, \ldots, r$. If $m > 0$, take*

$$\bar{D}(s,q) = \begin{cases} D(s,q) & \text{if } d_0(q) > 0; \\ D(s,q)(s-1) & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} f_1(s,q) &= s^{m-1}\bar{D}(s,q); \\ f_2(s,q) &= s^{m-2}\bar{D}(s,q); \\ &\cdots \quad \cdots \\ f_m(s,q) &= \bar{D}(s,q). \end{aligned}$$

When $r > 1$, continue with

$$\begin{aligned} f_{m+1}(s,q) &= s^m\bar{D}(s,q); \\ f_{m+2}(s,q) &= s^{m+1}\bar{D}(s,q); \\ &\cdots \quad \cdots \\ f_{m+r-1}(s,q) &= s^{m+r-2}\bar{D}(s,q). \end{aligned}$$

Step 3 *: Apply Stepwise Hurwitz Theorem to recursively select the α_i. If $r \leq 1$, take*

$$\bar{D}_c(s) = \alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m.$$

When $r > 1$, let

$$\begin{aligned} \bar{D}_c(s) = &\ \alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m \\ &+ \alpha_{m+1}s^m + \alpha_{m+2}s^{m+1} + \cdots \\ &+ \alpha_{m+r-1}s^{m+r-2}. \end{aligned}$$

Then, $D_c(s)$ is given by

$$D_c(s) = \begin{cases} \bar{D}(s)(s-1) & \text{if } m > 0 \ \&\ d_0(q) > 0; \\ \bar{D}(s) & \text{otherwise.} \end{cases}$$

Proof. It is easy to verify that the specified dimensions guarantee that the controller $C(s)$ is proper. Hence, it suffices to show that the f_k sequence, constructed via the procedure above, is robustly Hurwitz inducible. We first consider the case where $m = 1$ and $d_0(q)$ is negatively invariant. In the design procedure above, for this case, we first modify $D(s,q)$ by multiplying the factor $(s-1)$. It is straightforward to check that the resulting denominator $\bar{D}(s,q)$ has a positively invariant zeroth degree coefficient. Thus, this case is reduced to the case where $m > 0$ and $d_0(q)$ is positively invariant but with a new degree $\bar{n} = n+1$ and relative degree $\bar{r} = r+1$. Suppose the modified uncertain transfer function $s^m N(s,q)/\bar{D}(s,q)$ can be robustly stabilized by a controller $N_c(s)/\bar{D}_c(s)$ with $\deg N_c(s) = m+\bar{r}-2$ and $\deg \bar{D}_c(s) = \max\{m-1,\ m+\bar{r}-2\}$. Then it follows that the original uncertain transfer function $N(s,q)/D(s,q)$ can be robustly stabilized by $N_c(s)/D_c(s)$ with $D_c(s) = \bar{D}_c(s)(s-1)$. Because $\bar{r} = r+1$ and $m \geq 1$, $\deg N_c(s) = \deg D_c(s) = m+r-1$. Hence, in the sequel, we only need to consider the cases $m = 0$ and the case $m > 0$ with $d_0(q) > 0$.

Note that $i_0 = m$ and $s^{-i_0} f_0(s,q)$ is robustly Hurwitz with a positively invariant zeroth degree coefficient. For the case $m = 0$, we have $i_k = 0$ for all k and $I_{k+1} = I_k + 1$. By Theorem 3.1, it follows that the f_k sequence is robustly Hurwitz inducible. For the case $m > 0$, we claim that $i_1 = i_0 - 1$ and $I_1 = I_0 + 1$. The first part of the claim is easy to see because $f_1(s,q)$ has a factor s^{m-1} whereas $f_0(s,q)$ has a factor s^m and both $D(s,q)$ and $N(s,q)$ have positively invariant zeroth degree coefficients. To prove $I_1 = I_0 + 1$, by noting that both $D(s,q)$ and $N(s,q)$ have positively invariant highest degree coefficients, it follows that

$$\begin{aligned} I_0 &= \deg s^m N(s) + \deg N_c(s) \\ &= (n-r) + (m+r-2) = n+m-2 \end{aligned}$$

and

$$I_1 = (m-1) + \deg D(s,q) = (m-1) + n = I_0 + 1.$$

Next, for $k = 1, 2, \ldots, m-1$, it is easy to verify that the specified dimensions guarantee that $I_{k+1} = I_k$ and $i_{k+1} = i_k - 1$. A particular consequence of this fact is that $i_m = 0$. Finally, if $m+r-2 > m-1$, we have

$$I_{m+1} = I_m + 1; \quad i_{m+1} = i_m = 0;$$

$$I_{m+2} = I_{m+1} + 1; \quad i_{m+2} = i_{m+1} = 0.$$

Continuing in an identical manner for any indices above I_{m+2}, by Theorem 3.1, the f_k sequence is again robustly Hurwitz inducible.

5.3 Pseudo-Stable Uncertain Plants: Recalling the discussion in Section 1, we consider a proper transfer function of the form

$$G(s,q) = \frac{N(s,q)}{s^m D(s,q)}$$

where $m > 0$, $D(s,q)$ is an n-th order robustly Hurwitz polynomial. Without loss of generality, we assume that $D(s,q)$ has positively invariant coefficients. Finally, the uncertain polynomial $N(s,q)$ is assumed to have a sign-invariant zeroth degree coefficient. The degree, $v(q)$, of $N(s,q)$ is allowed to vary with q, provided that $G(s,q)$ remains proper. Since the denominator has all its roots at the origin and in the open left half plane, we refer to the plant as being *pseudo-stable*. The result below follows easily from Theorem 3.1.

5.4 Theorem: *The pseudo-stable uncertain plant $G(s,q)$ above is robustly stabilizable via output feedback. Furthermore, a robustly stabilizing proper controller $C(s) = N_c(s)/D_c(s)$ can be chosen to be stable satisfying*

$$\deg N_c(s) = \deg D_c(s) = m-1$$

and the controller can be designed using the following procedure:

Step 1 *: Choose $D_c(s)$ to be any $(m-1)$-th order Hurwitz polynomial and let the Hurwitz core be $f_0(s,q) = s^m D(s,q) D_c(s,q)$.*

Step 2 *: For $k = 1, 2, \ldots, m$, let $f_k(s,q) = s^{m-k} N(s,q) S_N$, where S_N is the sign of the zeroth degree coefficient of $N(s,q)$.*

Step 3 *: Apply Stepwise Hurwitz Theorem to recursively design the α_i. Then, let*

$$N_c(s) = S_N(\alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m)$$

5.5 Remark: In Theorems 5.2 and 5.4, we specified the order of the stabilizing controller. It is easy to construct examples show that stabilizing controllers may not exist in general if the order is lower than those given in the theorems.

6. Stabilizable State-Space Structures

In this section, we show that the pseudo-minimum phase uncertainty structure given in the previous section covers a large class of uncertain systems in the state-space framework. These systems admit a so-called the *Stepwise Augmentation Structure* which can be generated recursively using the so-called *down augmentations* and *up augmentations*. Such structures, first introduced in [1], were called the *admissible shuffles*. Later in [17], the term *anti-symmetric stepwise configuration* was used to describe a similar class of systems. For such systems, it is shown in [1], [17] that a robust linear, time-invariant state feedback stabilizer can be constructed. Such structures were also studied recently in [5] in the context of output regulation control via state feedback. The purpose of this section is to prove that a large class of such structures is robustly stabilizable via output feedback, provided that a suitably chosen output is available.

In the construction to follow, we begin with an uncertain system

$$\begin{aligned} \dot{x} &= A(q)x + b(q)u \\ y &= c^T(q)x \end{aligned}$$

where $q \in Q$ represents uncertain parameters as before, $A(q)$ is an $n \times n$ continuous matrix function, $b(q)$ and $c(q)$ are $n \times 1$ continuous vector functions, and u, x and y are the input, state and output of the system, respectively. We call $\Sigma = (A(q), b(q), c(q))$ a *generating system.*

6.1 Down-augmented Systems: Given a generating system $\Sigma = (A(q), b(q), c(q))$, the system

$$\begin{aligned} \dot{x} &= A(q)x + b(q)x_{n+1}; \\ \dot{x}_{n+1} &= \beta^T(q)x + \alpha(q)x_{n+1} + \theta(q)u; \\ y &= c^T(q)x \end{aligned}$$

with $n+1$ state variables is said to be a *down augmentation* of Σ if the added vectors and scalars $\alpha(q)$, $\beta(q)$ and $\theta(q)$ depend continuously on q and $\theta(q)$ is sign-invariant. We call x_{n+1} the *augmenting state variable.*

6.2 Up-augmented Systems: Given a generating system $\Sigma = (A(q), b(q), c(q))$, the system

$$\begin{aligned} \dot{x}_0 &= \beta^T(q)x; \\ \dot{x} &= A(q)x + b(q)(\alpha(q)x_0 + u); \\ y &= c^T(q)x \end{aligned}$$

with $n+1$ state variables is said to be an *up augmentation* of Σ if the added vector and scalar $\alpha(q)$ and $\beta(q)$ depend continuously on q and the first entry of $\beta(q)$ is sign-invariant. In this case, x_0 is called the *augmenting state variable.*

6.3 Stepwise Augmentation Structure: Let $\Sigma = (A(q), b(q), c(q))$ be a generating system with a robustly minimum phase transfer function. Then, a system is said to be a *stepwise augmentation structure* if it is obtained from Σ via a sequence of up and down augmentations, and in addition, if up augmentations are involved, the $A(q)$-matrix of the augmented system is nonsingular for all $q \in Q$.

6.4 Examples: To illustrate the stepwise augmentation structure, we list some of the uncertain systems which fit into this framework. Using the notation

$$M(q) \doteq [A(q) \mid b(q)]$$

we consider the four possible structures for $M(q)$ associated with 4-th order systems

$$\left[\begin{array}{cccc|c} * & \theta & 0 & 0 & 0 \\ \hline * & * & \theta & 0 & 0 \\ * & * & * & \theta & 0 \\ * & * & * & * & \theta \end{array}\right]; \quad \left[\begin{array}{cccc|c} 0 & \theta & * & * & 0 \\ 0 & 0 & \theta & * & 0 \\ 0 & 0 & 0 & \theta & 0 \\ \hline * & * & * & * & \theta \end{array}\right];$$

$$\left[\begin{array}{cccc|c} 0 & \theta & * & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 \\ \hline * & * & * & \theta & 0 \\ * & * & * & * & \theta \end{array}\right]; \quad \left[\begin{array}{cccc|c} 0 & \theta & * & * & 0 \\ 0 & 0 & \theta & 0 & 0 \\ \hline 0 & 0 & * & \theta & 0 \\ * & * & * & * & \theta \end{array}\right]$$

where $*$ denotes entries that are arbitrary functions of q and θ denotes the entries which are sign-invariant. For each matrix, the underlined state variable corresponds to the generating system. For example, for the third matrix $M(q)$ above, the generating system is described by

$$\dot{x} = \theta(q)u$$

The sequences of augmentations for the structures above are respectively down-down-down, down-up-up, down-up-down and down-down-up. In all of the examples above, the generating system is a scalar system of the form

$$\begin{aligned} \dot{x}_k &= a(q)x_k + \theta(q)u; \\ y &= x_k \end{aligned}$$

which is clearly robustly minimum-phase. It is also possible to give examples which is somewhat more complicated in the sense that the order of the generating system is higher than one.

6.5 Theorem: *Let $\Sigma = (A(q), b(q), c(q))$ be a generating system. Then, a down augmentation does not introduce any new zeros and each up augmentation introduces at most one zero at $s = 0$. Furthermore, if m up augmentations are involved and the final A-matrix for the augmented system is nonsingular for all $q \in Q$, then the augmented system has exactly m new zeros at $s = 0$.*

6.6 Corollary: *A stepwise augmentation structure is robustly pseudo-minimum phase, and thus robustly stabilizable via output feedback.*

Sketch of Proof of Theorem 6.5: We suppress the dependence of the system on q and denote the transfer function of the generating uncertain system by $G(s) = N(s)/D(s)$. Taking Laplace transforms and expressing the transfer function $\beta^T(sI - A)^{-1}b$ as $N_\beta(s)/D(s)$, a calculation leads to

$$Y(s) = \frac{N(s)}{D(s)(s-\alpha) - N_\beta(s)} U(s).$$

Hence, the down augmentation does not introduce any new zeros. Now, the transfer function of the up-augmented system is similarly computed. We obtain

$$Y(s) = \frac{sN(s)}{sD(s) - \alpha N_\beta(s)} U(s).$$

Hence, at most one new zero at $s = 0$ can be introduced by each up augmentation. Finally, if m up augmentations are involved (regardless of the number of down augmentations), the numerator of the augmented transfer function will be $s^m N(s, q)$. The new factor s^m can not be cancelled if the denominator of the augmented transfer function has a sign-invariant zeroth degree coefficient. This is guaranteed if the A-matrix of the augmented system is nonsingular for all $q \in Q$.

Proof of Corollary 6.6: By definition, the generating system of a stepwise augmentation structure is robustly minimum phase. By Theorem 6.5, down augmentations do not introduce new zeros and, if m up augmentations are involved, m new zeros at $s = 0$ are introduced because the $A(q)$-matrix of the augmented system is nonsingular for all $q \in Q$. This implies that the denominator of the augmented transfer function has a sign-invariant zeroth degree coefficient. Therefore, the transfer function of a stepwise augmentation structure is a robustly pseudo-minimum phase system, and thus robustly stabilizable via output feedback, according to Theorem 5.2.

7. Conclusion and Future Research

In this paper, we introduced the Stepwise Hurwitz Property as a means for extending a number of robust stabilization results from the full state feedback case to the output feedback case. Via the techniques introduced in this paper, it becomes possible to address large classes of uncertain systems falling into the pseudo-minimum phase or pseudo-stable categories. The results of this paper suggest some directions for future research. Most notably, the recursive design approach offered in this paper is a frequency domain approach, which is applicable to time-invariant parameters. If the uncertain parameters are time-varying, an analogous recursive method in the state-space domain is needed. In this regard, the concept of quadratic stabilization, which employs a parameter-independent quadratic Lyapunov function, is particularly useful. In fact, the state feedback design methods in [1], [19], [17] and [21], involve uncertainty structures similar to the Stepwise Augmentation Structure but with time-varying uncertainties. It would be important to investigate the extent to which our frequency domain approach also has a Lyapunov function interpretation. This sort of Lyapunov function interpretation would also be a stepping stone to output feedback stabilization of nonlinear systems with similar structures.

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