43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

A Separation Principle for Robust Stabilization of Linear Systems

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***Abstract*— The well-known Separation Principle plays a vital role in output feedback control of linear systems. The lack of a suitable Separation Principle for uncertain linear systems makes output feedback control for these systems a challenging problem. This paper is concerned with the problem of robust output feedback stabilization for a class of uncertain linear systems admitting the so-called Stepwise Augmentation Structure. This structure is generated via a sequence of up and down augmentations, including the well-known lower and upper triangular structures as special cases. We provide a recursive, state-space design approach for constructing quadratically stabilizing output feedback controllers. This design approach is based on a new Separation Principle for uncertain linear systems, which we believe is a powerful tool for robust control.**

I. Introduction

Robust stabilization of uncertain systems has been an active research area for many years. Numerous results are available for state feedback stabilization; see, e.g., see [1]-[6] for the case of linear systems and [7]-[13] for the case of nonlinear systems. However, most design approaches for state feedback control do not readily admit modifications to handle the output feedback case. Consequently, design methods for robust output feedback stabilization are scarce. Noticeable examples are [14]-[17] for linear uncertain systems with an minimum-phase transfer function and [18]-[19] where lower triangular state space uncertainty structures are considered.

In a recent paper [20], we proposed a frequency domain design approach to robust output feedback stabilization for a class of single-input-single-output uncertain linear systems with a pseudo-minimum phase transfer function. The numerator polynomial of a pseudo-minimum phase transfer function consists of a robust Hurwitz polynomial and a s^m factor, $m \geq 0$. Although this structure may appear to be a simple generalization of minimum phase systems, it requires a conceptually different design approach. Indeed, the commonly used high-gain approach for minimum phase uncertain systems fails to work for pseudo-minimum phase systems. Furthermore, despite its simple appearance, the structure above covers a large class of state space uncertain systems admitting the so-called *Stepwise Augmentation Structure* which is generalized from a number of uncertainty structures in the literature; see [1], [4], [6] and [16]. This structure is generated via a sequence of up and down augmentations. Special cases of this structure include the well-known lower-triangular structure and upper-triangular structure.

The well-known *Separation Principle* plays an important role in the linear control theory because it allows us to simplify an output feedback control problem into a state feedback control problem and a state observer design problem. There are several versions of the Separation Principle. The first version states that an observer-based state feedback system has closed loop eigenvalues specified by the state feedback controller and those of the state observers. The second version applies to Linear Quadratic Gaussian (LQG) control problems and states that an optimal output feedback controller is formed by an optimal state feedback controller and an optimal state estimator [21]. A similar version is available for output feedback H_∞ control [22].

However, these Separation Principles break down in general for uncertain systems. The lack of a suitable Separation Principle for uncertain linear systems makes output feedback control for these systems a challenging problem. The first goal in this paper is to provide a new Separation Principle suitable for robust stabilization of uncertain linear systems. The Separation Principle basically states that, under suitable assumptions, quadratic stabilizability via output feedback is equivalent to quadratic stabilizability via state feedback and the so-called *quadratic detectability*.

Using the new Separation Principle, we provide a recursive, state-space design approach for constructing quadratically stabilizing output feedback controllers. Roughly speaking, we prove that any uncertain linear system admitting the Stepwise Augmentation Structure can be quadratically stabilized using a dynamic, linear output feedback controller.

Due to the page limit, many results are stated without proofs. The details can be found in [24].

II. Separation Principle

In this section, we show, under some mild conditions, that output feedback quadratic stabilization can be separated into state feedback quadratic stabilization and the so-called *quadratic detection*, which is a dual version of quadratic stabilization for observer design. Our result is based on a generalization of the so-called *Elimination Lemma* which is a powerful algebraic tool for linear control design.

Definitions: The set of $n \times n$ symmetric matrices is denoted by $S^{n\times n}$ and the set of $n \times n$ symmetric and positive (resp. negative) definite matrices is denoted by $S_+^{n\times n}$ (resp. $S_-^{n\times n}$). An orthogonal complement of a matrix

N is a matrix $N_\perp$ with maximal rank such that $N_\perp N = 0$. When N is a nonsingular (square) matrix, $N_\perp$ is void.

Given an uncertain system:

$$\begin{aligned}\dot{x}(t) &= A(q(t))x(t) + B(q(t))u(t);\\ y(t) &= C^T(q(t))x(t),\end{aligned}$$

where $q(t) = [q_1(t)\ q_2(t)\ \dots\ q_v(t)]$ represents measurable time-varying uncertain parameters, $q(t) \in Q$ for all $t \in I\!R$, Q is a compact set. The system is said to be *quadratically stable* if there exists $P \in S_+^{n\times n}$ such that

$$A^T(q)P + PA(q) < 0, \quad \forall q \in Q.$$

The system is said to be *quadratically stabilizable via state feedback* if there exists a matrix K such that $A(q) + B(q)K$ is quadratically stable. The system is said to be quadratically stabilizable via output feedback if there exists an output feedback controller

$$\begin{aligned}\dot{z}(t) &= A_c z(t) + B_c y(t)\\ u(t) &= C_c z(t) + D_c y(t)\end{aligned}$$

such that the closed loop system is quadratically stable. The system is said to be *quadratically detectable* if there exists a matrix L such that $A(q) + LC^T(q)$ is quadratically stable.

The first main result of this paper is given below.

Theorem 1: (Separation Principle for Quadratic Stabilization) Given an n-th order uncertain system Σ described by

$$\begin{aligned}\dot{x}(t) &= A(q(t))x(t) + Bu(t);\\ y(t) &= C^T x(t),\end{aligned}$$

where $q(t) = [q_1(t)\ q_2(t)\ \dots\ q_v(t)]$ belong to a compact set Q for all $t \in I\!R$ and $A(q)$ is a continuous matrix function. Then, we have the following results:

(i) *Σ is quadratically stabilizable via state feedback if and only if there exists $S \in S_+^{n\times n}$ such that*

$$B_\perp(A(q)S + S^T A^T(q))B_\perp^T < 0.$$

(ii) *Σ is quadratically detectable if and only if there exists $P \in S_+^{n\times n}$ such that*

$$C_\perp(A^T(q)P + PA(q))C_\perp^T < 0;$$

(iii) *Σ is quadratically stabilizable via output feedback if and only if it is quadratically stabilizable via state feedback and quadratically detectable.*

(iv) *Σ is quadratically stabilizable via an k-th order output feedback controller for $k < n$ if and only if it is quadratically stabilizable via state feedback and quadratically detectable and in addition, the associated matrices P and S satisfy the following rank constraint:*

$$\begin{bmatrix} P & I_n \\ I_n & S \end{bmatrix} \geq 0 \quad \text{and} \quad \text{rank} \begin{bmatrix} P & I_n \\ I_n & S \end{bmatrix} \leq n + k.$$

III. Proof of Theorem 1

This section, devoted to the proof of the Separation Principle, can be skipped by readers solely interested in the application.

Lemma 1: (Elimination Lemma [23]) Given $G = G^T \in I\!R^{n\times n}$, $M \in I\!R^{n\times m}$, $N \in I\!R^{n\times r}$, there exists $K \in I\!R^{m\times r}$ such that

$$G + MKN^T + NK^T M^T < 0$$

if and only if the following two elimination conditions *hold:*

$$M_\perp G M_\perp^T < 0; \quad N_\perp G N_\perp^T < 0,$$

where $M_\perp$ and $N_\perp$ are any orthogonal complements of M and N, respectively[1].

Lemma 2: Given two full-rank matrices $M \in I\!R^{n\times m}$ and $N \in I\!R^{n\times r}$ with $n \geq m$ and $n \geq r$, there exists a nonsingular matrix $U \in I\!R^{n\times n}$, called transformation matrix *such that*

$$UM = \begin{bmatrix} I_k & 0 \\ 0 & I_{m-k} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} V_1; \quad UN = \begin{bmatrix} I_k & 0 \\ 0 & 0 \\ 0 & I_{r-k} \\ 0 & 0 \end{bmatrix} V_2$$

for some $0 \leq k \leq \min\{m,\ r\}$ and nonsingular matrices $V_1 \in I\!R^{m\times m}$ and $V_2 \in I\!R^{r\times r}$.

Lemma 3: (Robust Elimination Lemma) Given an $n \times n$ matrix function $G(q) = G^T(q)$ which is continuous in q, $q = [q_1\ q_2\ \dots\ q_v]$ belonging to a compact bounding set Q, and two full-rank matrices $M \in I\!R^{n\times m}$ and $N \in I\!R^{n\times r}$ with $m \leq n$ and $r \leq n$, there exists $K \in I\!R^{m\times r}$ such that

$$G(q) + MKN^T + N^T KM < 0, \quad \forall q \in Q$$

if and only if the following two (robust) elimination conditions *hold:*

$$M_\perp G(q) M_\perp^T < 0; \quad N_\perp G(q) N_\perp^T < 0, \quad \forall q \in Q.$$

Furthermore, suppose the condition above is satisfied, a solution to K is given by

$$V_1 K V_2^T = \begin{bmatrix} \tilde{K}_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where V_1 and V_2 are resulted from a transformation matrix U on M and N as specified in Lemma 2.2, and $\tilde{K}_1 \in S_-^{k\times k}$ is such that

$$2\tilde{K}_1 - \tilde{G}_{11}(q)\tilde{G}_{12}(q)\tilde{G}_{22}^{-1}(q)\tilde{G}_{21}(q) < 0, \quad \forall q \in Q$$

with

$$\tilde{G}(q) \doteq \begin{bmatrix} \tilde{G}_{11}(q) & \tilde{G}_{12}(q) \\ \tilde{G}_{21}(q) & \tilde{G}_{22}(q) \end{bmatrix} \doteq UG(q)U^T$$

[1]When M or N is a nonsingular matrix, the corresponding inequality is void. This convention will be adopted throughout the paper.

and k is also as specified in Lemma 2.

Lemma 4: (Completion Lemma [23]) Suppose $X, Y \in S_+^{n\times n}$ and k is a positive integer. Then the following are equivalent:

(i) *There exist $X_2, Y_2 \in I\!R^{n\times k}$ and $X_3, Y_3 \in S_+^{k\times k}$ such that*

$$\begin{bmatrix} X_3 & X_2^T \\ X_2 & X \end{bmatrix} > 0; \quad \begin{bmatrix} X_3 & X_2^T \\ X_2 & X \end{bmatrix}^{-1} = \begin{bmatrix} Y_3 & Y_2^T \\ Y_2 & Y \end{bmatrix};$$

(ii) *We have the following* rank constraint*:*

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0; \quad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + k.$$

Furthermore, if the condition in (ii) *holds, a solution to* (i) *is given by*

$$Y_3 = I_k; \quad Y_2 = -W; \quad X_3 = I_k + W^T X W; \quad X_2 = XW,$$

where $W \in I\!R^{n\times k}$ is a factorization of $Y - X^{-1}$, i.e.,

$$WW^T = Y - X^{-1}.$$

Proof of Theorem 1: Result (i) follows from [3]. Result (ii) holds because it is a dual version of (i). It is easy to see that, by scaling down P or S properly, the rank constraint above is automatically satisfied when $k = n$, and this scaling does not affect the conditions for quadratic observability or state feedback quadratic stabilizability. Therefore, Result (iii) follows from Result (iv) which is what we proceed to prove next.

Denoting

$$\bar{x} = \begin{bmatrix} x \\ z \end{bmatrix}; \quad \bar{K} \doteq \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix};$$

$$\bar{A}(q) \doteq \begin{bmatrix} 0_k & 0 \\ 0 & A(q) \end{bmatrix}; \bar{B} \doteq \begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix}; \bar{C} \doteq \begin{bmatrix} I_k & 0 \\ 0 & C \end{bmatrix},$$

then the closed loop system is given by

$$\dot{\bar{x}}(t) = (\bar{A}(q(t)) + \bar{B}\bar{K}\bar{C}^T)\bar{x}(t).$$

Therefore, the given uncertain system is quadratically stabilizable via a k-th order dynamic output feedback controller if and only if there exists $\bar{K}$ and $\bar{P} = \bar{P}^T > 0$ such that

$$(\bar{A}(q) + \bar{B}\bar{K}\bar{C}^T)^T\bar{P} + \bar{P}(\bar{A}(q) + \bar{B}\bar{K}\bar{C}^T) < 0, \quad \forall q \in Q.$$

According to the Robust Elimination Lemma, the above is equivalent to

$$\bar{B}_\perp(\bar{A}(q)\bar{P}^{-1} + \bar{P}^{-1}\bar{A}^T(q))B_\perp^T < 0, \quad \forall q \in Q,$$

and

$$\bar{C}_\perp(\bar{P}\bar{A}(q) + \bar{A}^T(q)\bar{P})C_\perp^T < 0, \quad \forall q \in Q.$$

Note that

$$\bar{B}_\perp = [0 \quad B_\perp]; \quad \bar{C}_\perp = [0 \quad C_\perp].$$

Denoting

$$P = [0 \quad I_n]\bar{P}\begin{bmatrix} 0 \\ I_n \end{bmatrix}; \quad S = [0 \quad I_n]\bar{P}^{-1}\begin{bmatrix} 0 \\ I_n \end{bmatrix}$$

and simplifying the two inequalities above, we get

$$B_\perp(A(q)S + S^T A^T(q))B_\perp^T < 0;$$
$$C_\perp(A^T(q)P + PA(q))C_\perp^T < 0.$$

Finally, the constraints on P and S are derived using the Completion Lemma. ■

Remark: The Separation Principle above implies that output feedback quadratic stabilizability can be separated into state feedback quadratic stabilizability and quadratic detectability. In fact, it also implies that the controller design can be separated from the construction of the Lyapunov matrix.

Controller Synthesis Procedure: *Suppose the conditions in Result* (iv) *of Theorem 2.5 are satisfied for some $P, S \in S_+^{n\times n}$. Then,*

(i) *A Lyapunov matrix $\bar{P}$ for the closed-loop system is constructed by completing a $(n+k)\times(n+k)$ symmetric and positive definite matrix with*

$$[0 \quad I_n]\bar{P}\begin{bmatrix} 0 \\ I_n \end{bmatrix} = P; \quad [0 \quad I_n]\bar{P}^{-1}\begin{bmatrix} 0 \\ I_n \end{bmatrix} = S,$$

which can always be done, according to the Completion Lemma.

(ii) *Defining*

$$\bar{G}(q) \doteq \bar{P}\begin{bmatrix} 0_k & 0 \\ 0 & A(q) \end{bmatrix}; \quad \bar{M} \doteq \bar{P}\begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix};$$

$$\bar{N} \doteq \begin{bmatrix} I_k & 0 \\ 0 & C \end{bmatrix}; \quad \bar{K} \doteq \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix},$$

then $\bar{K}$ can be found by applying the Robust Elimination Lemma.

IV. Stepwise Augmentation Structure

In this section, we introduce a class of single-input-single-output uncertain linear systems satisfying the so-called *Stepwise Augmentation Structure*. This structure is generated via a series of down augmentations and up augmentations. Using Theorem 1, we will show that the Stepwise Augmentation Structure is quadratically stabilizable via output feedback.

Notation: The following will be adopted in the rest of this paper. The uncertain parameter vector q is allowed to be time-varying but its time dependence is suppressed. The bounding set Q is compact. An asterisk (*) term denotes an arbitrary function of q, and an θ term denotes any sign-invariant function of q. If a system involves multiple asterisk (or θ) terms, it is understood that they are in general different uncertain functions.

Augmented Systems: For the construction to follow, we begin with an n-th order uncertain system

$$\begin{array}{rcl} \dot{x} & = & A(q)x + bu \\ y & = & c^T x \end{array}$$

and an auxiliary vector $v \in \mathbb{R}^{n\times 1}$. We require $b = [0\ 0\ \ldots\ 0\ 1]^T$ and decompose

$$A(q) = \left[\begin{array}{cc} \tilde{A}(q) & \tilde{b}(q) \\ \tilde{c}^T(q) & \tilde{d}(q) \end{array}\right].$$

We call $\Sigma = (A(q), b, c, v)$ a *generating system.*

Given a generating system $\Sigma = (A(q), b, c, v)$ as above, the system $\Sigma^+ \doteq (A^+(q), b^+, c^+, v^+)$ is said to be a *down augmentation* of Σ if

$$A^+(q) = \left[\begin{array}{ccc} \tilde{A}(q) & \tilde{b}(q) & 0 \\ [* & *]v_\perp^T & \theta \\ \tilde{c}^T(q) & \tilde{d}(q) & * \end{array}\right]; \quad b^+ = \left[\begin{array}{c} 0 \\ b \end{array}\right];$$

$$c^+ = [c\ \ 0]^T; \quad v^+ = [v\ \ 0]^T.$$

Similarly, given a generating system $\Sigma = (A(q), b, c, v)$ above, $\Sigma^+ \doteq (A^+(q), b^+, c^+, v^+)$ is called an *up augmentation* of Σ if

$$A^+(q) = \left[\begin{array}{ccc} 0 & \tilde{c}^T(q) & \tilde{d}(q) \\ 0 & \tilde{A}(q) & \tilde{b}(q) \\ \theta & * & * \end{array}\right]; \quad b^+ = \left[\begin{array}{c} 0 \\ b \end{array}\right];$$

$$c^+ = [0\ \ c]^T; \quad v^+ = [1\ 0\ \ldots\ 0]^T$$

Note that the auxiliary vector v is used only in the augmentation process and is redundant in the description of the augmented system. For this reason, we will often suppress v in the augmented system. When considering state feedback design or quadratic detection, we may also suppress c or b as well.

Stepwise Augmentation Structure: An uncertain system $\Sigma = (A(q), b, c)$ is said to admit a *Stepwise Augmentation Structure* if it is obtained via a sequence of up and down augmentations from a generating system $\Sigma_0 \doteq (A_0(q), b_0, c_0, v_0)$ of the form

$$A_0(q) \doteq \left[\begin{array}{cc} \tilde{A}_0(q) & \tilde{b}_0(q) \\ \tilde{c}_0^T(q) & \tilde{d}_0(q) \end{array}\right]; \quad b_0 \doteq \left[\begin{array}{c} 0 \\ 1 \end{array}\right]; \quad c_0 \doteq \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

where $\tilde{A}_0(q)$ is quadratically stable, and if no up augmentations are involved, $v_0 = 0$ and the remaining entries of $A_0(q)$ are arbitrary; otherwise, v_0 is such that $v_0^T v_0 = 1$ and

$$A_0(q)v_0 = \left[\begin{array}{c} 0 \\ \theta \end{array}\right].$$

Remark: The matrix $\tilde{A}_0(q)$ is used to represent possible stable zero dynamics in the system Σ. We note that the requirement on the existence of v_0 is a mild one. In fact, for a given $\tilde{A}_0(q)$, the condition on v_0 amounts to some constraints on the other entries of $A_0(q)$. Indeed, given any $\tilde{A}_0(q) \in \mathbb{R}^{n_0\times n_0}$ and any nonzero $\tilde{v}_0 \in \mathbb{R}^{n_0\times 1}$, we can simply choose

$$v_0 \doteq \frac{1}{\sqrt{\tilde{v}_0^T\tilde{v}_0^T + 1}}\left[\begin{array}{c} \tilde{v}_0 \\ 1 \end{array}\right]; \quad \tilde{b}_0(q) \doteq -\tilde{A}_0(q)\tilde{v}_0;$$

$$[\tilde{c}_0^T(q)\ \tilde{d}_0(q)] \doteq \theta v_0^T + [*](v_0)_\perp$$

to satisfy the required conditions on $A_0(q)$, where $[*]$ is .

Examples: To illustrate the stepwise augmentation structure, we list four possible structures for $A(q)$ associated with 4-th order systems

$$\left[\begin{array}{cccc} * & \theta & 0 & 0 \\ \hline * & * & \theta & 0 \\ * & * & * & \theta \\ * & * & * & * \end{array}\right]; \quad \left[\begin{array}{cccc} 0 & \theta & * & * \\ 0 & 0 & \theta & * \\ 0 & 0 & 0 & \theta \\ \hline \theta & * & * & * \end{array}\right];$$

$$\left[\begin{array}{cccc} 0 & \theta & * & 0 \\ 0 & 0 & \theta & 0 \\ \hline 0 & * & * & \theta \\ \theta & * & * & * \end{array}\right]; \quad \left[\begin{array}{cccc} 0 & \theta & * & * \\ 0 & 0 & \theta & 0 \\ \hline 0 & 0 & * & \theta \\ \theta & * & * & * \end{array}\right],$$

where the underlined row corresponds to the state variable where augmentations originate. The sequences for augmentations for the structures above are: down-down-down, down-up-up, down-up-down and down-down-up. The examples above are all void of zero dynamics. But they can be easily generalized to include them.

We have the second main result of this paper as follows:

Theorem 2: An n-th order Stepwise Augmentation Structure is quadratically stabilizable via an $(n - n_0 - 1)$-th order output feedback controller, where n_0 is the order of the quadratically stable uncertain matrix $\tilde{A}_0(q)$.

The rest of this paper is devoted to the proof of Theorem 2. But due to the page limit, we only show that an n-th order quadratically stabilizing controller exists.

V. State Feedback Quadratic Stabilization

In this section, we study the problem of state feedback quadratic stabilization.

Lemma 5: If $\tilde{A}_0(q)$ is quadratically stable with $\tilde{S}_0$ being its associated inverse Lyapunov matrix, then the uncertain structure

$$\Sigma_0 \doteq \left(\left[\begin{array}{cc} \tilde{A}_0(q) & \tilde{b}_0(q) \\ \tilde{c}_0(q) & \tilde{d}_0(q) \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right)$$

with any $\tilde{b}_0(q), \tilde{c}_0(q)$ and $\tilde{d}_0(q)$ is quadratically stabilizable via state feedback and its associated inverse Lyapunov matrix can take the form

$$S_0 = \left[\begin{array}{cc} \tilde{S}_0 & 0 \\ 0 & s_1 \end{array}\right]$$

for any $s_1 > 0$.

Theorem 3: Suppose an uncertain system $\Sigma \doteq (A(q), b, c, v)$ with $b = [0\ 0\ \ldots\ 0\ 1]^T$ is quadratically stabilizable via state feedback with an associated inverse Lyapunov matrix S. Then, any down augmentation $\Sigma^+ \doteq (A^+(q), b^+, c^+, v^+)$ of Σ is quadratically stabilizable via state feedback and its associated inverse Lyapunov matrix can be chosen as

$$S^+ = \begin{bmatrix} S & -\gamma b \\ -\gamma b^T & s_1 + \gamma^2 b^T S^{-1} b \end{bmatrix}$$

for any $s_1 > 0$ and some γ with a suitably large magnitude and an appropriate sign.

Lemma 6: Suppose a Stepwise Augmentation Structure $\Sigma = (A(q), b, c)$ contains at least one up augmentation. Then, the property of $A_0(q)v_0 = [0\ \ \theta]^T$ is preserved after each down or up augmentation.

Theorem 4: given an uncertain structure $\Sigma \doteq (A(q), b, c, v)$ with $b = [0\ 0\ \ldots\ 0\ 1]^T$ and

$$b_\perp A(q) v = 0, \quad \forall q \in Q,$$

suppose it is quadratically stabilizable via state feedback with an associated inverse Lyapunov matrix S. Then, any up augmentation $\Sigma^+ \doteq (A^+(q), b^+)$ of Σ is quadratically stabilizable via state feedback and its associated inverse Lyapunov matrix can be chosen as

$$S^+ = \begin{bmatrix} s_1 + \gamma^2 v^T S^{-1} v & -\gamma v \\ -\gamma v^T & S \end{bmatrix}$$

for any $s_1 > 0$ and some γ with suitably large magnitude and appropriate sign.

VI. QUADRATIC DETECTABILITY

In this section, we study the problem of quadratic detection. We start by introducing the notions of *dual augmentations* and showing that by reordering the state variables, the Stepwise Augmentation Structure can be generated via a sequence of dual augmentations. We then develop some basic results for quadratic detectability of the Stepwise Augmentation Structure by using these dual augmentations.

Dual Augmentations: Given a generating system $\Sigma = (A(q), c)$ with $c = [1\ 0\ \ldots\ 0]^T$, $\Sigma^+ \doteq (A^+(q), c^+)$ with $c^+ = [c^T\ 0]^T$ is called an *dual down augmentation*

$$A^+(q) = \left[\begin{array}{c|c} A(q) & \begin{bmatrix} * \\ \vdots \\ * \\ \theta \end{bmatrix} \\ \hline [*\ 0\ \ldots\ 0] & 0 \end{array}\right]$$

or *dual up augmentation* if

$$A^+(q) = \left[\begin{array}{c|c} * & [\theta\ 0 \ldots\ 0] \\ \hline \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} & A(q) \end{array}\right].$$

Note that for a dual down augmentation, the last row of $A^+(q)(c^+)_\perp^T$ is all zero.

Theorem 5: Given an n-th order Stepwise Augmentation Structure $\Sigma = (A(q), b, c)$ with the generating system $\Sigma_0 = (A_0(q), b_0, c_0, v_0)$ as specified in Section 3.3, let the sequence of up and down augmentations be s and $x_1, x_2, \cdots, x_k$ be the up-augmented state variables. Then the transformed structure

$$\bar{\Sigma} \doteq (\bar{A}(q), \bar{c}) \doteq (TA(q)T^{-1}, Tc)$$

with

$$T = \begin{bmatrix} 0 & I_{n-k} \\ I_k & 0 \end{bmatrix}$$

has the decomposition

$$\bar{A}(q) = \begin{bmatrix} \tilde{A}_0(q) & \tilde{b}(q)\bar{c}_1^T \\ * & \bar{A}_1(q) \end{bmatrix}; \ \bar{c} = \begin{bmatrix} 0 \\ \bar{c}_1 \end{bmatrix}; \ \ \bar{c}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and that $\bar{\Sigma}_1 \doteq (\bar{A}_1(q), \bar{c}_1)$ is obtained from

$$\begin{aligned} \dot{x}_n &= *x_n \\ y &= x_n \end{aligned}$$

via a sequence of dual augmentations with the sequence which is the reverse of s.

The following three results study the quadratic detectability of $\bar{\Sigma}$.

Theorem 6: A Stepwise Augmentation Structure Σ is quadratically detectable if and only if the subsystem $\bar{\Sigma}_1$ as specified in Theorem 5 is quadratically detectable. Furthermore, suppose $\tilde{P}_0$ is a Lyapunov matrix associated with the quadratic stability of $\tilde{A}_0(q)$ and $\bar{P}_1$ is a Lyapunov matrix associated with the quadratic detectability of $\bar{\Sigma}_1$. Then,

$$P \doteq T^{-1}\bar{P}T$$

with

$$\bar{P} \doteq \begin{bmatrix} \tilde{P}_0 & 0 \\ 0 & \rho\bar{P}_1 \end{bmatrix}$$

is a Lyapunov matrix associated with the quadratic detectability of Σ for some suitably small $\rho > 0$.

Theorem 7: Suppose an uncertain structure $\Sigma \doteq (A(q), c)$ with $c = [1\ 0\ 0\ \ldots\ 0]^T$ is quadratically detectable with an associated Lyapunov matrix P. Then, *any dual up augmentation $\Sigma^+ \doteq (A^+(q), c^+)$ of Σ is quadratically detectable and its associated Lyapunov matrix can be chosen as*

$$P^+ = \begin{bmatrix} p_1 + \gamma^2 c^T P^{-1} c & -\gamma c^T \\ -\gamma c & P \end{bmatrix}$$

for any $p_1 > 0$ and some suitably large scalar γ with an appropriate sign.

Theorem 8: Given an uncertain structure $\Sigma \doteq (A(q), c)$ with $c = [1\ 0\ 0\ \ldots\ 0]^T$ and the constraint that the last row of $A(q)c_\perp^T$ is all zero, suppose it is quadratically detectable with Lyapunov matrix P. Then, any dual down augmentation $\Sigma^+ \doteq (A^+(q), c^+)$ of Σ is quadratically detectable and its associated Lyapunov matrix can be chosen as

$$P^+ = \begin{bmatrix} P & -\gamma b \\ -\gamma b^T & p_1 + \gamma^2 b^T P^{-1} b \end{bmatrix}; \quad b = [0\ 0\ \ldots\ 0\ 1]^T$$

for any $p_1 > 0$ and some suitable γ.

VII. Proof of Theorem 2

Proof: Using Theorem 1, we only need to establish that any Stepwise Augmentation Structure $\Sigma = (A(q), b, c)$ generated from $\Sigma_0 \doteq (A_0(q), b_0, c_0, v_0)$ is both quadratically stabilizable via state feedback and quadratically detectable. To show the former, we note from Lemma 5 that $\Sigma_0 \doteq (A_0(q), b_0, c_0, v_0)$ is quadratically stabilizable via state feedback. From Theorem 3, the quadratic stabilizability property is preserved under any down augmentation. Similarly, Lemma 6 and Theorem 4 imply that this property is also preserved under any up augmentation. Hence, $\Sigma = (A(q), b, c)$ is quadratically stabilizable via state feedback. Next, Theorem 5 implies that $\Sigma = (A(q), b, c)$ is quadratically detectable if and only if the transformed system $\bar{\Sigma}$ (see Theorem 5), which is obtained using a sequence of dual augmentations, is quadratically detectable. The quadratic detectability of the Σ_1 part of $\bar{\Sigma}$ follows from Theorems 7-8. Subsequently, the quadratic detectability of $\bar{\Sigma}$ follows from Theorem 6. ■

VIII. Conclusion

We have presented a new class of uncertain linear systems which admit the so-called Stepwise Augmentation Structure. Using a new Separation Principle for uncertain linear systems, we have shown that this class of uncertain systems can be quadratically stabilized via a dynamic output feedback controller. The result also leads to a recursive design method for constructing such a controller.

The new Separation Principle introduced in this paper is the key to our design approach. We believe that the use of such a result is the first attempt of systematic designs for output feedback control for uncertain linear systems. It would be very interesting to see whether this Separation Principle can be applied to other robust control problems. We also hope that, with an appropriate generalization, this Separation Principle would lead to useful tools for output feedback control of nonlinear systems, which is an important problem made challenging also because of the lack of a suitable Separation Principle.

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