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Finite-Level Quantized Feedback Control for Linear Systems

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Abstract—Recent studies have shown that a logarithmic quantizer can provide the coarsest quantization for quadratic stabilization of linear systems using quantized feedback. However, the coarsest quantizer has an infinite number of quantization levels, which is not implementable in practice. In this paper, we investigate the quantized feedback control problem for discrete-time linear systems using a finite-level logarithmic quantizer. We introduce a dynamic scaling method for the logarithmic quantizer and show that asymptotic stabilization can be achieved with a moderate number of quantization levels. Our approach is easily implementable. We also study the quantized feedback stabilization problem for systems with bounded stochastic noise inputs and show that the system state can converge to a bounded region using a finite-level logarithmic quantizer in conjunction with a proper dynamic scaling scheme.

I. INTRODUCTION

There has been a lot of resurgent interest in quantized feedback control recently. Unlike previous research where the interest was on the digital implementation of control systems, the recent interest focuses on the use of digital communication channels for feedback control. In this setting, the feedback signal is quantized and then coded for transmission. From the control design point of view, a fundamental problem is how to design a feedback controller and a quantizer jointly in order to achieve a given control objective.

The research on quantized feedback control can be categorized depending on whether the quantizer is static or dynamic. A *static quantizer* is a memoryless nonlinear function, whereas a *dynamic quantizer* uses memory and thus can be much more complex and potentially more powerful. Existing work using static quantizers includes, e.g., [1], [2], [3], [14]. For quadratic stabilization of a linear system using state feedback, it is shown in [1] that the optimal static quantizer is a *logarithmic quantizer*. This result is generalized in [3] to a number of output feedback problems using a sector bound approach, where logarithmic quantizers are also shown to be optimal.

When a dynamic quantizer is allowed, it is shown in [4] that stabilization of a SISO LTI system (in some stochastic sense) can be achieved using only a finite number of quantization levels, and the minimum number of quantization levels (also known as the minimum *feedback information rate*) is explicitly related to the unstable poles of the system. In this setting, the dynamic quantizer effectively consists of two

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parts: an *encoder* at the output end and a *decoder* at the input end.

Another type of dynamic quantizers uses *dynamic scaling* in conjunction with a static quantizer. That is, the input signal is pre-scaled so that its range is more suitable for quantization. The scaling parameter is dynamically adjusted (i.e., adjusted online). Noticeable work along this line includes [9]-[12]. In [9], it is pointed out that if a system is not excessively unstable, by employing a quantizer with various sensitivity a feedback strategy can be designed to bring the closed-loop state arbitrarily close to zero for an arbitrarily long time. The idea of quantizer with sensitivity is extended in [10] where it is shown that there exists a dynamic adjustment of the quantizer sensitivity and a quantized state feedback that asymptotically stabilizes the system. In the case of output feedback, a local (or semi-global) stabilization result is obtained. We note that dynamic scaling is a very popular idea in signal processing for reducing quantization errors [13] but the key difference here is that we also need to guarantee stability in control.

This paper is primarily inspired by the work of [4] but also motivated by its limitations. Although it is shown in [4] that stabilization of a linear system can be achieved by feeding back only a finite number of bits per sample and this number is typically very small, the encoder-decoder scheme used for proving this result is impractical and non-robust. It is impractical because the encoder-decoder pair is a very nonlinear operator which would typically result in a large overshoot, and it can be non-robust because even a very small amount of noise in the system can drive the closed-loop unstable. These problems are not present in a logarithmic quantizer based feedback controller because the quantization error can be modeled as a sector bounded uncertainty and the closed-loop system is essentially linear. The main drawback of the logarithmic quantizer is that it requires an infinite number of quantization levels.

In this paper, we propose a simple dynamic scaling method for a logarithmic quantizer based feedback controller. A dynamic *scaling factor* is simply adjusted up or down depending whether the input signal to the quantizer is "too small" or "too large". Using this dynamic scaling method, we show that a linear system can be asymptotically stabilized using a logarithmic quantizer with only a finite number of quantization levels. This number turns out to be very moderate (typically a few bits to a few bytes) and is usually very compatible to the minimum information rate given in [4]. The main advantage of the proposed scheme is that the system behaves as if there were an infinite number of logarithmic

quantization levels when the initial state is "moderate" in size, i.e., the state would converge exponentially. Only when the initial state is very large, a transient period of overshoot can be present. The region of exponential convergence can be easily increased by using more quantization levels, and the number of feedback information bits grows only at a $\log(\log(\cdot))$ rate when the size of this region increases. Since most digital communication channels can easily handle a few tens of bytes per sample, the proposed scheme should be very practical. We also show that the proposed scheme is robust in the sense it can tolerate additive noises in the system effectively.

II. INFINITE-LEVEL LOGARITHMIC QUANTIZATION

Consider the following system:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k, & (1) \\
y_k &= Cx_k, & (2)
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}$ is the control input, $y_k \in \mathbb{R}$ is the measured output, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$ are given. We will denote the transfer function from u_k to y_k by $G(z)$. Without loss of generality, we assume that A is unstable and (A, B, C) is a minimal realization.

The quantized feedback control problem is to design a feedback quantizer

$$
v_k = Q(y_k), \tag{3}
$$

which takes values in

$$
\mathcal{V} = \{\pm \mu_i : i = 0, \pm 1, \pm 2, \cdots\} \cup \{0\}, \tag{4}
$$

and a feedback controller of the form

$$
\begin{aligned}
\hat{x}_{k+1} &= A_c \hat{x}_k + B_c v_k, & \hat{x}_0 = 0, & (5) \\
u_k &= C_c \hat{x}_k + D_c v_k, & & (6)
\end{aligned}
$$

with $\hat{x}_k \in \mathbb{R}^n$, such that the closed-loop system is stable and that the so-called quantization density [1] is coarsest. The quantization density of $Q(\cdot)$ is defined as follows:

$$
\eta_Q = \limsup_{\epsilon \to 0} \frac{\#g[\epsilon]}{-\ln \epsilon}, \tag{7}
$$

where $\#g[\epsilon]$ denotes the number of quantization levels in the interval $[\epsilon, \ 1/\epsilon]$.

The quantized feedback control problem for the system (1)-(2) is generalized from a quantized state feedback control problem in [1] and has been studied in details in [3]. In particular, it is known [3] that the coarsest quantization density for quadratic stabilization of the system above is achieved by a logarithmic quantizer. Such a quantizer is described by

$$
\mathcal{V} = \{\mu_i = \rho^i \mu_0 : \ i = 0, \pm 1, \pm 2, \cdots\} \cup \{0\}, \ \mu_0 > 0, \tag{8}
$$

where $\rho \in (0, \ 1)$. Since a smaller ρ corresponds to a smaller η_Q, we can regard ρ as the quantization density instead. The

associated quantizer $Q(\cdot)$ is defined as follows:

$$
Q(y) = \begin{cases} \rho^i \mu_0, & \text{if } \frac{1}{1+\delta}\rho^i \mu_0 < y \le \frac{1}{1-\delta}\rho^i \mu_0 \\ 0, & \text{if } y = 0 \\ -Q(-y), & \text{if } y < 0, \end{cases} \tag{9}
$$

where

$$
\delta = (1 - \rho)(1 + \rho)^{-1}. \tag{10}
$$

It is further shown in [3] that the smallest ρ for which the system (1)-(2) can be quadratically stabilized via a quantized feedback controller (3), (5)-(6) is given by

$$
\begin{aligned}
\rho_{\inf} &= \frac{1 - \delta_{\sup}}{1 + \delta_{\sup}}, & (11) \\
\delta_{\sup}^{-1} &= \inf_{H(z)} ||(1 - H(z)G(z))^{-1}H(z)G(z)||_\infty, & (12)
\end{aligned}
$$

where $H(z)$ is the transfer function of the controller.

III. FINITE-LEVEL QUANTIZED FEEDBACK STABILIZATION

It is obvious that a logarithmic quantizer (9) has an infinite number of quantization levels. This is certainly not implementable practically. One simple approach is to truncate the quantizer using a large saturator and a small dead zone. This will allow the state of the system to converge to a small neighborhood, provided that the initial state is within a known bound. Due to the use of logarithmic quantization, the number of quantization levels required is far less than required by using linear quantization.

In this section, we show that it is possible to dynamically scale the input-output signals of the quantizer so that asymptotic stabilization can be achieved using a finite-level logarithmic quantizer, even without knowing the bound for the initial state. We define an N-level logarithmic quantization with quantization density $\rho > \rho_{\inf}$ as

$$
\mathcal{V} = \{\pm \rho^i \mu_0, i = 0, 1, 2, \cdots, N - 1\}, \ \mu_0 > 0. \tag{13}
$$

The associated quantizer $Q(\cdot)$ becomes:

$$
Q(y) = \begin{cases} \rho^i \mu_0, & \text{if } \frac{1}{1+\delta}\rho^i \mu_0 < y \le \frac{1}{1-\delta}\rho^i \mu_0, \\ & 0 \le i < N \\ \rho^{N-1} \mu_0, & \text{if } 0 \le y \le \frac{1}{1+\delta}\rho^{N-1} \mu_0, \\ \mu_0, & \text{if } y > \frac{1}{1-\delta}\rho^i \mu_0, \\ -Q(-y), & \text{if } y < 0. \end{cases} \tag{14}
$$

The basic idea of dynamic scaling is very simple: When the signal y_k is outside of the quantization range, we scale it back by a *scaling factor* (or *gain*) $g_k > 0$ before quantization. The quantized signal is then scaled back by g_k^{-1}. That is, we use

$$
v_k = g_k^{-1} Q(g_k y_k). \tag{15}
$$

Suppose an infinite-level logarithmic quantizer with density $\rho > \rho_{\inf}$ is adopted. Following the sector bound approach [3], we can write the closed-loop system of (1)-(2) and (5)-(6) as

$$
\begin{aligned}
\xi_{k+1} &= \bar{A}(\Delta_k)\xi_k, & (16) \\
y_k &= \bar{C}\xi_k, & (17)
\end{aligned}
$$

where $\xi = [x^T \ \hat{x}^T]^T$,

$$\bar{A}(\Delta_k) = \bar{A} + \bar{B}(1 + \Delta_k)\bar{C}, \tag{18}$$

$$\bar{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}, \quad \bar{C} = [C \ \ 0].$$

Also,

$$\Delta_k y_k = Q(y_k) - y_k, \quad |\Delta_k| \leq \delta, \tag{19}$$

represents the quantization error. Because (16) is quadratically stable, we have a quadratic Lyapunov function $V(\xi) = \xi^T P \xi$ with $P = P^T > 0$ such that

$$V(\xi_{k+1}) - V(\xi_k) = \xi_k^T (\bar{A}(\Delta_k)^T P \bar{A}(\Delta_k) - P)\xi_k < 0 \tag{20}$$

for all nonzero $\xi_k \in \mathbb{R}^{2n}$ and admissible Δ_k. It is shown in [3] that the above is equivalent to

$$\bar{A}(\Delta)^T P \bar{A}(\Delta) - P < 0, \quad \forall \ |\Delta| \leq \delta. \tag{21}$$

Using the continuity argument, the above is equivalent to

$$\bar{A}(\Delta)^T P \bar{A}(\Delta) - P \leq -\eta P, \quad \forall \ |\Delta| \leq \delta, \tag{22}$$

for some $0 < \eta < 1$.

We now assume that an N-level logarithmic quantizer with the same density ρ and dynamic scaling (15) is applied instead. We choose two positive scaling factors $0 < \gamma_1, \gamma_2 < 1$ such that

$$\gamma_1^2 \bar{A}^T P \bar{A} - P < -\eta_1 P, \tag{23}$$

$$\gamma_2^{-2}(1 + \tau)\bar{A}(\Delta)^T P \bar{A}(\Delta) - P < -\eta_2 P, \quad \forall \|\Delta\| \leq \delta, \tag{24}$$

for some $0 < \eta_1, \eta_2, \tau < 1$. The latter is done by choosing γ_2 close to 1 and τ close to 0 so that $\gamma_2^{-2}(1+\tau)(1-\eta) < 1$ and taking

$$\eta_2 = 1 - \gamma_2^{-2}(1 + \tau)(1 - \eta). \tag{25}$$

This ensures $\eta_2 > 0$ and makes (24) equivalent to (22).

We initialize g_0 to be any positive value and define g_{k+1} for any $k \geq 0$ as follows:

$$g_{k+1} = \begin{cases} g_k \gamma_1, & \text{if } |Q(g_k y_k)| = \mu_0, \\ g_k/\gamma_2, & \text{if } |Q(g_k y_k)| = \rho^{N-1}\mu_0, \\ g_k, & \text{otherwise.} \end{cases} \tag{26}$$

Because of the flexibility in g_0, we can normalize $\mu_0 = 1$ without loss of generality. We will also denote $\bar{\varepsilon} = \rho^{N-1}$. The choice of g_0 does not affect stabilizability, but choosing it according to an estimate of $\|x_0\|$ helps improve the transient performance.

To help analyze the quantized feedback system, we consider the scaled state defined by

$$z_k = g_k \xi_k \tag{27}$$

and the associated Lyapunov function $V(z) = z^T P z$. We have the following result:

Lemma 3.1: Suppose the scaled N-level logarithmic quantizer (14), (15) and (26) is applied. Then, for any initial state x_0 and any $k \geq 0$,

$$V(z_{k+1}) - V(z_k)$$
$$\leq \begin{cases} -\eta_3 V(z_k), & \text{if } |Q(\bar{C}z_k)| = 1, \\ -\eta V(z_k), & \text{if } \bar{\varepsilon} < |Q(\bar{C}z_k)| < 1, \\ -\eta_2 V(z_k) + \eta_4 \bar{\varepsilon}^2, & \text{if } |Q(\bar{C}z_k)| = \bar{\varepsilon}, \end{cases} \tag{28}$$

where

$$\eta_3 = \max\{\eta_1, \ 1 - \gamma_1^2(1 - \eta)\} > 0,$$
$$\eta_4 = \gamma_2^{-2}(1 + \tau^{-1})\bar{B}^T P \bar{B}. \tag{29}$$

Proof: The result for the case of $\rho^{N-1} < |Q(\bar{C}z_k)| < 1$ follows directly from (16), (22) and $g_{k+1} = g_k$. For the case of $|Q(\bar{C}z_k)| = 1$, $g_{k+1} = g_k \gamma_1$. It follows that

$$V(z_{k+1}) - V(z_k) = \gamma_1^2 (\bar{A}z_k + \bar{B}\sigma_k)^T P(\bar{A}z_k + \bar{B}\sigma_k) - z_k^T P z_k,$$

where $\sigma_k = \text{sign}(\bar{C}z_k)$. Denote

$$f(u) = \gamma_1^2 (\bar{A}z_k + \bar{B}u)^T P(\bar{A}z_k + \bar{B}u) - z_k^T P z_k.$$

From (23),

$$f(0) \leq -\eta_1 z_k^T P z_k.$$

Since $\sigma_k = Q(\bar{C}z_k)$, we have $\sigma_k = \theta u_1$ for some $0 < \theta \leq 1$, where $u_1 = (1 + \Delta_k)\bar{C}z_k$ is the unsaturated output of the quantizer. Also from (22), we get

$$f(u_1) = \gamma_1^2 z_k^T \bar{A}(\Delta_k)^T P \bar{A}(\Delta_k)z_k - z_k^T P z_k$$
$$\leq -(1 - \gamma_1^2(1 - \eta))V(z_k).$$

Since $f(u)$ is quadratic and convex (because $f(u) \to +\infty$ when $|u| \to \infty$), it is clear that

$$V(z_{k+1}) - V(z_k) = f(\sigma_k) \leq \max\{f(0), \ f(u_1)\}$$
$$= -\eta_3 V(z_k).$$

For the case of $|Q(\bar{C}z_k)| = \bar{\varepsilon}$, $g_{k+1} = g_k/\gamma_2$. From (13) and (16), we can write

$$x_{k+1} = \bar{A}(\Delta_k)x_k + \bar{B}g_k^{-1}\varepsilon_k,$$

where $|\varepsilon_k| \leq \bar{\varepsilon}$. It follows that

$$V(z_{k+1}) - V(z_k)$$
$$= \gamma_2^{-2}(\bar{A}(\Delta_k)z_k + \bar{B}\varepsilon_k)^T P(\bar{A}(\Delta_k)z_k + \bar{B}\varepsilon_k)$$
$$\quad - z_k^T P z_k$$
$$= \gamma_2^{-2} z_k^T \bar{A}(\Delta_k)P\bar{A}(\Delta_k)z_k - z_k^T P z_k$$
$$\quad + \gamma_2^{-2}(2\varepsilon_k \bar{B}^T P \bar{A}(\Delta_k)z_k + \varepsilon_k^2 \bar{B}^T P \bar{B})$$
$$\leq \gamma_2^{-2}(1 + \tau)z_k^T \bar{A}(\Delta_k)P\bar{A}(\Delta_k)z_k - z_k^T P z_k$$
$$\quad + \gamma_2^{-2}(1 + \tau^{-1})\bar{\varepsilon}^2 \bar{B}^T P \bar{B}$$
$$= -\eta_2 z_k^T P z_k + \eta_4 \bar{\varepsilon}^2.$$

This completes the proof. $\square$

From Lemma 3.1, it is clear that $V(z_k)$ converges to a bounded region. This bound can be computed by solving

$$0 = -\eta_2 V_\infty + \eta_4 \bar{\varepsilon}^2,$$

which gives

$$V_\infty = \eta_2^{-1} \eta_4 \bar{\varepsilon}^2. \tag{30}$$

Lemma 3.1 leads to the following result:

Corollary 3.1: Suppose the scaled N-level logarithmic quantizer (14), (15) and (26) is applied. Then, for any initial state x_0, $z_k = g_k \xi_k$ converges exponentially to the ellipsoid

$$Z_\infty = \{z : \ z \in \mathbb{R}^{2n}, V(z) \le V_\infty\}. \tag{31}$$

From (30) and the corollary above, it is clear that we can choose N to be sufficiently large so that, when k is sufficiently large, $Q(\bar{C}z_k)$ will no longer be saturated. This is achieved by choosing N such that

$$|\bar{C}z| < 1 \quad \forall z^T P z \le \eta_2^{-1} \eta_4 \rho^{2(N-1)}.$$

Solving this gives $N \ge N_0$, where

$$N_0 = 1 + \frac{\log(\eta_2^{-1}\gamma_2^{-2}(1+\tau^{-1})\bar{B}^T P \bar{B} \bar{C} P^{-1} \bar{C}^T)}{2\log(\rho^{-1})}. \tag{32}$$

The analysis above yields the following main result:

Theorem 3.1: Suppose the scaled N-level logarithmic quantizer (14), (15) and (26) is applied with $N \ge N_0$ in (32). Then, the state ξ_k converges to zero asymptotically.

Proof: From Corollary 3.1, z_k converges to Z_∞ exponentially. This property and the choice of N_0 imply that $Q(\bar{C}z_k)$ will no longer be saturated after a finite number of steps, say k_0 steps. This means that g_k will be non-decreasing for $k \ge k_0$. Note that whenever $g_{k+1} = g_k$, $V(z_k)$ decreases exponentially. If this continues for enough number of steps, $|\bar{C}z_k|$ be less than $\bar{\varepsilon}$, forcing g_{k+1} to increase by factor of $1/\gamma_2$. This means that g_k cannot converge to a constant. Hence, $g_k \to \infty$ as $k \to \infty$. Since z_k is bounded for $k > k_0$, we conclude that $\xi_k \to 0$ as $k \to \infty$. $\square$

Remark 3.1: A typical behavior of the system is as follows: If the initial state is very large, the feedback signal tends to be saturated, forcing g_k to decrease fast. This would result in a period of overshoot. Once g_k is sufficiently small, saturation will stop and the state decays exponentially. When the state is sufficiently small, g_k will increase gradually, causing the quantizer to bounce back and forth between the dead zone and logarithmic region. During this phase, the state also decays exponentially, but at a lower rate.

Remark 3.2: We note that the number of bits required to code an N-level logarithmic quantizer is

$$N_b = \lceil \log_2(2N) \rceil, \quad N \ge \lceil N_0 \rceil, \tag{33}$$

where $\lceil \cdot \rceil$ is the integer round-up function. This is because there are N levels for both positive and negative inputs.

We can also relate N_b to the region of exponential convergence. Since the state x_k converges exponentially inside V_∞ and N is a logarithmic function of V_∞, we know that *the number of required information bits grows at a $\log(\log(\cdot))$ rate as the region of exponential convergence for the state increases.*

IV. NUMBER OF QUANTIZATION LEVELS

In this section, we try to analyze the number of quantization levels needed for stabilization. Recall that for a given controller (5)-(6) and an infinite-level logarithmic quantizer with density $\rho > \rho_{\inf}$ that quadratically stabilizes the system (1)-(2), this number is bounded by N_0 in (32). This formula is complicated because N_0 depends on a number of design parameters ($\eta_2, \gamma_2, \tau, \rho$ and P) which are interconnected.

By minimizing N_0 with respect to τ and η_2, we obtain (without showing the details)

$$\eta_2 = 1 - \gamma_2^{-1}\sqrt{1-\eta}, \quad \tau = \frac{\gamma_2}{\sqrt{1-\eta}} - 1, \tag{34}$$

which yields

$$N_0 = 1 + \frac{2\log(\gamma_2 - \sqrt{1-\eta}) - \log(\bar{B}^T P \bar{B} \bar{C} P^{-1} \bar{C}^T)}{2\log(\rho)}. \tag{35}$$

Since $\gamma_2 < 1$ is required, N_0 is minimized by taking γ_2 very close to 1. Although this minimizes N_0, it will make g_k increase very slowly which in turn make ξ_k converge to 0 very slowly. A good choice for γ_2 should balance the convergence rate of ξ_k and the number of quantization levels; see Example 1 in Section VI. Once γ_2 is chosen, the parameters η and δ need to be optimized jointly. We propose the following simple numerical search method for optimizing N_0: Start with any $0 < \delta < \delta_{\sup}$ and $0 < \eta < 1$. Find a controller to satisfy the control objectives. If a satisfactory controller is found, compute the corresponding matrix P and then N_0. Repeat the above for different values of δ and η and retain the smallest value of N_0.

V. ROBUSTNESS AGAINST ADDITIVE NOISES

Next, we consider the scenario where the system (1)-(2) is subject to some bounded additive noise, i.e., we consider the following system instead:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{36}$$
$$y_k = Cx_k, \tag{37}$$

where $||w_k|| \le \bar{w}$ for some constant $\bar{w} > 0$.

In this case, it is obviously not feasible to drive the state x_k to zero. Instead, we want to drive the state to a bounded region. Because of this, there is no need to drive g_k to infinity. It turns out that by saturating g_k at some upper bound, we can guarantee that the state is driven to a bounded region. This is detailed in the next result (without proof).

Theorem 5.1: Consider the system (36)-(37) and the dynamically scaled logarithmic controller as given before. We modify the scaling factor g_k by saturating it at some $\bar{g} > 1$. Then, both the closed-loop system state ξ_k and the scaled state $z_k = g_k \xi_k$ are bounded when $k \to \infty$.

VI. ILLUSTRATIVE EXAMPLES

In this section, we use several examples to illustrate the proposed dynamic scaling method.

Example 1: We consider a first order system:

$$x_{k+1} = ax_k + u_k, \tag{38}$$
$$y_k = x_k, \tag{39}$$

where $a > 1$. It turns out that we can have a relatively simple expression for N_0. Indeed, to stabilize the system using a logarithmic quantizer (9) with density ρ, the controller $H(z) = h$ a constant because of full state feedback. The closed-loop system is given by

$$x_{k+1} = (a + h(1 + \Delta_k))x_k, \quad |\Delta_k| \leq \delta,$$

where δ relates to ρ as in (10). Since it is a first order system, we take $V(x_k) = x_k^2$, which gives

$$V(x_{k+1}) = (a + h(1 + \Delta_k))^2 x_k^2 \leq (|a + h| + \delta|h|)^2 x_k^2$$

with the right-hand side being the worst-case value. Minimizing it gives $h = -a$ and

$$V(x_{k+1}) \leq \delta^2 a^2 V(x_k). \tag{40}$$

This gives the upper bound for δ to be a^{-1}.

Now, for any $\delta < a^{-1}$, η in (22) is given by $\eta = 1 - \delta^2 a^2$. Applying it to (35), we obtain

$$N_0 = 1 + \frac{\log(\gamma_2 a^{-1} - \delta)}{\log(1 - \delta) - \log(1 + \delta)}, \quad \delta < a^{-1}, \tag{41}$$

which can be minimized numerically. The result is shown in Figure 1, where two curves for the required bit rate (33), one for $\gamma_2 = 1$ and another for $\gamma_2 = 0.9$, are compared with the minimum bit rate $\lceil \log_2(a) \rceil$ given in [4]. We see that the difference is only a few bits even when a is taken up to 100.

Example 2: The second example we consider aims at demonstrating the convergence rate and robustness of the dynamic scaling method. Consider the system (1)-(2) with

$$A = \begin{bmatrix} 2.7 & -2.41 & 0.507 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C = [1 \quad -0.5 \quad 0.04].$$

The system is unstable with two unstable open-loop poles at $1.2 \pm i0.5$ but without unstable zero and the relative degree is 1. It follows from [3] that

$$\delta_{\sup} = |1.2 \pm i0.5|^{-2} = 0.5917, \quad \rho_{\inf} = 0.2565.$$

Choosing $\delta = 0.2$ and optimizing N_0 using a search on η, we obtain $\eta = 0.41$ and the corresponding optimal controller

$$A_c = \begin{bmatrix} 0 & 0.5367 & 1.2080 \\ 0.1342 & -0.2684 & -0.7382 \\ 0.1929 & 0.4194 & 0.1678 \end{bmatrix},$$
$$B_c = [\, 1.0000; 1.0000; 0.3750 \,],$$
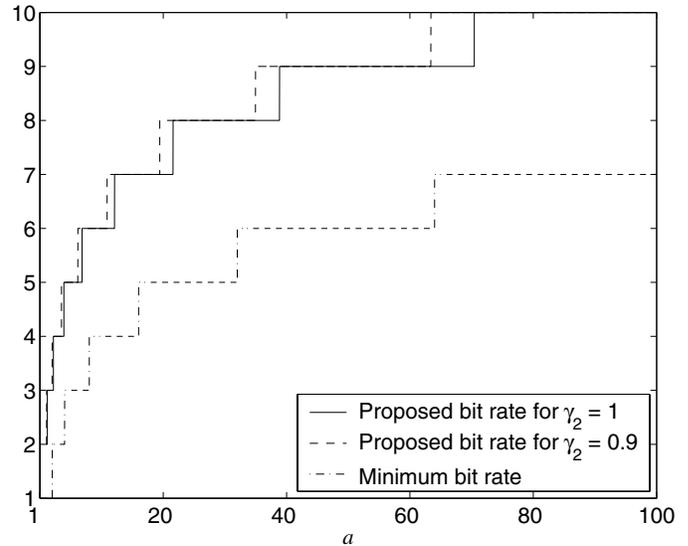$$C_c = [0.12049 \quad 0.2293 \quad -0.1431],$$
$$D_c = -1.4955.$$
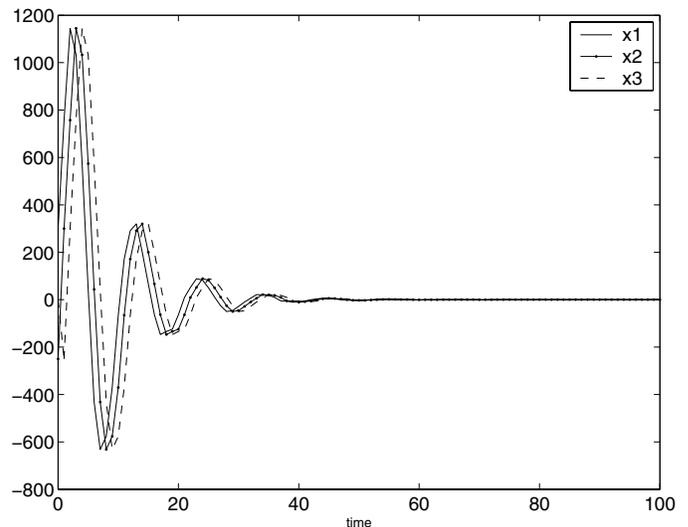

Fig. 1. Bit Rate Comparison for a First Order System

Fig. 2. State response of the closed-loop system

Since γ_2 is lower bounded by $\sqrt{1 - \eta} = 0.7681$, we choose $\gamma_2 = 0.8$. This gives $N_0 = 11.015$. Since $N = 12$ and $N = 16$ give the same bit rate (5 bits), we set $N = 16$. Note that the minimal bit rate required for stabilizing this system is 1 bit [4].

It can be easily verified that (23) is satisfied if we choose $\gamma_1 = 0.2$. Let the initial state of the controller be $\hat{x}_0 = [0 \ 0 \ 0]^T$ and the minimal level of the quantizer be 1 (correspondingly, $\mu_0 = 1/\rho^{N-1}$). For the initial state $x_0 = [250 \ -200 \ 0]^T$ and $g_0 = 1$, the state response of the closed-loop system is shown in Figure 2. The scaling gain g_k is shown in Figure 3.

If we have a good estimate $\tilde{x}_0$ of the initial state x_0, we
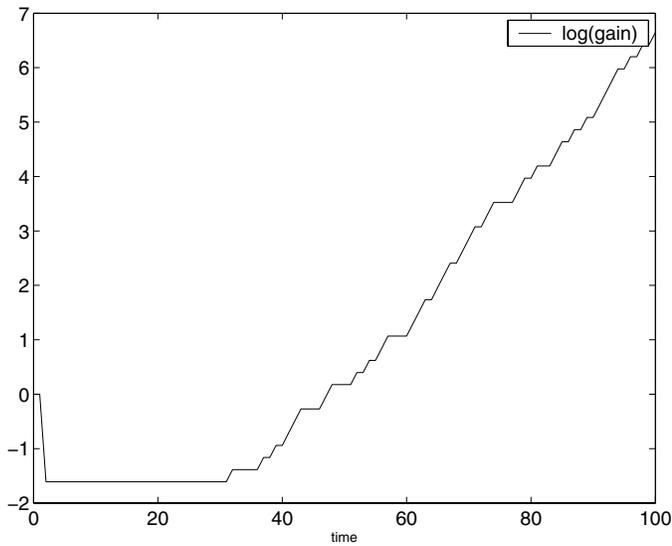
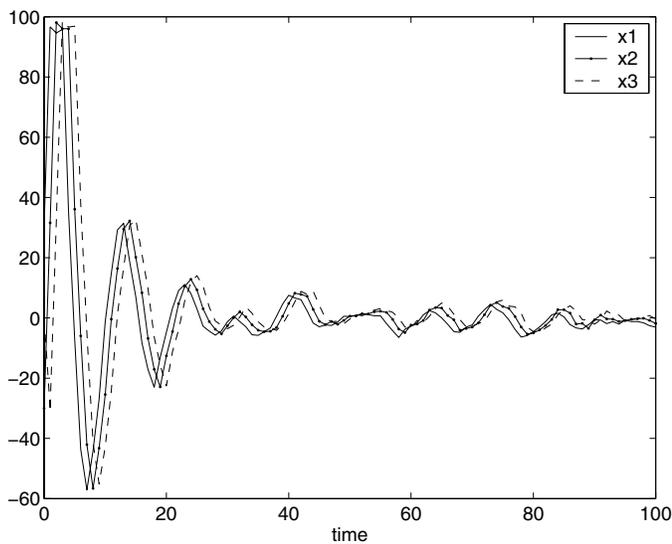Fig. 3.   The scaling factor g_k for the noise-free case

Fig. 4.   State response of the closed-loop system under input noise

may set the initial scaling gain g_0 to improve the transient performance. For example, if we set $g_0 = 1/|Cx_0|$ for the given initial condition, the transient overshoot is reduced by approximately 35%.

Next, we study the robustness of the closed-loop system. Let w_k in (36) be a saturated Gaussian white noise with zero mean and $|w_k| \leq 100$ and covariance matrix of $3I$. We set $\bar{g} = 40$. The state response of the closed-loop system with $x_0 = [30 \quad -30 \quad 0]^T$ is shown in Figure 4. It can be observed that the the final state converges to a bounded region. The scaling factor g_k has a similar initial response as in the noise free case but gets bounded at the steady state.

VII. Conclusion

We have proposed a simple dynamic scaling method for quantized feedback control. Using this, we show that asymptotic stabilization for a linear system can be achieved using a logarithmic quantizer with a very moderate number of quantization levels. The proposed control scheme is easily implementable and has nice convergence and robust properties. We emphasize that the proposed dynamic scaling method can be applied to a much wider range of quantized feedback control problems.

References

[1] N. Elia and K. Mitter, "Stabilization of Linear Systems with Limited Information," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1384-1400, September, 2001.

[2] N. Elia, "Design of hybrid systems with guaranteed performance," *Proc. 39th IEEE Conf. Decision and Control*, pp. 993-998, 2000.

[3] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1698-1711, Nov. 2005.

[4] G. N. Nair and R. J. Evans, "Exponential stabilization of multi-dimensional linear systems," *Automatica*, vol. 39, pp. 585-593, 2003.

[5] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: tightest attainable bounds," *Systems and Control Letters*, vol. 41, pp. 49-56, 2000.

[6] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints I: state estimation problems," *Transactions on Automatic Control*, vol. 42, no. 9, pp. 1294-1299, September, 1997.

[7] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: stabilization with limited information feedback," *Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049-1053, May, 1999.

[8] J. Baillieul, "Feedback designs in information-based control," *Stochastic Theory and Control Workshop*, Kansis, pp. 35-57, Springer, 2001.

[9] D. F. Delchamps, "Stabilizing a linear system with qunatized state feedback," *IEEE Transactions on Automatic Control*, vol. 35, no. 8, pp. 916-924, 1990.

[10] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279-1289, July, 2000.

[11] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056-1068, July, 2004.

[12] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1196-1201, July, 2004.

[13] A Mitra, M. Chakraborty and H. Sakai, "A block floating-point treatment to the LMS algorithm: efficient realization and a roundoff eror analysis," *IEEE Trans. on Signal Processing,* vol. 53, no. 12, 2005.

[14] M. Fu and S. Hara, "Quantized feedback control for sampled-data systems,", *IFAC World Congress*, Prague, July 2005.

[15] M. C. de Oliveira, J. C. Geromel and J. Bernussou, "An LMI optimization approach to multiobjective controller design for discrete-time systems," *Proc. the 38th CDC,* Phoenix, Dec. 1999.

[16] C. Scherer, P. Gahinet and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Contr.,* vol. 42, no. 7, pp. 896-911, 1997.

[17] L. Xie and Y. C. Soh, "Guaranteed cost control of uncertain discrete-time systems," *Control Theory and Advanced Technology*, Japan, vol. 10, no. 4, Part 2, pp. 1235-1251, 1995.