The Sector Bound Approach to Quantized Feedback Control

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***Abstract*—This paper studies a number of quantized feedback design problems for linear systems. We consider the case where quantizers are static (memoryless). The common aim of these design problems is to stabilize the given system or to achieve certain performance with the coarsest quantization density. Our main discovery is that the classical sector bound approach is nonconservative for studying these design problems. Consequently, we are able to convert many quantized feedback design problems to well-known robust control problems with sector bound uncertainties. In particular, we derive the coarsest quantization densities for stabilization for multiple-input–multiple-output systems in both state feedback and output feedback cases; and we also derive conditions for quantized feedback control for quadratic cost and H_∞ performances.**

***Index Terms*—H_∞ control, linear quadratic control, quadratic stabilization, quantized feedback, sector bound approach.**

I. Introduction

CONTROL using quantized feedback has been an important research area for a long time. Even as early as in 1956, Kalman [1] studied the effect of quantization in a sampled data system and pointed out that if a stabilizing controller is quantized using a finite-alphabet quantizer, the feedback system would exhibit limit cycles and chaotic behavior. Most of the work on quantized feedback control concentrates on understanding and mitigation of quantization effects; see, e.g., [2]–[4].

A simple classical approach to analysis and mitigation of quantization effects is to treat the quantization error as uncertainty or nonlinearity and bound it using a sector bound. By doing so, robustness analysis tools, such as absolute stability theory (see [5] and [6]), can be applied to study the quantization effect. Further, control parameters can be optimized to minimize the quantization effect. We will call this the *sector bound method*.

There is a new line of research on quantized feedback control where a quantizer is regarded as an information coder. The fundamental question of interest is how much information needs to be communicated by the quantizer in order to achieve a certain control objective. Noticeable works include [7]–[20]. In [16], the problem of quadratic stabilization of discrete-time single-input–single-output (SISO) linear time-invariant (LTI) systems using quantized feedback is studied. The quantizer is assumed to be static and time-invariant (i.e., memoryless and with fixed quantization levels). It is proved in [16] that for a quadratically stabilizable system, the quantizer needs to be *logarithmic* (i.e., the quantization levels are linear in logarithmic scale). Furthermore, the coarsest quantization density is given explicitly in terms of the system's unstable poles. Note that the required quantizer has an infinite number of quantization levels because of its time-invariance nature. When a finite number of quantization levels are available, the so-called *practical stability* is obtained where there is a region of attraction in the state and the steady state converges to a small limit cycle. One can think of many ways to scale the dynamic range of the quantizer to increase the region of attraction and reduce the size of the limit cycle.

When the quantizer is allowed to be dynamic and time-varying, it is obviously advantageous to scale the quantization levels dynamically so that the region of attraction is increased and the steady state limit cycle is reduced. This is indeed the basic idea behind [10]–[15]. In fact, it is shown in [12] that stabilization of a SISO LTI system (in some stochastic sense) can be achieved using only a finite number of quantization levels. In addition, the minimum number of quantization levels (also known as the minimum *feedback information rate*) is explicitly related to the unstable poles of the system, under the assumption of noise free communications. In this setting, the dynamic quantizer effectively consists of two parts: an *encoder* at the output end and a *decoder* at the input end. The problem of minimum feedback information rate is studied in more details in [13] by analyzing the structures of the encoder and decoder. We do caution that many results on quantized feedback with dynamic quantizers may be impractical due to three problems: 1) Most results are for stabilization only rather than for performance control; 2) the transient response is typically very poor due to the lack of good control design algorithms; 3) as pointed out in [15], the capacity results are in general not valid for practical communications channels which are not noise free.

The most pertinent work to this paper is [16]. In fact, this paper stems from the following motivations. First, the results in [16] (also those in [12]) are for SISO systems and for stabilization only. We want to know how to generalize their results to multiple-input–multiple-output (MIMO) systems and to control design for performances. Secondly, the technique used in [16], although being novel, does not seem to have a simple interpretation. This is perhaps why the generalization of their results appears to be difficult.

Manuscript received June 8, 2004; revised May 8, 2005. Recommended by Associate Editor D. Miller.

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Digital Object Identifier 10.1109/TAC.2005.858689

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In this paper, we first review the key result in [16] which is on quadratic stabilization of SISO linear systems using quantized state feedback. We show that coarsest quantization density can be simply obtained using the sector bound method. This not only gives a simpler interpretation of the result, but also provides the basis for generalization of the result. Further, the coarsest quantization density is directly related to an H_∞ optimization problem, which is better than relating it to an "expensive" control problem as done in [16] because the optimal H_∞ control shares the same linear feedback gain with that of the quantized feedback having the coarsest quantization density. Second, we study the output feedback stabilization of SISO systems. Two cases are considered: observer-based quantized state feedback and dynamic feedback using quantized output. We show that the coarsest quantization density in the former case is the same as in quantized state feedback, whereas the latter case is related to a different H_∞ optimization problem and in general requires a finer quantization density. Third, we generalize the quadratic stabilization problem to MIMO systems and show that quadratic stabilization with a set of logarithmic quantizers is the same as quadratic stabilization for an associated system with sector-bounded uncertainty. Because the latter problem has been well studied, the technical difficulty for the first problem is clearly revealed. A sufficient condition is then given, in terms of an H_∞ optimization problem, for the quantizers to render a quadratic stabilizer. As in the SISO case, both state feedback and output feedback are considered. Finally, we generalize the results to performance control problems. Both linear quadratic performance and H_∞ performance problems are studied and conditions are given for a set of quantizers to render a given performance level.

II. Stabilization Using Quantized State Feedback

In this section, we revisit the work of Elia and Mitter [16] on stabilization using quantized state feedback and reinterpret their result using the sector bound method.

The simplest and most fundamental case considered in [16] is the problem of quadratic stabilization for the following system:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, x is the state and u is the control input. We assume that A is unstable and (A, B) is stabilizable and consider quantized state feedback in the following form:

$$u(k) = f(v(k)) \tag{2}$$
$$v(k) = g(x(k)). \tag{3}$$

In the above, $g(\cdot)$ is the *unquantized feedback law*, and $f(\cdot)$ is a quantizer which is assumed to be symmetric, i.e., $f(-v) = -f(v)$. Note that the quantizer is static and time-invariant.

The set of (distinct) quantized levels is described by

$$\mathcal{U} = \{\pm u_i, i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\}. \tag{4}$$

Each of the quantization level (say u_i) corresponds to a segment (say V_i) such that the quantizer maps the whole segment to this quantization level. In addition, these segments form a partition of $\mathbf{R}$, i.e., they are disjoint and their union equals to $\mathbf{R}$.

Denote by $\#g[\epsilon]$ the number of quantization levels in the interval $[\epsilon,\ 1/\epsilon]$. The density of the quantizer $f(\cdot)$ is defined as follows:

$$\eta_f = \limsup_{\epsilon\to 0} \frac{\#g[\epsilon]}{-\ln \epsilon}. \tag{5}$$

With this definition, the number of quantization levels of a quantizer with a nonzero, finite quantization density grows logarithmically as the interval $[\epsilon,\ 1/\epsilon]$ increases. A small η_f corresponds to a coarse quantizer. A finite quantizer (i.e., a quantizer with a finite number of quantization levels) has $\eta_f = 0$, and a linear quantizer has $\eta_f = \infty$.

A quantizer is called *logarithmic* if it has the form

$$\mathcal{U} = \{\pm u_{(i)} : u_{(i)} = \rho_i u_{(0)}, i = \pm 1, \pm 2, \ldots\} \cup \{\pm u_{(0)}\} \cup \{0\} \quad 0 < \rho < 1, u_{(0)} > 0. \tag{6}$$

The associated quantizer f is defined as follows:

$$f(v) = \begin{cases} u_i, & \text{if } \frac{1}{1+\delta}u_i < v \le \frac{1}{1-\delta}u_i,\ v > 0 \\ 0, & \text{if } v = 0 \\ -f(-v), & \text{if } v < 0 \end{cases} \tag{7}$$

where

$$\delta = \frac{1-\rho}{1+\rho}. \tag{8}$$

It is easily verified that $\eta_f = 2/\ln(1/\rho)$ for the logarithmic quantizer. This means that the smaller the ρ, the smaller the η_f. For this reason, we will abuse the terminology by calling ρ (instead of n_f) the *quantization density* in the rest of this paper. The logarithmic quantizer is illustrated in Fig. 1. In contrast, a nonlogarithmic quantizer is illustrated in Fig. 2.

For the quadratic stabilization problem, a quadratic Lyapunov function $V(x) = x^T P x$, $P = P^T > 0$, is used to assess the stability of the feedback system. That is, the quantizer must satisfy

$$\nabla V(x) = V(Ax + Bf(g(x))) - V(x) < 0 \qquad \forall\, x \neq 0. \tag{9}$$

The coarsest quantizer is the one which minimizes η_f subject to (9). But the coarsest quantizer is in general not attainable because the constraint in (9) is a strict inequality.

The required density of the quantizer depends on $V(x)$ (or P), $f(\cdot)$ and $g(\cdot)$. This raises the key question: What is the coarsest density, $\rho_{\rm inf}$, among all possible P and $g(\cdot)$? It is shown in [16] that the answer is the logarithmic quantizer with $\rho_{\rm inf}$ given by

$$\rho_{\rm inf} = \frac{\prod_i |\lambda_i^u| - 1}{\prod_i |\lambda_i^u| + 1} \tag{10}$$

where λ_i^u are the unstable eigenvalues of A.

We see from Figs. 1 and 2 that a quantizer can be bounded by a sector. For a logarithmic quantizer, the sector bound is described by a single parameter δ which is related to the quantization density by (8). In contrast, for a nonlogarithmic quantizer, two parameters, δ^- and δ^+, are needed to describe the sector in general. For both finite quantizers and linear quantizers, a default output value, u_0, is needed when the input is smaller

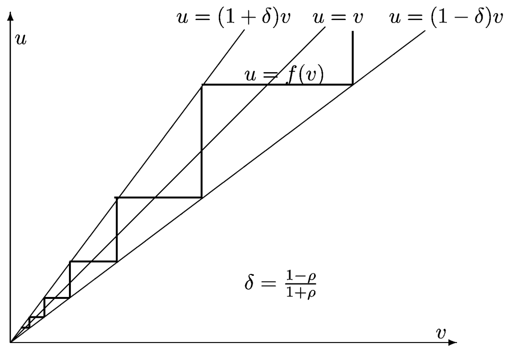

Fig. 1. Logarithmic quantizer.

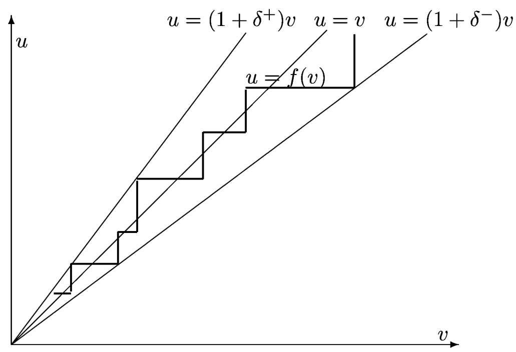

Fig. 2. Nonlogarithmic quantizer.

than some minimal threshold (in magnitude). If $u_0 = 0$, then $\delta^- = -1$; otherwise, $\delta^+ = \infty$.

In the theorem that follows, we use the sector bound method to study the quantized state feedback problem for the system (1). In particular, we reveal a strong connection between the quantized state feedback stabilization problem and a state feedback quadratic stabilization problem with sector bound uncertainty. This connection leads to an alternative proof for the coarsest quantization density result (10).

Theorem 2.1: The following results hold.

1) If the system (1) is quadratically stabilizable via quantized state feedback (2)–(3), then the coarsest quantization density can be approached by taking a logarithmic quantizer and a linear unquantized feedback law.
2) Given a logarithmic quantizer with quantization density ρ, the system (1) is quadratically stabilizable via quantized linear state feedback if and only if the following uncertain system:

$$x(k+1) = Ax(k) + B(1+\Delta)v(k), \qquad \Delta \in [-\delta, \delta] \quad (11)$$

is quadratically stabilizable via linear state feedback, where δ and ρ are related by (8).
3) The largest sector bound for (11) to be quadratically stabilizable via linear state feedback is given by

$$\delta_{\sup} = \frac{1}{\prod_i |\lambda_i^u|}. \quad (12)$$

Consequently, the coarsest quantization density $\rho_{\inf}$ for (1) is given by (10).

Four lemmas are needed for the proof of Theorem 2.1.

Lemma 2.1: Consider quadratic stabilization for the system in (1) using quantized state feedback (2)–(3) and a given Lyapunov matrix $P = P^T > 0$. Then, the coarsest quantization density can be approached by taking a linear state feedback

$$g(x) = Kx \quad (13)$$

and a logarithmic quantizer.

The proof is given in Appendix.

Lemma 2.2: Given a constant vector $K \in \mathbf{R}^{1\times n}$, a constant matrix $\Omega_0 \in \mathbf{R}^{n\times n}$, a vector function $\Omega_1(\cdot) : \mathbf{R} \to \mathbf{R}^{n\times 1}$, a scalar $\delta > 0$, and a scalar function $\Delta(\cdot) : \mathbf{R} \to [-\delta, \delta]$ with the following property: For any $\Delta_0 \in [-\delta, \delta]$, there exists $v_0 \neq 0$ such that $\Delta(v_0) = \Delta_0$. Define the following matrix function:

$$\Omega(\cdot) = \Omega_0 + \Omega_1(\cdot)K + K^T\Omega_1^T(\cdot). \quad (14)$$

Then

$$x^T\Omega(\Delta(Kx))x < 0 \qquad \forall\, x \neq 0,\ x \in \mathbf{R}^n \quad (15)$$

if and only if

$$\Omega(\Delta_0) < 0 \qquad \forall\, \Delta_0 \in [-\delta, \delta]. \quad (16)$$

Proof: It is obvious that (16) implies (15). To see the converse, we assume (15) holds but (16) fails. Then, there exists some $x_0 \neq 0$ and $\Delta_0 \in [-\delta,\ \delta]$ such that

$$x_0^T\Omega(\Delta_0)x_0 \geq 0. \quad (17)$$

We claim that $Kx_0 \neq 0$. Indeed, if $Kx_0 = 0$, then

$$x_0^T\Omega(\Delta(Kx_0))x_0 = x_0^T\Omega_0 x_0 = x_0^T\Omega(\Delta_0)x_0 \geq 0 \quad (18)$$

by (14) and (17), which contradicts (15). So, $Kx_0 \neq 0$. Because of the property of $\Delta(\cdot)$, there exists a scalar $\alpha \neq 0$ such that $\Delta(\alpha K x_0) = \Delta_0$. Define $x_1 = \alpha x_0 \neq 0$. Then

$$x_1^T\Omega(\Delta(Kx_1))x_1 = \alpha^2 x_0^T\Omega(\Delta_0)x_0 \geq 0$$

which violates (15). Hence, (15) implies (16). □

Lemma 2.3: Consider the uncertain system in (11). Define

$$G_c(z) = K(zI - A - BK)^{-1}B. \quad (19)$$

Then, the supreme of δ for which quadratic stabilization is achievable is given by

$$\delta_{\sup} = \frac{1}{\inf_K \|G_c(z)\|_\infty}. \quad (20)$$

Proof: It is well-known [22]–[24] that the quadratic stabilization for (11) is achievable if and only if

$$\delta < \frac{1}{\inf_K \|G_c(z)\|_\infty}.$$

Taking the limit in the above yields (20). □

Lemma 2.4: The solution to (20) is given by (12).

The proof is given in Appendix.

Proof of Theorem 2.1: Part 1) follows from Lemma 2.1. Part 3) follows from Part 2) and Lemmas 2.3–2.4. To show Part 2), by taking into consideration of Part 1), we can assume $g(x) = Kx$ for some K and that $f(\cdot)$ is a logarithmic quantizer with quantization density ρ for some ρ. Define the quantization error by

$$e = u - v = f(v) - v = \Delta(v)v. \tag{21}$$

Then

$$\Delta(v) \in [-\delta, \delta] \tag{22}$$

with δ in (8). We can model the quantized feedback system as the following uncertain system:

$$x(k+1) = Ax(k) + B(1+\Delta(Kx))Kx(k). \tag{23}$$

The corresponding quadratic stabilization condition becomes

$$\begin{aligned}\nabla V(x) &= V((A+B(1+\Delta(Kx))K)x) - V(x) < 0\\ &\forall\, x \neq 0.\end{aligned} \tag{24}$$

Define

$$\nabla P(\Delta) = (A+B(1+\Delta)K)^T P(A+B(1+\Delta)K) - P \tag{25}$$

where Δ is independent of the state. Note that the inverse mapping of $\Delta(v)$ in (21) is a multi-branch continuous function (except at $v = 0$). Hence, for any $\Delta_0 \in [-\delta,\ \delta]$, there exists some $v_0 \neq 0$ such that $\Delta(v_0) = \Delta_0$. By Lemma 2.2, (24) is equivalent to

$$\nabla P(\Delta) < 0 \qquad \forall\, \Delta \in [-\delta, \delta].$$

However, the latter is the condition for the system (11) to be quadratically stabilizable via linear state feedback. □

Remark 2.1: It is shown in [16] that the coarsest quantization density is related to the solution to the so-called "expensive" linear quadratic control problem

$$\begin{aligned}&\min_K \quad \sum_{k=0}^{\infty} |u(k)|^2\\ &\text{subject to} \quad \text{closed-loop stability with}\\ &\qquad\qquad u(k) = Kx(k).\end{aligned} \tag{26}$$

More specifically, the optimal ρ can be solved using the solution to the Riccati equation for the "expensive" control problem. However, the optimal control gain K for the quantization problem is different from the optimal control gain for the "expensive" control problem (This is also pointed out in [16]). From the proofs above, we see that it may be more aesthetically pleasing to interpret the coarsest quantization problem as an H_∞ optimization problem (20) because they share the same optimal control gain.

Remark 2.2: We have seen that logarithmic quantizers are essential for quadratic stabilization via quantized feedback if a coarse quantization density is required. Nonlogarithmic quantizers such as finite quantizers and linear quantizers are unsuitable. For this reason, we will consider logarithmic quantizers only in the rest of this paper.

III. Stabilization Using Quantized Output Feedback

We now show how to generalize the technique for state feedback to quantized output feedback. Consider the following system:

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k)\end{aligned} \tag{27}$$

where A and B are the same as before and $C \in \mathbf{R}^{1\times n}$.

We consider two possible basic configurations for quantized output feedback which may lead to other more complicated settings.

4) **Configuration I:** The control signal is quantized but the measurement is not.
5) **Configuration II:** The measurement is quantized but the control signal is not.

In both configurations, we assume that the controller is linear time-invariant with a finite order. It turns out that the two configurations result in different quantization density requirements.

Configuration I: This is an easy case which has an interesting result.

Theorem 3.1: Consider the system (27) with quantized control input. Suppose (A, C) is an observable pair. Then, the coarsest quantization density for quadratic stabilization by state feedback can also be achieved by output feedback. In particular, the corresponding output feedback controller can be chosen as an observer-based controller

$$\begin{aligned}x_c(k+1) &= Ax_c(k) + Bu(k) + L(y(k) - Cx_c(k))\\ v(k) &= Kx_c(k)\\ u(k) &= f(v(k))\end{aligned} \tag{28}$$

where $f(\cdot)$ is the quantizer as before, K is the state feedback gain designed for any achievable quantization density via quantized state feedback, and L is any gain which yields (28) a deadbeat observer.

Proof: Let K be any state feedback gain that achieves any given quantization density. Choose L such that the observer is deadbeat, i.e., $e(k) = x(k) - x_c(k) \neq 0$ only for a finite number of steps N. This can be always done because (A, C) is observable. Then, after N steps, the output feedback controller is the same as state feedback controller. Hence, the system is quadratically stabilized after N steps. Finally, it is a simple fact (although we do not give the details) that if a (nonlinear) system is quadratically stable after N steps and that the state is bounded in the first N steps [which clearly holds for the system (28)], it is quadratically stable. □

Configuration II: In this case, the controller is in the form

$$\begin{aligned}x_c(k+1) &= A_c x_c(k) + B_c f(y(k))\\ u(k) &= C_c x_c(k) + D_c f(y(k))\end{aligned} \tag{29}$$

where $f(\cdot)$ is the quantizer as before.

It is straightforward to verify that the closed-loop system is given by

$$\bar{x}(k+1) = \mathcal{A}(\Delta(y(k))\bar{x}(k) \tag{30}$$

where $\bar{x} = [x^T x_c^T]^T$, $\Delta(\cdot)$ is the same as in (22) and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$
$$\hat{I} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \hat{C} = [C \quad 0] \quad \bar{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{31}$$

and

$$\mathcal{A}(\Delta) = \bar{A} + \bar{B}\bar{K}(\bar{C} + \hat{I}\Delta\hat{C}). \tag{32}$$

The problem of concern is to find the coarsest quantizer for quadratic stabilization of the closed-loop system. This can be solved by generalizing the idea for the state feedback case. The result is given here.

Theorem 3.2: Consider the system (27). For a given quantization density $\rho > 0$, the system is quadratically stabilizable via a quantized controller (29) if and only if the following auxiliary system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ v(k) &= (1+\Delta)Cx(k) \quad |\Delta| \le \delta \end{aligned} \tag{33}$$

is quadratically stabilizable via

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c v(k) \\ u(k) &= C_c x_c(k) + D_c v(k) \end{aligned} \tag{34}$$

where δ and ρ are related by (8).

The largest sector bound $\delta_{\sup}$ (which gives $\rho_{\inf}$) is given by

$$\delta_{\sup} = \frac{1}{\inf_{\bar{K}} \|\bar{G}_c(z)\|_\infty} \tag{35}$$

where $\bar{K}$ is defined in (31) and

$$\bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z) \tag{36}$$

where $G(z) = C(zI - A)^{-1}B$ and $H(z) = D_c + C_c(zI - A_c)^{-1}B_c$.

Further, if $G(z)$ has relative degree equal to 1 and no unstable zeros, then the coarsest quantization density for quantized state feedback can be reached via quantized output feedback.

Proof: The proof is similar to the proof of Theorem 2.1. The sector bound for the quantization error is done as in (21)–(22). For the given ρ, the quadratic stability of the closed-loop system (27)–(29) requires the existence of some $\bar{P} = \bar{P}^T > 0$ such that

$$\bar{x}^T[\mathcal{A}(\Delta(y))^T\bar{P}\mathcal{A}(\Delta(y)) - \bar{P}]\bar{x} < 0 \tag{37}$$

for all $\bar{x} \neq 0$ and $y = Cx = \hat{C}\bar{x}$. Using Lemma 2.2, the above is equivalent to

$$\mathcal{A}(\Delta)^T\bar{P}\mathcal{A}(\Delta) - \bar{P} < 0 \qquad \forall\, |\Delta| \le \delta.$$

The latter is the same as requiring the system (33)–(34) to be quadratically stable. Since the transfer function of (33) is $G(z)(1+\Delta)$ and that of (34) is $H(z)$, the closed-loop system (33)–(34) is the same as a closed-loop system with the open-loop block equal to $\bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z)$ and feedback block equal to Δ. It follows that the solution to $\delta_{\sup}$ comes from the equivalence between quadratic stability and H_∞ optimization [22]–[24].

Suppose $G(z)$ has relative degree 1 and no unstable zeros. Write $G(z) = b(z)/a(z)$. From the proof of Theorem 2.1, we know that the state feedback case corresponds to H_∞ optimization of $G_c(z)$ in (109). If we choose $H(z) = k(z)/b(z)$. Then, $\bar{G}_c(z)$ in (36) becomes $G_c(z)$. Hence, the quantization density for the quantized state feedback can be achieved by quantized output feedback. □

Now, we give an example to show that using quantized output requires a higher quantization density than using quantized state feedback.

Example 3.1: The system is given by (27) with $G(z) = C(zI - A)^{-1}B = (z-3)/z(z-2)$. Using quantized state feedback, $\delta = 2$ and $\rho = (2-1)/(2+1) = 0.3333$. For quantized output feedback, computing (35) yields $\delta = 10$ and $\rho = (10-1)/(10+1) = 0.8182$.

Remark 3.1: In [16], output feedback control design is done in two steps. In step 1), coarsest quantization is solved for state estimation, which is a dual problem to the state feedback stabilization problem. In step 2), the separation principle is applied, i.e., optimal state feedback is combined with optimal state estimation. The main result is that logarithmic quantization is sufficient for output feedback stabilization.

The drawback of the approach in [16] is that the physical meaning of the state estimation quantizer is not clear. Indeed, the problem of quantized state estimation is formulated to be

$$e(k+1) = Ae(k) + Lf_e(Ce(k)) \tag{38}$$

where $e(k) = x(k) - x_c(k)$ is the state estimation error and $f_e(\cdot)$ is the state estimation quantizer. What is unsatisfactory in this formulation is that the quantizer needs to know both $y(k)$ and its estimate $Cx_c(k)$. If the control signal is generated at the measurement end, there is obviously no need to use quantized $y(k)$. If the control signal is generated elsewhere using a quantized $y(k)$, it is difficult to imagine why its estimate needs to be sent back to the measurement end to form $Ce(k)$ for quantization. Hence, the validity of this formulation seems to be questionable.

IV. Stabilization of MIMO Systems Using Quantized Feedback

Quantized feedback control for MIMO systems has been studied in a number of papers; see, e.g., [17]–[19]. In these papers, state feedback is assumed but the input is multidimensional. Instead of using a separate quantizer for each input, the approach in [17]–[19] uses a single quantizer by jointly quantizing the multiple-input space. The main advantage of this approach is that a much coarser quantization density is required in comparison with separate quantization, as shown in [17]. The main disadvantage of this approach, however, is that it is a centralized process. That is, the quantizer requires the information about the whole input vector or even the whole state. This may not be practical in many output feedback control problems where different output channels need to be quantized separately without additional communications among them. Another disadvantage of the joint quantization approach is that the partitioning of the input or state space is computationally intensive. In fact, the results in [17]–[19] are limited to two-input systems only.

In this section, we generalize the quantization results in Sections II and III to MIMO systems with multiple quantizers. For simplicity, the number of quantizers is assumed to equal to the number of inputs, although this can be easily relaxed. The quantizers are assumed to be static and independent. As in the SISO case, two configurations are treated. Configuration I considers quantized inputs, whereas in Configuration II quantized outputs are used.

Configuration I: The system is still as in (27) except that we now allow $u \in \mathbf{R}^m$, $y \in \mathbf{R}^r$. Suppose

$$f(v) = \mathrm{diag}\{f_1(v_1), f_2(v_2), \ldots, f_m(v_m)\} \tag{39}$$

where v_j is the jth component of v and $f_j(\cdot)$ is a quantizer of the form (6) but with quantization level $0 < \rho_j < 1$. The feedback law for v is assumed to be linear, i.e.,

$$v = Kx \tag{40}$$

where K is the feedback gain matrix which may or may not be subject to some structural constraints. For example, in the case of decentralized static output feedback design, the output y may be partitioned into m sub-vectors, $y_1, y_2, \ldots, y_m$, with $y_i = C_i x$ for some constant matrix C_i, $i = 1, 2, \ldots, m$. In this case, $v_i = \bar{K}_i C_i x$ for some row vector $\bar{K}_i$, $i = 1, 2, \ldots, m$, i.e.,

$$K = \mathrm{col}\{\bar{K}_1 C_1, \bar{K}_2 C_2, \ldots, \bar{K}_m C_m\}.$$

Because we have more than one quantizer, the notion of coarsest quantization is not well-defined. Instead, we ask the following question: Given a vector of quantization levels $\rho = [\rho_1\ \rho_2\ \cdots\ \rho_m]$, does there exist an quantized feedback controller that quadratically stabilizes the system (27)? The main result is given here.

Theorem 4.1: Given the system in (27) and a quantization level vector ρ, consider the following auxiliary system:

$$x(k+1) = Ax(k) + B(I + \Delta(k))v(k) \tag{41}$$

where $|\Delta_j(k)| \le \delta_j$ for all $j = 1, 2, \ldots, m$ and k, and δ_j are converted from ρ_j using (8), and $v(k)$ is a control input. Suppose the auxiliary system is quadratically stablizable via the state feedback law (40), then (27) is quadratically stabilizable via quantized state feedback with the same state feedback law. Conversely, suppose the system (27) is quadratically stabilizable via quantized state feedback with the state feedback law (40) and, in addition, suppose $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \ne j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (41) with $|\Delta_j(k)| \le \delta_j - \epsilon$ is quadratically stabilizable via the same state feedback law (40).

Furthermore, the auxiliary system is quadratically stabilizable via state feedback (40) if

$$\|\Lambda \Gamma K (zI - A - BK)^{-1} B \Gamma^{-1}\|_\infty < 1 \tag{42}$$

for some diagonal scaling matrix $\Gamma > 0$, where

$$\Lambda = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}. \tag{43}$$

In particular, any K that renders (42) is a solution to either quadratic stabilization problem.

Remark 4.1: It is obvious that if a given ρ does not satisfy the condition that $\ln \rho_i / \ln \rho_j$ are irrational for $i \ne j$, we can make it so by perturbing the ρ_j slightly. Therefore, if a quantized state feedback controller is designed to quadratically stabilize (27) for a given ρ and we require this controller to remain quadratically stabilizing when ρ is perturbed slightly, then it is necessary that the auxiliary system (41) with $|\Delta_j(k)| \le \delta_j - \epsilon$ is quadratically stabilizable via state feedback for some arbitrarily small $\epsilon > 0$.

Three technical lemmas are required for the proof of the result above.

Lemma 4.1: For the quantizer (6) and any $|\Delta| \le \delta$, the inverse function for $\Delta(v)$ is not unique, and is given by

$$\ln \frac{v}{u^{(0)}} = i \ln \rho - \ln(\Delta + 1), \qquad i = 0, \pm 1, \pm 2, \ldots. \tag{44}$$

Proof: The results follow directly from the definition of $\Delta(v)$ in (22). □

Lemma 4.2: Let $f_j(\cdot), j = 1, 2, \ldots, m$ be a set of quantizers as in (6) but with (possibly different) values $u_j^{(0)}$ and $0 < \rho_j < 1$. Suppose the ratios $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $1 \le i, j \le m, i \ne j$ (This condition is void if $m = 1$). Then, given any pairs of vectors (v, Δ^0) with $v_j \ne 0$ and $|\Delta_j^0| \le \delta_j$, $j = 1, 2, \ldots, m$, and any scalar $\epsilon > 0$ (arbitrarily small), there exists a scalar $\alpha > 0$ such that

$$|\Delta_j(\alpha v_j) - \Delta_j^0| < \epsilon, \qquad j = 1, 2, \ldots, m \tag{45}$$

where $\Delta_j(\cdot)$ is as defined in (21)–(22). That is, as α varies from 0 to ∞, the vector $[\Delta_1(\alpha v_1)\ \cdots\ \Delta_m(\alpha v_m)]^T$ covers the hyperrectangle $[-\delta_1, \delta_1] \bigoplus \cdots \bigoplus [-\delta_m, \delta_m]$ densely.

Proof: Note that each $\Delta_j(v)$ is periodic in $\ln(v/u_j^{(0)})$ with the period $\ln \rho_j$ and that within each period the mapping between $\ln(v/u_j^{(0)})$ and $\Delta_j(v)$ is one-to-one. Therefore, it suffices to show that as α varies, $[\mathrm{mod}(\ln \alpha v_1/u_1^{(0)}, \ln \rho_1)\ \ \cdots\ \ \mathrm{mod}(\ln \alpha v_m/u_m^{(0)}, \ln \rho_m)]^T$ covers $B = [0, \ln \rho_1] \bigoplus \cdots \bigoplus [0, \ln \rho_m]$ densely. This is equivalent to that $\gamma = [\mathrm{mod}(\ln \alpha, \ln \rho_1)\ \cdots\ \mathrm{mod}(\ln \alpha, \ln \rho_m)]^T$ covers B densely.

Let $\beta = [\beta_1, \ldots, \beta_m]^T \in B$ be any given vector. We need to find α such that γ is arbitrarily close to β. The assumption that $\ln \rho_i / \ln \rho_j$ are irrational implies that quantizers $f_i(\cdot)$ and $f_j(\cdot)$, $i \ne j$, do not share a common period (in the logarithmic scale), which is the key to the analysis that follows. If $m = 1$, we can simply take

$$\ln \alpha = \beta_1 + i_1 \ln \rho_1 \tag{46}$$

as a solution with any integer i_1. If $m = 2$, we keep $\ln \alpha$ as in (46) but let i_1 vary. Because $f_1(\cdot)$ and $f_2(\cdot)$ do not share a common period, as the integer i_1 varies from $-\infty$ to ∞, $\mathrm{mod}(\ln \alpha,\ \ln \rho_2)$ will cover the set $[0, \ln \rho_2]$ densely. Let I_1 and I_2 be the infinite sequences of i_1 and the corresponding i_2, respectively, which make the corresponding set of $\mathrm{mod}(\ln \alpha,\ \ln \rho_2)$ sufficiently close to β_2. For $m = 3$, because $f_1(\cdot)$, $f_2(\cdot)$ and $f_3(\cdot)$ do not share a common period pair-wise, there is an infinite sequence $\tilde{I}_1$ for i_1 (a subsequence of I_1) which generates the corresponding infinite sequence $\tilde{I}_2$ for i_2 (a subsequence of I_2) and infinite sequence I_3 for i_3 such that $\mathrm{mod}(\ln\alpha, \ln\rho_3)]^T$ is also sufficiently close to β_3. This process can continue for $m > 3$. Hence, we have proved the needed result. □

Lemma 4.3: Let $f_j(\cdot)$, $j = 1,\ldots,m$, be a set of quantizers satisfying the conditions in Lemma 4.2. Given constant matrices $K \in \mathbf{R}^{m\times n}$ and $\Omega_0 = \Omega_0^T \in \mathbf{R}^{n\times n}$, and a matrix function $\Omega_1(\cdot) : \mathbf{R}^m \to \mathbf{R}^{n\times m}$, define

$$\Omega(\cdot) = \Omega_0 + \Omega_1(\cdot)K + K^T\Omega_1^T(\cdot). \tag{47}$$

Suppose $\Omega(\cdot)$ is strictly convex. Then

$$x^T\Omega(\Delta(Kx))x < 0 \qquad \forall\, x \neq 0,\ \ x \in \mathbf{R}^n \tag{48}$$

if

$$\Omega(\Delta) < 0 \qquad \forall\, |\Delta_j| \le \delta_j,\ \ j = 1,\ldots,m. \tag{49}$$

Conversely, (48) implies

$$\Omega(\Delta) < 0 \qquad \forall\, |\Delta_j| \le \delta_j - \epsilon,\ \ j = 1,\ldots,m \tag{50}$$

for any $\epsilon > 0$.

Proof: It is obvious that (49) implies (48). To see the converse, we assume (48) holds but (49) fails. Then, there exists some $x_0 \neq 0$ and $\Delta^0 = (\Delta_1^0 \cdots \Delta_m^0)$ with $|\Delta_j^0| \le \delta_j$, $j = 1,\ldots,m$, such that

$$x_0^T\Omega(\Delta^0)x_0 \ge 0. \tag{51}$$

If such Δ^0 is only a boundary point, i.e., $|\Delta_i^0| = \delta_i$ for some i, then, (50) holds for any $\epsilon > 0$. In the sequel, we assume that Δ^0 is an interior point.

We claim that $Kx_0 \neq 0$. Indeed, if $Kx_0 = 0$, then

$$x_0^T\Omega(\Delta(Kx_0))x_0 = x_0^T\Omega_0x_0 = x_0^T\Omega(\Delta^0)x_0 \ge 0 \tag{52}$$

by (47) and (51), which contradicts (48). So, $Kx_0 \neq 0$.

Because of the strict convexity of $\Omega(\cdot)$, there exists Δ^1 with $|\Delta_j^1| \le \delta_j - \epsilon_1, j = 1,\ldots,m$, for some small $\epsilon_1 > 0$ such that

$$x_0^T\Omega(\Delta^1)x_0 > 0. \tag{53}$$

Because this is continuous in x_0, we may perturb x_0 slightly such that (53) still holds and every element of Kx_0 is nonzero.

Now, using Lemma 4.2, we know that $\Delta(\alpha Kx_0)$ covers $[-\delta_1,\delta_1]\bigoplus\cdots\bigoplus[-\delta_m,\delta_m]$ densely as α varies from $-\infty$ to ∞. Hence, there exists $\alpha \neq 0$ such that

$$x_0^T\Omega(\Delta(\alpha Kx_0))x_0 > 0.$$

Define $x_1 = \alpha x_0$, we get

$$x_1^T\Omega(\Delta(Kx_1))x_1 > 0$$

which contradicts (48). That is, Δ^0 cannot be an interior point. Hence, (48) implies (50).

Proof of Theorem 4.1: The "equivalence" between the quantized feedback problem and the quadratic stabilization problem for the auxiliary system (41) follows from Lemma 4.3. The H_∞ condition for the latter comes from [23]. □

Configuration II: If the output measurements are quantized directly, we have the following result.

Theorem 4.2: Given the system in (27) and a quantization level vector ρ, consider the following auxiliary system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k)\\ v(k) &= (I + \Delta(k))y(k) \end{aligned} \tag{54}$$

where $|\Delta_j(k)| \le \delta_j$ for all $j = 1,2,\ldots,m$ and k, and δ_j are converted from ρ_j using (8), and $v(k)$ is the output available for feedback. Suppose the auxiliary system is quadratically stablizable, then (27) is quadratically stabilizable via (29). Conversely, suppose the system (27) is quadratically stabilizable via (29) and, in addition, suppose $\ln\rho_i/\ln\rho_j$ are irrational numbers for all $i \neq j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (54) with $|\Delta_j(k)| \le \delta_j - \epsilon$ is quadratically stabilizable.

Furthermore, the auxiliary system is quadratically stabilizable if the following state feedback H_∞ control has a solution $H(z)$ for some diagonal scaling matrix $\Gamma > 0$:

$$\|\Lambda\Gamma(I - G(z)H(z))^{-1}G(z)H(z)\Gamma^{-1}\|_\infty < 1 \tag{55}$$

where Λ is given in (43). In particular, any $H(z)$ that renders (42) is a solution to either quadratic stabilization problem.

Proof: The "equivalence" between the quantized feedback problem and the quadratic stabilization problem for the auxiliary system (54) follows from Lemma 4.3. The proof for the relation to H_∞ optimization is similar to the proof of Theorem 3.2. □

V. Quantized Quadratic Performance Control

The purpose of this section is to extend the results in the previous sections to include a quadratic performance objective.

Consider the system in (27). Suppose the output $y(k)$ needs to be quantized. We now want to design a controller in (29) such that the following performance cost function:

$$\begin{aligned} J(x(0)) &= \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k)\\ Q &= Q^T \ge 0 \quad R = R^T > 0 \end{aligned} \tag{56}$$

is minimized in the sense that follows:

$$\min EJ(x_0). \tag{57}$$

In this, $x(0)$ is assumed to be a white noise with covariance $Ex(0)x^T(0) = \sigma^2 I$ for some $\sigma > 0$.

Because the state of the closed-loop system is $\bar{x}(k)$, we may rewrite the performance cost as

$$J(\bar{x}(0)) = \sum_{k=0}^{\infty} \bar{x}^T(k)\bar{Q}\bar{x}(k) + u(k)^TRu(k) \tag{58}$$

where

$$\bar{x}(0) = \begin{bmatrix} x(0)\\ 0 \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} Q & 0\\ 0 & 0 \end{bmatrix}. \tag{59}$$

Suppose we want the closed-loop system to be quadratically stable. Let $V(\bar{x}) = \bar{x}^T \bar{P} \bar{x}$, $\bar{P} = \bar{P}^T > 0$, be the associated Lyapunov function. Define

$$\nabla V(\bar{x}(k)) = V(\bar{x}(k+1)) - V(\bar{x}(k)). \quad (60)$$

Then, using (30), the performance cost is given by

$$\begin{aligned} J(\bar{x}(0)) &= \bar{x}^T(0)\bar{P}\bar{x}(0) + \sum_{k=0}^{\infty} \nabla V(\bar{x}(k)) \\ &\quad + \bar{x}^T(k)\bar{Q}\bar{x}(k) + u(k)^T R u(k) \\ &= \bar{x}^T(0)\bar{P}\bar{x}(0) + \sum_{k=0}^{\infty} \bar{x}^T(k)\bar{\Omega}(\Delta(y(k)))\bar{x}(k) \end{aligned} \quad (61)$$

where

$$\begin{aligned} \bar{\Omega}(\Delta) &= \mathcal{A}(\Delta)^T \bar{P} \mathcal{A}(\Delta) - \bar{P} + \bar{Q} \\ &\quad + (\bar{C} + \hat{I}\Delta)\hat{C})^T \bar{K}^T \hat{I} R \\ &\quad \times \hat{I}^T \bar{K}(\bar{C} + \hat{I}\Delta)\hat{C}). \end{aligned} \quad (62)$$

For the case without quantization, i.e., $\Delta(\cdot) = 0$, it is well-known (and easy to see from above) that the optimal solution for $\bar{K}$ is such that $\bar{x}^T(k)\bar{\Omega}(0)\bar{x}(k) = 0$ for all k, which leads to $J(\bar{x}(0)) = \bar{x}^T(0)\bar{P}\bar{x}(0)$ and minimization of $\mathrm{tr}\bar{P}$. In the presence of the quantizer, we can formulate the performance control problem as follows: Given a performance bound $\gamma > 0$ and $\rho > 0$, find $\bar{P}$, $\bar{K}$, if exist, such that

$$\mathrm{tr}(\bar{P}) < \gamma \quad (63)$$

subject to

$$\bar{x}^T \bar{\Omega}(\Delta(\hat{C}\bar{x}))\bar{x} < 0 \qquad \forall\, \bar{x} \neq 0. \quad (64)$$

This problem will be called *quantized quadratic performance control (QQPC)* problem. The solution to this problem is related to the so-called *guaranteed-cost control (GCC)* problem for the auxiliary system (27) and (54), i.e., we want to find $\bar{P}$, $\bar{K}$ such that (63) holds subject to

$$\bar{\Omega}(\Delta) < 0 \qquad \forall\, |\Delta_j| \leq \delta_j \quad (65)$$

where δ_j and ρ_j are related by (8).

Theorem 5.1: Consider the system in (27), the performance cost in (56), the controller structure in (29), some performance bound $\gamma > 0$ and quantization level vector $0 < \rho < 1$. Suppose the GCC problem has a solution. Then, there exists a solution to the QQPC problem. Conversely, if the QQPC problem has a solution and in addition (when $m > 1$), $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \neq j$, then, given any (arbitrarily small $\epsilon > 0$), the GCC problem for (65) has a solution for $|\Delta_j(k)| \leq \delta_j - \epsilon$.

Proof: The proof is similar to that of Theorem 4.1. The key is to show the relationship between (64) and (65). Obviously, (65) implies (64). The fact that (64) implies (65) but with $|\Delta_j| \leq \delta_j - \epsilon$ is proved using Lemma 4.3. The details are omitted here. □

When quantized state feedback is used instead, we have the following result.

Theorem 5.2: Consider the system (1) with $B \in \mathbf{R}^{n \times m}$ and quantized state feedback as in (2)–(3), where $f(\cdot) = [f_1(\cdot), \ldots, f_m(\cdot)]^T$ with given quantization levels $0 < \rho_1, \ldots, \rho_m < 1$. Given the performance cost function in (56) and a performance bound $\gamma > 0$, the QQPC problem becomes to finding $P = P^T > 0$ and K, if exist, such that

$$\mathrm{tr}P < \gamma \quad (66)$$

subject to

$$x^T \Omega(\Delta(v)) x < 0 \qquad \forall\, x \neq 0 \quad (67)$$

where $v = Kx$ and

$$\begin{aligned} \Omega(\Delta) = (A + B(I+\Delta)K)^T P (A + B(I+\Delta)K) \\ -P + Q + K^T(I+\Delta)R(I+\Delta)K. \end{aligned} \quad (68)$$

The related GCC problem becomes to finding $P = P^T > 0$ and K, if exist, such that (66) holds subject to

$$\Omega(\Delta) < 0 \qquad \forall\, |\Delta_j| \leq \delta_j. \quad (69)$$

Further, the GCC problem has a solution if the following LMIs:

$$\mathrm{tr}\tilde{P} < \gamma \quad \begin{bmatrix} -\tilde{P} & I \\ I & -S \end{bmatrix} \leq 0 \quad (70)$$

$$\begin{bmatrix} -S & * & * & * & * \\ AS + BW & -S + B\Lambda\Gamma\Lambda B^T & * & * & * \\ W & \Lambda\Gamma\Lambda B^T & -\tilde{R} & * & * \\ W & 0 & 0 & -\Gamma & * \\ Q^{1/2}S & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (71)$$

have a solution for some $\tilde{P} = \tilde{P}^T$, $S = S^T$, W and a diagonal scaling matrix Γ, where $\tilde{R} = R^{-1} - \Lambda S \Lambda$, Λ is given in (43), and * denotes the symmetric part in the matrix. Also, P and K are related to S and W as follows:

$$P = S^{-1} \quad K = WP. \quad (72)$$

Proof: The relationship between the QQPC and GCC problems is easy to check. We proceed to verify (71) as a sufficient condition for the GCC problem. Indeed, (69) holds if and only if

$$\begin{bmatrix} -P + Q & * & * \\ A + B(I+\Delta)K & -P^{-1} & * \\ (I+\Delta)K & 0 & -R^{-1} \end{bmatrix} < 0 \quad (73)$$

for all $|\Delta_j| \leq \delta_j$. Using (72), this becomes

$$\begin{bmatrix} -S + SQS & * & * \\ AS + B(I+\Delta)W & -S & * \\ (I+\Delta)W & 0 & -R^{-1} \end{bmatrix} < 0 \quad (74)$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} -S + SQS & * & * \\ AS + BW & -S & * \\ W & 0 & -R^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ I \end{bmatrix} \Delta [W\ 0\ 0] \\ + \begin{bmatrix} W^T \\ 0 \\ 0 \end{bmatrix} \Delta [0\ B^T\ I] < 0. \end{aligned} \quad (75)$$

Taking $\Gamma > 0$ to be any diagonal scaling matrix, (75) holds if

$$\begin{bmatrix} -S+SQS & * & * \\ AS+BW & -S & * \\ W & 0 & -R^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ B\Lambda \\ \Lambda \end{bmatrix} \Gamma[0\ \Lambda B^T\ \Lambda] + \begin{bmatrix} W^T \\ 0 \\ 0 \end{bmatrix} \Gamma^{-1}[W\ 0\ 0] < 0 \quad (76)$$

which is equivalent to (71) using Schur complement. □

In the single-input case, we have a better solution that follows.

Corollary 5.1: In the single-input case, the aforementioned QQPC problem for a given performance bound γ and quantization density $0 < \rho < 1$ has a solution if and only if there exists a solution $(\tilde{P}, S)$ to (70) and

$$\begin{bmatrix} S & SA^T & SA^T & SQ^{1/2} \\ AS & (1-\delta^2)^{-1}(S+BR^{-1}B^T) & 0 & 0 \\ AS & 0 & \delta^{-2}S & 0 \\ Q^{1/2}S & 0 & 0 & I \end{bmatrix} > 0. \quad (77)$$

In this situation, the solution to (P, K) is given by

$$P = S^{-1} \quad K = -(R + B^T PB)^{-1} B^T PA. \quad (78)$$

Proof: In view of Theorem 5.2, it suffices to show that $\Omega(\Delta) < 0$ for all $|\Delta| \le \delta$ if and only if (77)–(78) hold. It is straightforward to verify that $\Omega(\Delta) < 0$ can be rewritten as

$$M - \left[(1+\Delta)K + R_1^{-1}B^T PA\right]^T R_1 \cdot \left[(1+\Delta)K + R_1^{-1}B^T PA\right] > 0$$

where $R_1 = R + B^T PB$ and

$$M = P - A^T PA - Q + A^T PBR_1^{-1}B^T PA. \quad (79)$$

By the Schur complement, the previous inequality is equivalent to

$$\begin{bmatrix} M & (K + R_1^{-1}B^T PA)^T \\ K + R_1^{-1}B^T PA & R_1^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta [K \quad 0] + \begin{bmatrix} K^T \\ 0 \end{bmatrix} \Delta [0 \quad 1] > 0. \quad (80)$$

By applying the S-procedure [25], the above is equivalent to

$$\begin{bmatrix} M & K^T + A^T PBR_1^{-1} & K^T \\ K + R_1^{-1}B^T PA & R_1^{-1} - \delta^2\tau & 0 \\ K & 0 & \tau \end{bmatrix} > 0. \quad (81)$$

Applying the Elimination lemma to remove K, the above is equivalent to $\tau > 0$, $R_1^{-1} - \tau\delta^2 > 0$ and

$$\begin{bmatrix} M & A^T PBR_1^{-1} \\ R_1^{-1}B^T PA & R_1^{-1} + \tau(1-\delta^2) \end{bmatrix} > 0.$$

The optimal scaling is such that $\tau \to \delta^{-2}R_1^{-1}$. Hence, the previous conditions is the same as

$$\begin{bmatrix} M & A^T PBR_1^{-1} \\ R_B^{-1}B^T PA & \delta^{-2}R_1^{-1} \end{bmatrix} > 0. \quad (82)$$

Returning to (81), $\tau \to \delta^{-2}R_1^{-1}$ implies the solution for K in (78). Using (79) and Schur complement, (82) is the same as

$$P - A^T PA - Q + (1-\delta^2)A^T PBR_1^{-1}B^T PA > 0 \quad (83)$$

or yet

$$P - \delta^2 A^T PA - Q - (1-\delta^2)A^T(P^{-1} + BR^{-1}B^T)^{-1}A > 0.$$

Multiplying this from the left and right by S and applying Schur complement, we obtain (77). □

Remark 5.1: It is clear from (83) that if for some $\delta = \delta^* > 0$, (77) has a solution $S > 0$, then for any $0 < \delta \le \delta^*$, (77) admits a solution as well. Hence, the largest δ or the coarsest quantization density for a given performance γ can be obtained by maximizing δ subject to (70) and (77).

VI. Quantized H_∞ Control

Here, we extend the quantization results to H_∞ control. For simplicity, only quantized state feedback is considered. This problem in the single-input setting has been studied in [20]. Our main purpose is to show that the sector bound approach can be easily generalized to quantized feedback H_∞ control.

The system of interest is as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_1 w(k) \\ z(k) &= Cx(k) + Du(k) + D_1 w(k) \end{aligned} \quad (84)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $w \in \mathbf{R}^{m_1}$, $z \in \mathbf{R}^\ell$ and the control signal is in the form of (2)–(3). Given a quantization level vector ρ and H_∞ performance bound $\gamma > 0$, the design objective is to find K such that the induced L_2-gain from w to z is less than γ.

It is easy to verify that the closed-loop system is given by

$$\begin{aligned} x(k+1) &= [A + B(I + \Delta(v))K]x(k) + B_1 w(k) \\ z(k) &= [C + D(I + \Delta(v))K]x(k) + D_1 w(k). \end{aligned} \quad (85)$$

As in the quadratic performance control problem, we consider the following relaxed H_∞ control problem which will be called *quantized H_∞ performance control* (QHPC) problem: Find $P = P^T > 0$ and K such that

$$x^T \Pi(\Delta(Kx))x < 0 \qquad \forall\, x \neq 0 \quad (86)$$

where

$$\begin{aligned} \Pi(\Delta) = {} & A_\Delta^T PA_\Delta - P + \gamma^{-2}(A_\Delta^T PB_1 + C_\Delta^T D_1) \\ & \times [I - \gamma^{-2}(D_1^T D_1 + B_1^T PB_1)]^{-1} \\ & \times (B_1^T PA_\Delta + D_1^T C_\Delta) + C_\Delta^T C_\Delta \end{aligned} \quad (87)$$

$$A_\Delta = A + B(I+\Delta)K \quad C_\Delta = C + D(I+\Delta)K \quad (88)$$

and $\gamma^2 I > D_1^T D_1 + B_1^T PB_1$.

The motivation for formulating the previous problem is as follows: When $\Delta = 0$, $\Pi(0) < 0$ is a necessary and sufficient condition for the H_∞ norm of the transfer function from w to z to be less than γ. When $\gamma \to \infty$, (86) recovers the condition for stabilization. When γ is finite and Δ represents a sector bound uncertainty, the condition $\Pi(\Delta) < 0$ corresponds to the *robust H_∞ control problem* studied in [23].

Theorem 6.1: Consider the given system (84), controller structure (2)–(3), quantization level vector ρ and a H_∞ performance bound γ. Suppose there exist $P = P^T > 0$ and K such that (86) holds, then the induced L_2-norm from w to z is less than γ.

Further, for any $P = P^T > 0$ and K, (86) holds if $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j$, where δ_j are related to ρ_j by (6). Conversely, if (86) holds, $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j - \epsilon$, where $\epsilon > 0$ is arbitrarily small.

In addition, there exist $P = P^T > 0$ and K such that $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j$ if $\gamma^2 I > D_1^T D_1$ and the following LMI:

$$\begin{bmatrix} -S & * & * & * \\ \bar{A}S + \bar{B}W & -S_1 & * & * \\ C + DW & D\Gamma\Lambda^2\bar{B}^T & -\gamma\bar{R}_1^{-1} + D\Gamma\Lambda^2 D^T & * \\ W & 0 & 0 & -\Gamma \end{bmatrix} < 0 \tag{89}$$

has a solution for $S = S^T, W$ and diagonal scaling matrix Γ, where

$$\begin{aligned} S_1 &= S - \gamma^{-1}\bar{B}_1\bar{B}_1^T - \bar{B}\Gamma\Lambda^2\bar{B}^T \\ \bar{A} &= A + \gamma^{-2}\bar{B}_1\bar{D}_1^T C \quad \bar{B} = B + \gamma^{-2}\bar{B}_1\bar{D}_1^T D \\ \bar{B}_1 &= B_1(I - \gamma^{-2}D_1^T D_1)^{-(1/2)} \\ \bar{D}_1 &= D_1(I - \gamma^{-2}D_1^T D_1)^{-(1/2)} \\ \bar{R}_1 &= I + \gamma^{-2}\bar{D}_1\bar{D}_1^T \end{aligned}$$

and Λ and the relationship between (S, W) and (P, K) are the same as in Theorem 5.2. In the single-input case, the LMI in (89) is also necessary.

Proof: The relationship between quantized H_∞ control and robust H_∞ control can be checked as before. We now show that the existence of a solution to (89) provides a sufficient condition for quantized H_∞ control. First, it is straightforward to verify that (87) can be rewritten as

$$\hat{A}_\Delta^T P \hat{A}_\Delta - P + C_\Delta^T \bar{R}_1 C_\Delta + \gamma^{-2}\hat{A}_\Delta^T P B_1 \cdot \left[I - \gamma^{-2}(D_1^T D_1 + B_1^T P B_1)\right]^{-1} B_1 P \hat{A}_\Delta < 0 \tag{90}$$

where $\hat{A}_\Delta = \bar{A} + \bar{B}(I + \Delta)K$. Applying the matrix inversion lemma, (90) can be rewritten as

$$\hat{A}_\Delta^T \left(P^{-1} - \gamma^{-2}\bar{B}_1\bar{B}_1^T\right)^{-1} \hat{A}_\Delta - P + C_\Delta^T \bar{R}_1 C_\Delta < 0.$$

Setting $S = \gamma P^{-1}$ and applying Schur complement, the above is equivalent to

$$\begin{bmatrix} -S^{-1} & * & * \\ \bar{A} + \bar{B}K & -S + \gamma^{-1}\bar{B}_1\bar{B}_1^T & * \\ C + DK & 0 & -\gamma\bar{R}_1^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B} \\ D \end{bmatrix} \Delta[K\ 0\ 0] + \begin{bmatrix} K^T \\ 0 \\ 0 \end{bmatrix} \Delta^T[0\ \bar{B}^T\ D^T] < 0. \tag{91}$$

Using the S-procedure [25], the above holds when

$$\begin{bmatrix} -S^{-1} & * & * & * \\ \bar{A} + \bar{B}K & -S_1 & * & * \\ C + DK & D\Gamma\Lambda^2\bar{B}^T & -\gamma\bar{R}_1^{-1} + D\Gamma\Lambda^2 D^T & * \\ K & 0 & 0 & \Gamma \end{bmatrix} < 0 \tag{92}$$

for some positive diagonal scaling matrix Γ. Note that in the single-input case, Γ is a scalar and the conversion above is also necessary [25]. Multiplying (92) from the left and the right by $\mathrm{diag}\{S, I, I, I\}$ and noting $W = KS$, we obtain (89). □

Corollary 6.1: As $\gamma \to \infty$, the LMI condition (89) is equivalent to the condition (42) in Theorem 4.1.

Proof: When $\gamma \to \infty$, $\bar{A} \to A$, $\bar{B} \to B$, $\bar{R}_1 \to I$ and (89) implies

$$\begin{bmatrix} -S & * & * \\ AS + BW & -S + B\Lambda\Gamma\Lambda B^T & * \\ W & 0 & -\Gamma \end{bmatrix} < 0.$$

By Schur complement and letting $K = WS^{-1}$, we have

$$(A + BK)^T(S - B\Lambda\Gamma\Lambda B^T)^{-1}(A + BK) - S^{-1} + K^T\Gamma^{-1}K < 0.$$

This is equivalent to [21]

$$\|\Gamma^{-1/2}K(zI - A - BK)^{-1}B\Lambda\Gamma^{1/2}\|_\infty < 1.$$

Letting $\Gamma_1^{-1} = \Gamma^{1/2}\Lambda$ and noting that both Λ and Γ are diagonal matrices, the above can be rewritten as

$$\|\Lambda\Gamma_1 K(zI - A - BK)^{-1}B\Gamma_1^{-1}\|_\infty < 1$$

which is (42) in Theorem 4.1. □

Remark 6.1: As we mentioned earlier, the quantized H_∞ control problem has been studied in [20]. We now comment on the connection between Theorem 6.1 and a related result in [20] (Theorem 5.1: Discrete-time). The problem formulation in [20] is more restrictive because it treats the single input case and assumes $C^T D = 0$, $D^T D = I$ and $D_1 = 0$. The coarsest quantization density ρ given in [20] can be written as

$$\rho = \frac{\alpha - 1}{\alpha + 1}$$

where α is the optimal solution to the following problem:

$$\inf \alpha \text{ subject to } \Sigma(X) > \Sigma_0 \quad \alpha > 1, \ \ X = X^T > 0 \tag{93}$$

where

$$\Sigma(X) = \begin{bmatrix} \alpha^2 X - AXA^T & -AXA^T & AXC^T \\ -AXA^T & \frac{\alpha^2}{\alpha^2 - 1}X - AXA^T & AXC^T \\ CXA^T & CXA^T & -CXC^T \end{bmatrix}$$
$$\Sigma_0 = \begin{bmatrix} \alpha^2 B_1 B_1^T & 0 & 0 \\ 0 & \frac{\alpha^2}{\alpha^2 - 1}(B_1B_1^T - \gamma^2 B_2 B_2^T) & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix}.$$

It can be shown that, by setting $\alpha = \delta^{-1}$, this condition is equivalent to the condition in Theorem 6.1, when specialized under the assumptions on C, D and D_1 (The proof is similar to that for Corollary 5.1). That is, Theorem 6.1 generalizes the result in [20]. Moreover, we provide a clear interpretation of the result in the single-input case, i.e., the condition in (89), or $\Sigma(x) > \Sigma_0$, is necessary and sufficient for the QHPC problem.

VII. Conclusion

We have shown that the classical sector bound method can be used to study quantized feedback control problems in a nonconservative manner. Various cases have been considered: quantized state feedback control, quantized output feedback control, MIMO systems, and control with performances. In all these problems, the key result is that quantization errors can be converted into sector bound uncertainties without conservatism. By doing so, quantized feedback control problems become well-known robust control problems.

For quadratic stabilization of SISO systems (using either quantized state feedback or quantized output feedback), complete solutions are available by solving related H_∞ optimization problems. For MIMO systems or SISO systems with a performance control objective, the resulting robust control problems usually do not have simple solutions, thus sufficient conditions on quantization densities are derived. These conditions are expressed either in terms of H_∞ optimization or linear matrix inequalities. Note that these conditions are for a given set of quantization densities. However, because these conditions are convex in the sector bounds associated with the quantization densities, optimal quantization densities can be easily computed numerically.

Finally, we note that the use of the sector bound method also explains why it is difficult to find the coarsest quantization densities in the cases of MIMO stabilization and/or performance control problems. More precisely, the difficulties are the same as finding nonconservative solutions to the related robust control problems, which are known to be very difficult.

Appendix I
Proof of Lemma 2.1

We will prove that the coarsest quantization density can be achieved by taking

$$K = K_m = -\frac{B^TPA}{B^TPB} \tag{94}$$

and the logarithmic quantizer (6) with

$$\rho = \rho_m = \frac{1-\delta_m}{1+\delta_m} \tag{95}$$

where

$$\delta_m = \frac{1}{\sqrt{K_m M^{-1} K_m^T}} < 1 \tag{96}$$

and

$$M = \frac{A^TPBB^TPA - (B^TPB)(A^TPA - P)}{(B^TPB)^2} > 0. \tag{97}$$

To prove this, let the quantized state feedback (2)–(3) quadratically stabilize (1) with $V(x) = x^TPx$ as the associated Lyapunov function. We rewrite (9) as

$$\begin{aligned}\nabla V(x) &= x^T(A^TPA - P)x + 2x^TA^TPBu + B^TPBu^2\\ &= \{-x^TMx + (u - K_mx)^2\}(B^TPB)\end{aligned} \tag{98}$$

where K_m and M are given in (94) and (97), respectively. Noting that A is unstable and (A, B) is stablizable, B must be a nonzero column vector and hence $B^TPB > 0$. Since x^TPx is a Lyapunov function for (1), we must have $M > 0$. It follows from (98) that $\nabla V(x) < 0$ if and only if $u \in (u_1,\ u_2)$ with

$$u_1 = K_mx - \sqrt{x^TMx} \quad u_2 = K_mx + \sqrt{x^TMx}.$$

We can always decompose $M^{1/2}x$ into [16]

$$M^{1/2}x = \alpha M^{-1/2}K_m^T + z \tag{99}$$

where α is a scalar and z is a vector orthogonal to $M^{-1/2}K_m^T$. With this decomposition, we can rewrite u_1 and u_2 as

$$\begin{aligned}u_1 &= u_1(\alpha, z) = \alpha K_mM^{-1}K_m^T - \sqrt{\alpha^2K_mM^{-1}K_m^T + z^Tz};\\ u_2 &= u_2(\alpha, z) = \alpha K_mM^{-1}K_m^T + \sqrt{\alpha^2K_mM^{-1}K_m^T + z^Tz}.\end{aligned}$$

Note that $\delta_m < 1$ because otherwise $0 \in (u_1(\alpha, z), u_2(\alpha, z))$ for any nonzero x, which would imply that $u = 0$ quadratically stabilizes the system (1), violating the assumption that A is unstable.

Let $\mathcal{U}$ in (4) be the set of quantization levels corresponding to the quantized state feedback (2)–(3). Then, for any (α, z) in (99), there exists $u_i \in \mathcal{U}$ such that

$$u_i \in (u_1(\alpha, z), u_2(\alpha, z)). \tag{100}$$

In particular, when $z = 0$, there must exist some $u_i \in \mathcal{U}$ for each α such that

$$u_i \in (u_1(\alpha, 0), u_2(\alpha, 0)). \tag{101}$$

Due to the fact that $(u_1,\ u_2)$ is minimal when $z = 0$, we know that if u_i of $\mathcal{U}$ satisfies (101), it automatically satisfies (100). That is, the quantized state feedback (2)–(3) quadratically stabilizes (1) if and only if the corresponding u_i of $\mathcal{U}$ satisfies (101).

Now for each u_i, we can define the range of α values, (α_i^-, α_i^+) for which (101) holds. For $u_i > 0$, we have

$$\alpha_i^- = \frac{u_i\delta_m^2}{1+\delta_m} \quad \alpha_i^+ = \frac{u_i\delta_m^2}{1-\delta_m}$$

where δ_m is defined in (96). It is obvious that the coarsest selection for u_i, $i = 0, \pm1, \pm2, \ldots$, is to take $\{u_i\}$ such that $\alpha_{i+1}^+ = \alpha_i^-$ for all i. This means

$$\frac{u_{i+1}}{u_i} = \rho_m = \frac{1-\delta_m}{1+\delta_m}. \tag{102}$$

Comparing (102) with (6), we see that the optimal u_i must be logarithmic with $\rho = \rho_m$ when $u_i > 0$. For $u_i < 0$, it can be shown similarly that the same conclusion holds. Therefore, the optimal quantizer must be logarithmic.

It remains to argue that the coarsest quantization density can be approached by taking $v(x) = K_m x$. It suffices to consider the case $K_m x > 0$ as the case $K_m x < 0$ is similar. Let $f(\cdot)$ be any logarithmic quantizer with quantization density $\rho < \rho_m$. By choosing $u_{(0)}$ appropriately, we can guarantee that

$$(1-\delta_m)K_m x < f(K_m x) < (1+\delta_m)K_m x \qquad \forall\, x \in \mathbf{R}^n.$$

Since

$$\begin{aligned} K_m x &= K_m M^{-1/2}(M^{1/2}x) \\ &= K_m M^{-1/2}(\alpha M^{-1/2}K_m^T + z) \\ &= \alpha K_m M^{-1} K_m^T \end{aligned}$$

we know that $\alpha > 0$ when $K_m x > 0$. It follows that

$$\begin{aligned} u_1(\alpha,0) &= \alpha K_m M^{-1} K_m^T - \alpha\sqrt{K_m M^{-1} K_m^T} \\ &= (1-\delta_m)\alpha K_m M^{-1} K_m^T \\ &= (1-\delta_m)K_m x. \end{aligned}$$

Similarly, we have

$$u_2(\alpha,0) = (1+\delta_m)K_m x.$$

Using the fact that

$$(u_1(\alpha,0),\ u_2(\alpha,0)) \in (u_1(\alpha,z), u_2(\alpha,z))$$

we conclude that $f(K_m x)$ quadratically stabilizes (1). Since ρ can be arbitrarily close to ρ_m, the coarsest quantization density can be approached by taking $v(x) = K_m x$. □

APPENDIX II
PROOF OF LEMMA 2.4

Without loss of generality, we take (A, B) to be of the form:

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{103}$$

where $A_s \in \mathbf{R}^{n_1 \times n_1}$ has all its eigenvalues inside the unit disk and $A_u \in \mathbf{R}^{n_2 \times n_2}$ has all its eigenvalues either on or outside the unit circle, and (A_u, B_2) is of a controllable canonical form.

We first claim the following.

Claim 1: Suppose A is unstable and there exists a K such that $A + BK$ is stable and

$$\gamma > \|K(zI - A - BK)^{-1}B\|_\infty. \tag{104}$$

Then, $\gamma > \det(A_u) = \Pi_i|\lambda_i^u|$.

To prove the claim, we first note that $A + BK$ is stable and (104) holds if and only if [21]

$$\begin{aligned} &I - \gamma^{-2}B^T P B > 0; \quad (A+BK)^T P(A+BK) - P \\ &\quad + \gamma^{-2}(A+BK)^T PB \cdot (I - \gamma^{-2}B^T PB)^{-1} \\ &\quad \times B^T P(A+BK) + K^T K < 0 \end{aligned}$$

for some $P = P^T > 0$. The two inequalities shwon previously are equivalent to

$$\begin{aligned} P^{-1} - \gamma^{-2}BB^T > 0; (A+BK)^T(P^{-1} - \gamma^{-2}BB^T)^{-1} \\ \times (A+BK) - P + K^T K < 0. \end{aligned}$$

Defining $Q = (P^{-1} - \gamma^{-2}BB^T)^{-1}$, the last two previous inequalities become $Q > 0$ and

$$(A+BK)^T Q(A+BK) - P + K^T K < 0. \tag{105}$$

Denoting

$$\begin{aligned} \Phi &= (B^T QB + I)^{-1}B^T QA + K; \\ \Pi &= \Phi^T(B^T QB + I)\Phi \geq 0 \end{aligned}$$

the inequality (105) can be rewritten as

$$A^T QA - A^T QB(B^T QB + I)^{-1}B^T QA - P + \Pi < 0.$$

Using $\Pi \geq 0$ and

$$(Q^{-1} + BB^T)^{-1} = Q - QB(B^T QB + I)^{-1}B^T Q$$

it follows that

$$A^T(Q^{-1} + BB^T)^{-1}A - P < 0.$$

Using Schur complement, the above is equivalent to

$$AP^{-1}A^T - (Q^{-1} + BB^T) < 0.$$

Using $Q^{-1} = P^{-1} - \gamma^{-2}BB^T$ and defining $X = P^{-1}$, we get $X > \gamma^{-2}BB^T$ and

$$AXA^T < X + (1 - \gamma^{-2})BB^T. \tag{106}$$

Note that if A is stable, the above inequality exists a solution $X = X^T > \gamma^{-2}BB^T$ for any $\gamma > 0$.

Now, let X be partitioned in conformity with (103)

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^T & X_2 \end{bmatrix}.$$

Then, (106) with $X > \gamma^{-2}BB^T$ implies

$$A_u X_2 A_u^T - X_2 - (1 - \gamma^{-2})B_2 B_2^T < 0 \tag{107}$$

and $X_2 > \gamma^{-2}B_2 B_2^T$.

Now, using the fact that for any two symmetric matrices U and V with $0 \leq U < V$, $\det(U) < \det(V)$, (107) leads to

$$\det(A_u X_2 A_u^T) < \det(X_2 + (1 - \gamma^{-2})B_2 B_2^T).$$

Since $B_2 = [0\ 0\ \cdots\ 1]^T$, it follows that

$$|\det(A_u)|^2 \det(X_2) < \det(X_2) + (1 - \gamma^{-2})\det(X_{2,(n_2-1)}) \tag{108}$$

where $X_{2,(n_2-1)}$ is the upper left $(n_2 - 1) \times (n_2 - 1)$ block of X_2. Also, since $X_2 > \gamma^{-2}B_2 B_2^T$, we have

$$\det(X_2) - \gamma^{-2}\det(X_{2,(n_2-1)}) > 0$$

or, equivalently

$$\det(X_{2,(n_2-1)}) < \gamma^2 \det(X).$$

Substituting this into (108) and noting that $\det(X_2) > 0$, we have

$$|\det(A_u)|^2 < 1 + (1-\gamma^{-2})\gamma^2 = \gamma^2.$$

Thus, we have verified Claim 1.

We now show that the solution to (20) is indeed given by (12). To this end, we consider three cases.

Case 1) $a(z)$ is strictly anti-stable. In this case, simply take (A, B) be of a controllable canonical form, it is clear that

$$G_c(z) = \frac{k(z)}{a(z) - k(z)} \tag{109}$$

where $a(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n = |zI - A|$ and $k(z) = k_0 + k_1 z + \cdots + k_{n-1}z^{n-1}$ is the control polynomial.

We claim that choosing

$$K = \frac{a_0^2}{a_0^2 - 1}\left[a_0 - \frac{1}{a_0}, a_1 - \frac{a_{n-1}}{a_0}, \ldots, a_{n-1} - \frac{a_1}{a_0}\right] \tag{110}$$

leads to $\|G_c(z)\|_\infty = |a_0| = \prod_i |\lambda_i^u| > 1$. This together with Claim 1) implies that the solution in (110) is the optimal solution. The second claim above holds because (110) comes from solving the all-pass requirement for $G_c(z)$

$$a(z) - k(z) = \alpha z^n k(z^{-1}) \tag{111}$$

for some α. Replacing z by z^{-1}, (111) becomes

$$a(z^{-1}) - k(z^{-1}) = \alpha z^{-n} k(z). \tag{112}$$

Combining (111)–(112) yields

$$k(z) = \frac{a(z) - \alpha z^n a(z^{-1})}{1 - \alpha^2}. \tag{113}$$

Setting the nth order coefficient of $k(z)$ to zero results in $\alpha = 1/a_0$. It is straightforward to verify that (113) is the same as (110). It remains to show that $G_c(z)$ is stable, which is the same as showing that $k(z)$ is strictly antistable. To see this, we rewrite (113) as

$$k(z) = \frac{a(z)}{1-\alpha^2}\left(1 - \alpha\frac{z^n a(z^{-1})}{a(z)}\right).$$

Because $a(z)$ is antistable, $|\alpha| < 1$ and $|z^n a(z^{-1})/a(z)| \le 1$ for any $|z| \le 1$, $k(z) \ne 0$ for any $|z| \le 1$. Hence, $k(z)$ is strictly antistable.

Case 2) $a(z)$ is marginally anti-stable. In this case, we first replace $a(z)$ by $\tilde{a}(z)$ which is a strictly antistable polynomial obtained from $a(z)$ by slightly perturbing the marginal zeros. From Case 1), we can choose $k(z)$ such that $\|\tilde{G}_c(z)\|_\infty = |\tilde{a}_0|$, where $\tilde{a}_0$ and $\tilde{G}_c(z)$ are the perturbed versions of a_0 and $G_c(z)$. By the continuity of $\tilde{a}_0$ (with respect to the perturbation), it is clear that $\inf_K \|G_c(z)\| = |a_0|$.

Case 3) $a(z)$ has a stable factor. In this case, we can write $a(z) = a_s(z)a_u(z)$, where $a_s(z)$ and $a_u(z)$ are the stable and unstable factors. Taking $k(z) = a_s(z)k_u(z)$ yields

$$G_c(z) = \frac{k_u(z)}{a_u(z) - k_u(z)}.$$

Then, we have reverted to Case 2). Again, we obtain (12). □

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