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Brief Paper

# The Edge Theorem and Graphical Tests for Robust Stability of Neutral Time-delay Systems*

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**Key Words**—Robust stability; time-delay systems; robustness; uncertain systems; parametric perturbation.

**Abstract**—We consider the robust stability problem for a class of uncertain neutral time-delay systems where the characteristic equations involve a polytope $\mathcal{P}$ of quasipolynomials of neutral type. Given a stability region $D$ in the complex plane our goal is to find a constructive technique to verify the $D$-stability of $\mathcal{P}$ (i.e. to verify whether the roots of every quasipolynomial in $\mathcal{P}$ all belong to $D$). We first show that, under a certain assumption on the stability region $D$, $\mathcal{P}$ is $D$-stable if and only if the edges of $\mathcal{P}$ are $D$-stable. Hence, the $D$-stability problem of a higher dimensional polytope is reduced to the $D$-stability problem of a finite number of pairwise convex combinations of vertices. Based on this result, we then give an effective graphical test for checking the $D$-stability of a polytope of quasipolynomials of neutral type

## 1. *Introduction*

THE GENERAL problem of robust stability can be roughly formulated as follows: Given a family of linear systems $\mathcal{S}$ and a set $D$ in the complex plane, provide computationally tractable techniques for determining the $D$-stability of $\mathcal{S}$, i.e. checking whether the eigenvalues of the systems in $\mathcal{S}$ stay within $D$. The most pertinent results to the problem we are addressing here are those by Bartlett *et al.* (1988), Fu and Barmish (1989) and a recent paper by the authors (Fu *et al.*, 1989). Bartlett *et al.* 1988 presented what is now widely known as the "Edge Theorem." In Fu *et al.* (1989), this edge Theorem was generalized to handle the $D$-stability problem for a class of uncertain delay systems, and an effective graphical test was proposed. One fundamental assumption of Fu *et al.* (1989) is that the characteristic equation of the time-delay system should not have a neutral term. This assumption was technically needed so that the leading coefficient (see Section 2 for definition) does not vanish. But it fails for the interesting class of systems called neutral time delay systems (Hale, 1977; Kolmanovski and Nosov, 1986). These systems are analogous to such distributed parameter systems as the undamped wave equation and the lossless transmission line. In fact, the solution to the classical wave equation in one space dimension can be shown to satisfy a neutral differential-difference equation; see Kolmanovski and Nosov (1986 Chapter 1).

In this paper, we consider the $D$-stability problem for the polytope of quasipolynomials associated with an uncertain neutral time-delay system. Under a mild assumption on the set $D$, we prove that the Edge Theorem is also valid. That is, the polytope of quasipolynomials $\mathcal{P}$ of neutral type is $D$-stable if and only if the edges of $\mathcal{P}$ are $D$-stable. In addition, when the set $D$ is any open left half plane we provide a simple test which permits us to determine whether the neutral term introduces instability. More specifically, in order for $\mathcal{P}$ to be $D$-stable, it is necessary that the subpolytope $\mathcal{P}_N$ corresponding to the neutral terms is $D$-stable. Since this subpolytope is usually of much lower dimension than $\mathcal{P}$, it is advantageous to perform the simple test first because the failure of this test will eliminate further computation. Furthermore, both $\mathcal{P}$ and $\mathcal{P}_N$ can be examined by using the Edge Theorem and a graphical test similar to that given in (Fu *et al.*, 1989).

We consider the class of neutral time-delay systems described by

$$\sum_{i=0}^{l} F_i\dot{x}(t-\tau_i) - \sum_{i=0}^{l} A_i x(t-\tau_i) = 0 \qquad (1)$$

where the trajectory vector $x(t) \in \mathbf{R}^n$, $A_i$ and $F_i$ are real (or complex) system matrices with $F_0$ nonsingular, and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_l$ represent the time delays. The characteristic equation of (1) is given using an $n$th order quasipolynomial of the form

$$p(s) \doteq \det\left(\sum_{i=0}^{l} e^{-\tau_i s}(sF_i - A_i)\right) = 0$$

where $p(s)$ can be written as

$$p(s) = \sum_{i=0}^{n}\left(\sum_{k=0}^{N} a_{ik}e^{-h_k s}\right)s^{n-i} \qquad (2)$$

and $a_{ik} = \alpha_{ik} + j\beta_{ik}$; $\alpha_{ik}$, $\beta_{ik} \in \mathbf{R}$ are constants, $a_{00} \neq 0$ is called the leading coefficient, and $0 = h_0 < h_1 < h_2 < \cdots < h_N$ are integer combinations of $\tau_i$. The term $\sum_{k=0}^{N} a_{0k}e^{-h_k s}$ is called the neutral term.

*Definition* 1. Given a set $D$ in the complex plane, the delay system (1) is called $D$-stable if the zeros of the characteristic quasipolynomial $p(s)$ in (2) stay in $D$. If so, $p(s)$ is called $D$-stable. In particular, $p(s)$ is called stable if there exists some $\epsilon > 0$ such that $p(s)$ is $D$-stable for $D = \{s : \mathrm{Re}\,(s) < -\epsilon\}$. [The latter case corresponds to exponential stability of solutions to (1) with suitable initial functions; (see Bellman and Cooke (1963) and Hale (1977).] □

Suppose the coefficients of $p(s)$ in (2) involve uncertain parameters, then it is of interest to determine the $D$-stability of the system for all admissible parameter perturbations.

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Mathematically, we consider a family of $n$th order (real or complex quasipolynomials

$$\mathscr{P} \doteq \left\{p(s) = \sum_{i=0}^{n}\left(\sum_{k=0}^{N} a_{ik}e^{-h_k s}\right)s^{n-i}: (a_{00}, a_{01}, \ldots, a_{0N}, \ldots, a_{nN}) \in \mathscr{F}\right\}, a_{00} \neq 0 \quad (3)$$

for some $\mathscr{F} \in \mathscr{S}^{(n+1)(N+1)}$ characterizing the parameter perturbations. Given a set $D$ in the complex plane, we want to determine the $D$-stability of $\mathscr{P}$, i.e. whether $p(s)$ is $D$-stable for all $p(s) \in \mathscr{P}$. For the family of quasipolynomials $\mathscr{P}$ in (3), we define the subpolytope of neutral terms

$$\mathscr{P}_N \doteq \left\{\sum_{k=0}^{N} a_{0k}e^{-h_k s}: (a_{00}, a_{01}, \ldots, a_{0N}, \ldots, a_{nN}) \in \mathscr{F}\right\}. \quad (4)$$

In this paper, we consider a special family of quasipolynomials for which $\mathscr{P}$ is a polytope generated by the convex combinations of a number of $n$th order quasipolynomials $p_1(s), p_2(s), \ldots, p_r(s)$ as in (2), i.e.

$$\mathscr{P} \doteq \operatorname{conv}\{p_1(s), p_2(s), \ldots, p_r(s)\} \quad (5)$$

and for which every member of $\mathscr{P}$ does not have vanishing leading coefficient $a_{00}$.

We denote by $E[X]$ the set of all edges of a polytope $X$; recall that an edge of a polytope is its one-dimensional face (Brondsted, 1983). The end points of an edge are called vertices.

*Remark* 1. The requirement that the leading coefficient $a_{00}$ of every member of $\mathscr{P}$ does not vanish is equivalent to the assumption that the set of the leading coefficients of the generators $p_i(s)$ are on one side of some line through the origin in the complex plane. For the case of real parameters, this requires that the leading coefficients of $p_i(s)$ are of the same sign. □

For the $n$th order quasipolynomial $p(s)$ given in (2), we denote its coefficient vector by

$$\mathbf{p} = [\alpha_{00}\,\beta_{00}\,\alpha_{01}\,\beta_{01}\cdots\alpha_{0N}\cdots\alpha_{nN}\,\beta_{nN}]^T. \quad (6)$$

Then, it is straightforward to show that $s$ is a zero of $p(s)$ if and only if

$$K(s)\mathbf{p} = 0$$

where

$$K(s) \doteq \begin{bmatrix} \operatorname{Re}(e^{-h_0 s}s^n) & \operatorname{Im}(e^{-h_0 s}s^n) \\ -\operatorname{Im}(e^{-h_0 s}s^n) & \operatorname{Re}(e^{-h_0 s}s^n) \\ \cdots & \cdots \\ \operatorname{Re}(e^{-h_N s}s^n) & \operatorname{Im}(e^{-h_N s}s^n) \\ -\operatorname{Im}(e^{-h_N s}s^n) & \operatorname{Re}(e^{-h_N s}s^n) \\ \cdots & \cdots \\ \operatorname{Re}(e^{-h_N s}) & \operatorname{Im}(e^{-h_N s}) \\ -\operatorname{Im}(e^{-h_N s}) & \operatorname{Re}(e^{-h_N s}) \end{bmatrix}^T \quad (7)$$

is a $2 \times 2(n+1)(N+1)$ real matrix. For the family of $n$th order quasipolynomials $\mathscr{P}$ given in (3) and $\xi$ in the complex plane, we define

$$Q(\mathscr{P}, \xi) \doteq \{K(\xi)\mathbf{p}: p(s) \in \mathscr{P}\}. \quad (8)$$

Note that for a polytope of quasipolynomials $\mathscr{P}$ and for each fixed $\xi$, $Q(\mathscr{P}, \xi)$ is a polytope in the complex plane, and that a polytope $\mathscr{P}$ as in (5) is $D$-stable if and only if $Q(\mathscr{P}, \xi)$ does not contain 0 for any $\xi \in D^c$ (the complement of $D$).

## 2. *D-stability criteria for a polytope of neutral time-delay systems*

In this section we provide an Edge Theorem (Theorem 1) and a graphical test (Theorem 2) for a polytope of neutral time-delay systems $\mathscr{P}$. For simplicity, the graphical test is stated for the (unshifted) open left half plane although the result applies to other $D$ regions (see Remark 3). In addition to the Edge Theorem, a simple necessary condition (Theorem 3) for checking the $D$-stability of $\mathscr{P}$ is given by considering the subpolytope $\mathscr{P}_N$ corresponding to the neutral terms. It is shown that in order for $\mathscr{P}$ to be $D$-stable, it is necessary that $\mathscr{P}_N$ be $D$-stable. Naturally, the $D$-stability of $\mathscr{P}_N$, which is a polytope containing only the $e^{-h_k s}$ terms, can be verified by using Theorem 2.

*Theorem* 1. Consider a polytope of $n$th order (real or complex) quasipolynomials $\mathscr{P}$ as in (5) and a set $D$ in the complex plane satisfying the following condition: There exists some real number $\alpha$ such that $D^c$ (the complement of $D$) contains the half plane $\operatorname{Re} s \geq \alpha$ and, for any point $x \in D^c$ and any $M > 0$, we can find a continuous path in $D^c$ connecting $x$ and some point $y$ with $|y| \geq M$ and $\operatorname{Re} y \geq \alpha$. Then, $\mathscr{P}$ is $D$-stable if and only if all the edges of $\mathscr{P}$ are $D$-stable.

The following lemma is essential in the proof of Theorem 1; see Fu *et al.* (1989) for the proof.

*Lemma* 1. Consider a polytope of quasipolynomials $\mathscr{P}$ as in (5) and $Q(\cdot, \cdot)$ defined in (8). Then, for any $\xi$ in the complex plane,

$$E[Q(\mathscr{P}, \xi)] \subset Q(E[\mathscr{P}], \xi) \quad (9)$$

where $E[\mathscr{P}]$ (resp. $E[Q]$) denotes the set of the edges of $\mathscr{P}$ (resp. $Q$).

*Proof of Theorem* 1. The necessity is obvious because $E[\mathscr{P}] \subset \mathscr{P}$. We proceed with the sufficiency by assuming, on the contrary, that there exists some $s_0 \in D^c$ such that $0 \in Q(\mathscr{P}, s_0)$. We need to show that there exists some $s_1 \in D^c$ such that $0 \in Q(E[\mathscr{P}], s_1)$. Indeed, because of the boundedness of $\mathscr{P}$, there exists some $M > 0$ such that $0 \notin Q(\mathscr{P}, s)$ for all $s$ with $|s| \geq M$ and $\operatorname{Re} s \geq \alpha$. This follows from the fact that

$$\sup\left\{\left|\frac{p(s)}{a_{00}s^n} - 1\right| : p(s) \in \mathscr{P}, \operatorname{Re} s \geq \alpha, |s| \geq M\right\}$$
$$= \sup\left\{\frac{1}{|a_{00}|}\left|\sum_{i=1}^{n}\left(\sum_{k=0}^{N} a_{ik}e^{-h_k s}\right)s^{-i} + \sum_{k=1}^{N} a_{0k}e^{-h_k s}\right| : p(s) \in \mathscr{P}, \operatorname{Re} s \geq \alpha, |s| \geq M\right\} \to 0$$

as $\alpha \to +\infty$. Since, by our assumption, $\alpha$ can be chosen arbitrarily large, it is seen that $p(s)$ approaches $a_{00}s^n$ in some right half plane and therefore it does not vanish in $\{s: \operatorname{Re} s \geq \alpha\}$ for $\alpha$ sufficiently large. Now let $\Gamma \subset D^c$ be any continuous path connecting $s_0$ and some point $s_2$ with $|s_2| \geq M$ and $\operatorname{Re} s_2 \geq \alpha$. For every $\xi \in \Gamma$, we define

$$d(\xi) \doteq \begin{cases} \min\{|q_\xi| : q_\xi \in E[Q(\mathscr{P}, \xi)]\} & \text{if } 0 \notin Q(\mathscr{P}, \xi) \\ -\min\{|q_\xi| : q_\xi \in E[Q(\mathscr{P}, \xi)]\} & \text{if } 0 \in Q(\mathscr{P}, \xi). \end{cases}$$

By the continuity of $\Gamma$, the minimum function, and the vertices with respect to $\xi$, we know that $d(\cdot)$ is continuous on $\Gamma$. Since $d(s_2) > 0$ and $d(s_0) \leq 0$, there must exist some $s_1 \in \Gamma$ such that $d(s_1) = 0$, i.e. $0 \in E[Q(\mathscr{P}, s_1)]$. Using Lemma 1, we conclude that $0 \in Q(E[\mathscr{P}], s_1)$. □

*Remark* 1. It can be seen that the Edge Theorem (Theorem 1) is extendable to a polyhedron of polynomials as well as a polyhedron of quasipolynomials using the same proof above. A polyhedron can be defined as the union of finitely many polytopes. □

*Theorem* 2. Consider a polytope of $n$th order (real or complex) quasipolynomials $\mathscr{P}$ as in (5). We use $E_1, E_2, \ldots, E_t$ to denote the edges of $\mathscr{P}$ and $p_{k0}(s)$ and $p_{k1}(s)$ to denote the vertic quasipolynomials of $E_k$. Then, $\mathscr{P}$ is stable if and only if the following two conditions hold for every $E_k$, $1 \leq k \leq t$:

(i) The frequency response plot of $p_{k0}(j\omega)/(j\omega + 1)^n$ for all real $\omega$ including $\pm\infty$ does not encircle the origin;

(ii) The frequency response plot of $p_{k1}(j\omega)/p_{k0}(j\omega)$ for all real $\omega$ including $\pm\infty$ does not cross $(-\infty, 0]$ (the nonpositive part of the real axis).

*Proof.* The proof is essentially identical to that of Theorem 5.1 in Fu *et al.* (1989). □

*Remark* 2. The number of tests in (i) for stability of vertices can be reduced to checking just one arbitrarily chosen vertex $p_{10}(s)$. Then we can check the stability of those edges which contain $p_{10}(s)$ using the tests of form (ii). In the next step we test the stability of those edges which have a common vertex with one of the previous edges, etc. Since the set of edges of a polytope is connected, we can verify in this way the stability of all edges in a finite number of steps. □

*Remark* 3. Note that the graphical test given in Theorem 2 can be generalized to sets other than the open left half plane by using the argument principle. In general, if the set $D$ is an open set and the boundary of $D$ is a continuous path (or a finite collection of such paths in the case when $D$ is disconnected), then the graphical test can be carried over by replacing $j\omega$ with a point on the boundary and $(s+1)^n$ with $(s+d)^n$ for some arbitrary $d \in D$.

*Theorem* 3. Consider a polytope of $n$th order (real or complex) quasipolynomials $\mathscr{P}$ as in (5) and assume $D$ to be an arbitrary open left half plane. Let $\mathscr{P}_N$ be the polytope of neutral terms corresponding to $\mathscr{P}$ as given in (4). Then, in order for $\mathscr{P}$ to be $D$-stable, it is necessary that $\mathscr{P}_N$ be $D$-stable.

*Proof.* The proof follows directly from Lemma 2.3 in Datko (1978). □

### 3. *Conclusion*

This paper extends the robust stability results of Fu *et al.* (1989) to neutral time delay systems. Our main result shows that, under a mild assumption on the set $D$, a polytope of quasipolynomials of neutral type is $D$-stable if and only if the edges of the polytope are $D$-stable. In addition, the graphical test proposed in Fu *et al.* (1989) is extended to quasipolynomials of neutral type.

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