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On Separation Principle for Linear Quadratic Control with Input Saturation

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***Abstract*— In this paper, we study the validity of the separation principle for linear quadratic control of linear systems in the presence of control input saturation. Our result shows that the well-known separation principle for linear systems fails to hold in this case.**

***Index Terms*— Separation Principle, Linear Quadratic Control, Saturation Control.**

I. Introduction

The well-known *Separation Principle* plays an important role in the linear control theory because it allows us to simplify an output feedback control problem into a state feedback control problem and a state observer design problem. There are several versions of the Separation Principle. The first version, which can be found in most introductory control textbooks, states that an observer-based state feedback system has closed loop eigenvalues specified by the state feedback controller and those of the state observers. The second version applies to Linear Quadratic Gaussian (LQG) control problems by stating that an optimal output feedback controller is formed by an optimal state feedback controller and an optimal state estimator [1]. A similar version is available for output feedback H_∞ control [2]. Recently, a version of Separation Principle has also been developed for a class of uncertain linear systems [3].

In this paper, we study the validity of the Separation Principle for linear systems with input saturation. The study is done in the setting of classical linear quadratic control. We show via an example that the well-known Separation Principle breaks down due to the presence of the input saturation.

The implication of this result is important because for most applications input saturations are unavoidable. In turn, we need to be cautious about using Separation Principle based control designs.

II. Background

Consider the following linear system:

$$\begin{aligned} x(t+1) &\quad A\ (t) \quad B\ (t) \quad w(t), \quad x(\\ y(t) &\quad C\ (t) \quad v(t) \end{aligned} \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state at time t, $y(t) \in \mathbf{R}^r$ is the measured output, $w(t)$ and $v(t)$ are process noise and measurement noise, respectively, t takes non-negative integer values, and A and C are constant matrices. It is commonly assumed that $w(\cdot)$ and $v(\cdot)$ are independent zero-mean Gaussian white noises with variance matrices given by $\Gamma_w \geq 0$ and $\Gamma_v \geq 0$, respectively. Also consider the following infinite-horizon cost function:

$$J(u) \quad {}_{N\to\infty} \frac{1}{N}\mathcal{E}\sum_{t=0}^{N}(x^T(t)Q\ (t) \quad u^T(t)R\ (t) \qquad (2)$$

with some $Q = Q^T \geq 0$ and $R = R^T > 0$. It is assumed that $(A \quad ^{1/2})$ is a controllable pair and $(A \quad)$ is an observable pair.

Consider a linear dynamic controller of the following form:

$$\begin{aligned} \hat{x}(t+1) &\quad \hat{A}\hat{x}(t) \quad \hat{B}\ (t) \\ u(t) &\quad \hat{C}\hat{x}(t) \quad \hat{D}\ (t) \end{aligned} \qquad (3)$$

The well-known separation principle states that the optimal controller that minimises the cost function $J(u)$ is given by

$$u(t) \quad -K\hat{x}(t) \qquad (4)$$

where K is the optimal state feedback control gain and $\hat{x}(t)$ is the optimal estimate of the state $x(t)$ (obtained by a Kalman filter).

The purpose of this paper is to study if this separation principle can be extended to the case where the control input is saturated, i.e., when $u(t)$ in (1) is replaced with $\sigma(u(t)$, where $\sigma(\cdot)$ is a saturation function with saturation level equal to 1.

III. Noise Model

As mentioned earlier, the noises $w(t)$ and $v(t)$ are assumed to be Gaussian. This is a standard assumption used in most work, if not all, on linear quadratic control. The advantage of this assumption is that the state is also Gaussian when there is no input saturation.

In the presence of input saturation, the controlled system usually has a bounded region of stabilisability. This implies that it is no longer realistic to assume the noises to be Gaussian. For example, a Gaussian process noise has a non-zero probability to have a sufficiently large noise sample that drives the state out of the region of stabilisability, unless the noise space lies in the subspace of the state that is open-loop stable (or marginally stable). It can also be argued that a Gaussian measurement noise has a similar destabilising mechanism in the output feedback case, although this is more indirect to see. The observation above means that we should consider bounded noises in dealing with systems with input saturation.

The use of a non-Gaussian model for noises makes difficult to study the distribution of the state, even when the system is linear (i.e., without input saturation). For example, a uniformly distributed process noise does not yield a uniform distribution for the state. However, this problem does not cause difficulty in calculating the covariance matrix for the state. That is, the covariance matrix only depends on the covariance matrices of the noises. This is easily seen because the state vector

$$[x^T(\quad x^T(\quad x^T(\quad x^T(\quad \cdots]^T$$

is a linear function of the noise vectors

$$[w^T(\quad w^T(\quad \cdots]^T, \quad [v^T(\quad v^T(\quad \cdots]^T$$

A careful analysis shows that the property above is sufficient for the separate principle to be valid. The following is the formal statement:

The separation principle holds for the system (1) and the cost function (2) when the process noise and measurement noise are independent zero-mean white noises, regardless of their distributions.

In particular, the separation principle holds when the noises are norm-bounded.

IV. Breakdown of Separation Principle

Now, we give an example to show that the separation principle breaks down when the control input is saturated.

Our example involves the following scalar system:

$$\begin{array}{ll} x(t+1) & .05x(t) \quad \sigma(u(t) \quad w(t), \quad x(\\ y(t) & x(t) \quad v(t) \end{array} \tag{5}$$

where $w(t)$ (resp. $v(t)$) is uniformly distributed between $[-0.9, 0.9]$ (resp .$[-0.95, 0.95]$). It is assumed that $w(t)$ and $v(t)$ are independent. The variances of $w(t)$ and $v(t)$ are found to be $0.9^2/3$ and $0.9^2/3$, respectively.

The cost function to be considered is

$$J(u) \quad {}_{N\to\infty} \frac{1}{N}\mathcal{E}\sum_{t=0}^{N}(x^2(t) \quad .05\sigma^2(u(t) \tag{6}$$

We first consider the state feedback case. We argue that the optimal state feedback is given by

$$u(t) \quad u^*(t) \quad -x(t) \tag{7}$$

To see this, we first need to show that the state does not diverge under the control law (7) for any $w(t)$ bounded by $|w(t)| \leq 0.9$. Indeed, we claim that

$$|x(t)| \leq 2 \tag{8}$$

This can be easily checked by assuming $0 \leq x(t) \leq 2$ and verifying the $-0.9 \leq x(t+1) \leq 2$. Similarly, if $-2 \leq x(t) \leq 0$, we have $-2 \leq x(t) \leq 0.95$. Therefore, the claim in (8) holds.

To show the optimality of (7), we denote

$$J_N(u) \quad \mathcal{E}\sum_{t=0}^{N}(x^2(t) \quad .05\sigma^2(u(t)$$

and note that

$$\begin{aligned} J_N(u) \quad & \mathcal{E}(x^2(\quad - x^2(N+1) \\ & + \mathcal{E}\sum_{t=0}^{N}(x^2(t+1) - x^2(t) \quad x^2(t) \quad .05\sigma^2(u(t) \\ & = \ -\mathcal{E}x^2(N+1) \quad \mathcal{E}\sum_{t=0}^{N} f(x(t), u(t), w(t) \end{aligned}$$

where

$$f(x \qquad) \qquad .05x + \sigma(u) \quad w)^2 + 0.05\sigma^2(u)$$

with

$$\mathcal{E}f(x \qquad) \qquad .05x + \sigma(u) \ {}^2 + 0.05\sigma^2(u) \qquad .9^2/3$$

The optimal $u(N)$ that minimises $\mathcal{E}f(x(N), u(N), w(N)$ is obviously given by (7). The result is

$$\begin{aligned} & \mathcal{E}f(x(N), u(N), w(N) \\ = & \begin{cases} (\ .05 + 0.05^2)x^2(N) & \text{if } |x(N)| \leq 1 \\ (|1.05x(N)| - 1)^2 + 0.05 & \text{otherwise} \end{cases} \end{aligned}$$

In either case, $\mathcal{E}f(x(N), u(N), w(N)$ is a convex function of $x(N)$. Since $x(N)$ is linear in $\sigma(u(N-1)$ which in turn is convex in $u(N-1)$, we know that $\mathcal{E}f(x(N), u(N), w(N)$ is a convex function of $u(N-1)$. Also note that $\mathcal{E}f(x(N-1), u(N-1), w(N-1)$ is convex in $u(N-1)$, so the optimal value for $u(N-1)$ is once again given by (7). The process above can be applied recursively to show that the optimal solution at any t is indeed given by (7).

Next, we consider the output feedback case. The optimal control seems difficult to calculate. Instead, we consider the following simple control law:

$$u(t) \quad u(t) \quad -y(t) \tag{9}$$

We will argue that the state $x(t)$ is bounded for any $|w(t)| \leq 0.9$ and $|v(t)| \leq 0.95$ under this control law, implying that the cost $J(x \ \bar{u})$ is bounded. We then argue that the control law

$$u(t) \quad u(t) \quad -\hat{x}(t) \tag{10}$$

with an optimal state estimate $\hat{x}(t)$ is destablising with a non-zero probability, implying the cost $J(x \ \hat{u})$ is unbounded. These results together will prove that the separation principle fails in this example.

To see that the control law in (9) results in a bounded state, we claim that (8) is still valid. Indeed, we only need to consider $x(t) \geq 0$ and we divide it into two cases: $0 \leq x(t) \leq 1.95$ and $1.95 \leq x(t) < 2$. For $0 \leq x(t) \leq 1.95$, we have $-1 \leq \bar{u}(t) \quad -x(t) - v(t) \leq 0.95$ (i.e., no saturation), $x(t+1) > -1 - 0.9 > -2$ and

$$\begin{aligned} x(t+1) &\leq 1.05x(t) \quad -x(t) - v(t) \quad w(t) \\ &\leq \ 0.05 \times 1.95 + 0.95 + 0.9 < 2 \end{aligned}$$

For $1.95 \leq x(t) < 2$, we have $x(t+1) > -2$ as before and

$$x(t+1) \leq 1.05x(t) - 1 + 0.9 < 2$$

Hence, the bound (8) holds.

The optimal state estimator is of the following form:

$$\begin{array}{ll} \hat{x}(t) & x(t) \quad \ell(y(t) - \bar{x}(t) \ , \quad \bar{x}(\\ \bar{x}(t+1) & .05\hat{x}(t) \quad \sigma(u(t) \end{array} \qquad (11)$$

for some ℓ. This yields

$$\bar{x}(t{+}1) \quad .05\bar{x}(t) \quad .05\ell(y(t){-}\hat{x}(t) \quad \sigma(u(t) \ , \ \bar{x}(\qquad (12)$$

Defining $\bar{e}(t) \quad x(t) - \bar{x}(t)$, we have

$$\bar{e}(t+1) \quad .05(\ -\ell)e(t) \quad w(t) - 1.05\ell \ (t), \ \bar{e}($$

To keep $\bar{t}$ bounded, we need $|1 - \ell| \leq 1$. Denoting

$$q = \lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathcal{E}\bar{e}^2(t)$$

Then,

$$q = 1.05^2(\ -\ell)^2 q + 0.9^2/3 + 0.95^2 \times 1.05^2 \ell^2/3$$

or equivalently,

$$q = \frac{0.9^2/3 + 0.95^2 \times 1.05^2 \ell^2/3}{1 - 1.05^2(\ -\ell)^2}$$

The optimal ℓ is obtained by minimising q, resulting in

$$\ell = 0.6111$$

Consequently,

$$\bar{e}(t{+}1) \quad .4084\bar{e}(t) \ \ w(t){-}0.6416v(t), \ \ \bar{e}(\qquad (13)$$

Define the estimation error $e(t) \quad x(t) - \hat{x}(t)$. From (11), we have

$$e(t) \quad x(t) - \hat{x}(t) \quad .3889\bar{e}(t) - 0.6111v(t) \qquad (14)$$

Taking the optimal estimator (11) and the control law in (10), we consider $w(t) \quad .9$ and $v(t) \quad -0.95$ for all $0 \leq t \leq 100$ and notice that $\bar{e}(t)$ quickly converges to its final value of 2.5516. In fact, $\bar{e}(t) \approx 2.5516$ for all $t \geq 15$. It follows from (14) that $e(t) \approx 1.5729$ for all $t \geq 15$. That is, the controller (13) has a constant bias of 1.5729 after $t \geq 15$. It is checked (via computer simulation) that $x(t)$ reaches 38.15 at $t = 91$. Once $x(t) > 38$,

$$x(t+1) \geq 1.05x(t) - 1 - 0.9 > x(t)$$

for all $u(t)$ and $|w(t)| \leq 0.9$, and the divergence is unstoppable. In fact, the divergence is exponential.

Although a particular sequence of $(w(t), v(t) \ , \ t = 0, \cdots, 100$, is used to demonstrate the divergence of $x(t)$, we note that $x(t)$ is a continuous function of that sequence. That is, the divergence property still holds when the sequence of $(w(t), v(t)$ is perturbed slightly. This implies that there is a non-zero probability for $x(t)$ to diverge. Therefore, the cost function $J(u)$ is unbounded.

This concludes our painstaking exercise to prove the invalidity of the separation principle.

V. Concluding Remarks

Remark 1. Although in the example above we chose the destablising sequence of noise samples to start from $t = 0$, it is not difficult to see that such a destablising sequence can occur at any time. Because such a sequence has only a finite duration and with a non-zero probability, it is easy to see that the existence of such a finite destablising sequence in a random infinite sequence of noises actually has probability 1. That is, the cost function $J(u)$ has probability 1 to be unbounded.

Remark 2. The main reason for the separation principle to fail in the example above is that the Kalman filter, although giving a state estimate with a minimum covariance of estimation error, sooner or later the system will experience a sequence of noise samples such that the estimation error is large enough to drive the closed-loop system out of the stabilisability region.

The comment above suggests that it is necessary to consider the "worst-case" effect of the noises, when designing controllers with input saturation.

References

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