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GLOBAL QUADRATIC STABILIZATION OF A CLASS OF NONLINEAR SYSTEMS

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SUMMARY

The problem of quadratic stabilization for a class of nonlinear systems is examined in this paper. By employing a well-known Riccati approach, we develop a technique for designing a state feedback control law which quadratically stabilizes the system for all admissible uncertainties. This state feedback control law consists of linear and nonlinear feedback control terms. The linear feedback control term is generalized from a well-known H_∞ result, while the nonlinear term can be viewed as a correcting term for the presence of nonlinear bounded uncertainty. This stabilization result is extended to static output feedback and to systems for which the nonlinear uncertainty satisfies *generalized matching conditions*. Furthermore, we point out that in the presence of nonlinear uncertainty the global quadratic stability may be destroyed by some arbitrary small *mismatched uncertainty* in the matrix, and proceed to establish the region of semi-global quadratic stability of the controlled system. © 1998 John Wiley & Sons, Ltd.

1. INTRODUCTION

Robust stabilization of nonlinear systems has been an important research problem in recent years. Its origin can be traced back to Leitmann's paper [1] who introduced the *matching conditions* and a technique for robust stabilization of systems under these conditions. Subsequently, a great deal of work has been done to study various robust stabilization issues for *matched* nonlinearity and uncertainty; see References 2–4 for example. Most recently, the *generalized matching conditions*, also known as the *triangular structure*, have been used to capture a much larger class of nonlinearities and uncertainties using the so-called *back-stepping* design approach; see e.g., References 5–13.

A main drawback of the aforementioned results is that the closed-loop system may not be very robust against additional *mismatched* nonlinearity and/or uncertainty. Although there are a number of papers dealing with *mismatched uncertainties* (see, e.g., References 14–16) the results are not quite satisfactory in the sense that the additional uncertainty is not taken into account in the control design. That is, the controller is designed based on the matched uncertainty only, then the size of the allowable mismatched uncertainty is calculated depending on the *robustness margin* of the resulting closed-loop system. Also, this method works only for linear systems with sufficiently

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small mismatched uncertainty, and it would fail for nonlinear systems in general. As we will show, even an arbitrarily small size of mismatched uncertainty (in certain sense) will cause the closed-loop system to lose global stability.

Another weakness of the existing results on the triangular structure is that a subsystem is required to be stable in some sense (input-to-state stable typically); see, e.g. References 8–13. More specifically, systems satisfying the triangular structure (26) (see Section 3.3) or a variation of it are considered and the subsystem

$$\dot{x}_1 = f_1(x_1, t)$$

is assumed to be stable in some sense. This may not be a harmful assumption in the case where the functions $f_1(x_1, t)$ and $g_1(x_1, t)$ in (26) are known (i.e., they do not contain uncertainties) because by choosing a stabilizer $u_1(x_1, t)$ and a co-ordinate transformation ($z_2 = x_2 - g_1(x_1, t)u_1(x_1, t)$ being a part of it), the system (26) can be rewritten to have a new $f_1(x_1, t)$ which is stable. However, this technique does not apply when the system contains uncertainties, for two reasons: (1) it is not clear how to find a stabilizer for $f_1(x_1, t)$; and (2) The co-ordinate transformation may depend on the uncertainties.

In this paper, we consider the robust stabilization problem for nonlinear systems with both matched and mismatched nonlinearities and uncertainties. The matched nonlinearity is not restricted to the Lipschtz bounded, in fact, it can be bounded by almost any continuous nonlinear and time-varying function. The mismatched part is allowed to be of large size but restricted to be Lipschtz bounded (i.e., norm bounded) and in the autonomous part of the system. We show that this type of uncertain and nonlinear system can be quadratically stabilized via a fixed state feedback controller if and only if the same system with the norm-bounded uncertainty alone can be robustly stabilized. The latter task can be solved by using a standard H_∞ result [17]. That is, the robust stabilizability of the system with *mismatched* norm-bounded uncertainty can be determined by the solvability of an algebraic Riccati equation. When the algebraic Riccati equation has a desired solution, the robust controller can be designed by a simple procedure.

The aforementioned robust stabilization result is extended in two cases. The first extension is to convert the state feedback controller into a static output feedback controller under some additional conditions. The second extension is to relax the *matching conditions* to the *generalized matching conditions* by restricting the norm-bounded uncertainty to a subsystem.

Another related robust stabilization problem of interest is when the control input matrix is also subject to some mismatched uncertainty. We will show, via a simple example, that global stability is impossible to establish even when this additional mismatched uncertainty has an arbitrarily small size. Hence, one has to settle for semi-global stability. We provide an estimate of the size of the semi-global stability region (in the state space) in terms of the size of the additional mismatched uncertainty. Furthermore, we use this estimate to show that the global stability is restored when either the mismatched uncertainty in the input matrix completely vanishes, or when it is sufficiently small and the matched uncertainty or nonlinearity in the autonomous part of the system is Lipschitz bounded.

2. SYSTEM AND PRELIMINARIES

The class of systems to be considered in this paper are described by the following state equations:

$$\Sigma\colon \dot{x}(t) = (A + \Delta A(x, t))x(t) + (B + \Delta B(x, t))u(t) + Bf(x, t)$$
$$y(t) = Cx(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the control output, A, B and C are known constant matrices with appropriate dimensions, $f(x, t)$ is an $m \times 1$ vector representing the nonlinear uncertainties in the plant, $\Delta A(x, t)$ and $\Delta B(x, t)$ are matrix functions representing uncertainties in the matrices A and B.

The following structure for the uncertainties $\Delta A(x, t)$ and $\Delta B(x, t)$ will be assumed throughout this paper:

Assumption 1

$$\Delta A(x, t) = DF(x, t)E_1$$
$$\Delta B(x, t) = BJ(x, t)E_2 \tag{2}$$

where $F(x, t) \in \mathbb{R}^{k \times j}$ and $J(x, t) \in \mathbb{R}^{m \times g}$ are Carathéodory matrix functions* bounded by

$$F(x, t)^t F(x, t) \leqslant \xi \text{ for some } \xi \geqslant 0 \tag{3}$$

and

$$\max_{J(x,t);\,(x,t) \in \mathbb{R}^n \times \mathbb{R}} \| J(x, t)E_2 \| \leqslant \gamma \text{ for some } 0 \leqslant \gamma < 1 \tag{4}$$

and D, E_1 and E_2 are known real matrices which characterize the structures of the uncertainties. The nonlinear function $f(x, t)$ is also assumed to be a Carathéodory function and to satisfy the following assumption:

Assumption 2

There exists a positive scalar Carathéodory function $\rho(x, t)$ satisfying following conditions:

(1) $\| f(x, t) \| \leqslant \rho(x, t)$; $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$ where $\| \cdot \|$ denotes the Euclidean norm, and
(2) $\rho(x, t)$ contains terms which are quadratic or higher in x.
(3) Also, $\lim_{t \to \infty} \rho(\cdot, t) < \infty$, $\forall x \in \mathbb{R}^n$.

Remark 1

Condition 2 given in Assumption 2 is not too restrictive. Note that if function $\rho(x, t)$ does contain terms which are linear in x, we can rewrite $\rho(x, t) = \rho_1 x + \rho_0(x, t)$, where $\rho_0(x, t)$ is a function containing terms which are quadratic or higher in x, and $\rho_1 x$ can be absorbed into the ΔA term in the system equations.

The following linear system associated with (1) will be called the *nominal* system:

$$\dot{x}(x, t) = Ax(t) + Bu(t) \tag{5}$$

Remark 2

Two special cases of system (1) have been well studied. When $\Delta A(x, t) = 0$ or the matrix D in (2) is equal to B, then we have the so-called *matching conditions*. It is well known that an uncertain system with matching conditions can be robustly stabilized via a fixed state feedback controller if

* A function $V : \mathbb{R}^p \times \mathbb{R} \mapsto \mathbb{R}^q$ is called Carathéodory if: (i) $V(z, \cdot)$ is Lebesque measurable for each $z \in \mathbb{R}^p$; (ii) $V(\cdot, t)$ is continuous for each $t \in \mathbb{R}$; (iii) for each compact set $U \subset \mathbb{R}^p \times \mathbb{R}$, there exists a Lebesque integrable function $m_u(\cdot)$ such that $\| V(z, t) \| \leqslant m_u(t)$ for all $(z, t) \in U$. This type of function is needed primarily for ensuring the existence and continuity of the solution to a differential equation; see Reference 4 and references therein.

and only if the nominal system (5) is stabilizable, see References 1 and 2, for example. Further, when $f(x, t) = 0$, the robust stabilization problem can be solved by using an H_∞ control method; see Reference 17. That is, the robust stabilizability of the system via state feedback is equivalent to the solvability of a Riccati equation. What we intend to do in this paper is to develop a unified method to treat the general case.

Remark 3

We emphasize that the assumption on $\Delta B(x, t)$ is not too restrictive in the sense that even an arbitrarily small mismatched (i.e., unstructured) uncertainty may cause the system to lose global stabilizability, provided that $f(x, t)$ is not Lipschtz bounded. See Section 4 for example.

Definition 1

Σ is said to be *quadratically stabilizable* if there exists a continuous mapping $u(\cdot)$; $\mathbb{R}^n \mapsto \mathbb{R}^m$, with $u(0) = 0$, an $n \times n$ positive-definite symmetric matrix P, positive constants $\alpha > 0$ such that the following inequality holds

$$\mathscr{L}(x, t) = x^t P(A + \Delta A(x, t))x + x^t PB\, f(t, x) + x^t P(B + \Delta B(x, t))u(t) \leqslant -\alpha \| x \|^2 \tag{6}$$

for all pairs $(x, t) \in \mathbb{R}^{n+1}$ and any admissible uncertainties $\Delta A(\cdot)$, $\Delta B(\cdot)$ and $f(\cdot)$.

3. MAIN RESULTS

In this section, we present a state feeback stabilization result for the system (1) under Assumptions 1 and 2. This result will be extended to static output feedback under additional assumptions, and to systems with generalized matching conditions.

3.1. State feedback

Given the system (Σ) in (1), we are searching for a state feedback stabilizer of the following form:

$$u(t) = \phi_c(x, t) \tag{7}$$

where $\phi_c(x, t)$ is a Carathéodory function. We now state our main result.

Theorem 3.1

The system (Σ) satisfying Assumptions 1 and 2 is quadratically stabilizable via a nonlinear state feedback controller (7) if and only if there exists $\varepsilon > 0$ and a positive-definite symmetric matrix $Q \in \mathbb{R}^{n \times n}$, such that the following algebraic Riccati equation

$$\tfrac{1}{2}\{A^t P + PA + \varepsilon\xi PDD^t P - 2PBB^t P + \tfrac{1}{\varepsilon} E_1^t E_1\} + Q = 0 \tag{8}$$

has a positive definite symmetric solution P. If this is the case, then, a suitable stabilizing control law is given by

$$u(t) = -B^t P\left(1 + \frac{\gamma}{1-\gamma}\right)x(t) - \frac{1}{(1-\gamma)}\,\phi_c(x, t) \tag{9}$$

where

$$\phi_c(x, t) = \frac{B^t P x \rho^2(x, t) \| B^t P x \rho(x, t) \|^2}{\| B^t P x \rho(x, t) \|^3 + \varepsilon^{*3} \| x \|^6} \tag{10}$$

and

$$\varepsilon^* \leqslant \tfrac{1}{2}\lambda_{\min}[Q] \tag{11}$$

Remark 4

In order to guarantee the existence of partial derivatives, we need to choose $\phi_c(x, t)$ such that it does not contain Euclidean norms of first order. It can be shown that $\phi_c(x, t)$ is uniformly continuous if $\rho(x, t)$ is, and uniformly smooth at all $x \neq 0$ if $\rho(x, t)$ is. This requirement is essential for the subsequent discussion in Section 3.3.

Remark 5

Using condition 2 given in Assumption 2, it is easy to verify that $\phi_c(x, t)$ is in fact continuous at the origin.

Proof of Theorem 3.1. The necessity follows from Definition 1. To prove the sufficiency, we let $V(x) = \frac{1}{2}x^t Px$ be a Lyapunov candidate for system (Σ) with (9). Then the time derivative of $V(x(t))$ along (Σ) is given by

$$\begin{aligned}\dot{V}(x(t)) = {} & x^t P(A + DF(t)E_1 - Bb^t P)x - \frac{\gamma}{1-\gamma}x^t PB\{1 + J(x, t)E_2\}B^t Px + x^t PBf(t, x) \\ & - \frac{\gamma}{(1-\gamma)}x^t PB\phi_c(x, t) - \frac{\gamma}{(1-\gamma)}x^t PBJ(x, t)E_2\phi_c(x, t)\end{aligned} \tag{12}$$

Using the triangular inequality

$$x^t PDF(x, t)E_1 x \leqslant \tfrac{1}{2}x^t\{\varepsilon\xi PDD^t P + \tfrac{1}{2}E_1^t E_1\}x \tag{13}$$

for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, and using the bound on $J(x, t)E_2$ given in (4), the second right-hand term of (12) is bounded above by zero, i.e.,

$$\frac{\gamma}{1-\gamma}x^t PB\{1 + J(x, t)E_2\}B^t Px \leqslant 0 \tag{14}$$

Then, following (13) and (14) we obtain

$$\begin{aligned}\dot{V}(x(t)) \leqslant {} & \frac{1}{2}x^t[A^t P + PA]x + \frac{1}{2}\varepsilon\xi PDD^t Px + \frac{1}{2\varepsilon}x^t E_1^t E_1 x - x^t PBB^t Px \\ & + x^t PBf(x, t) - \frac{1}{(1-\gamma)}x^t PB\{1 + J(x, t)E_2\}\phi_c(x, t)\end{aligned} \tag{15}$$

Using (8), we have

$$\dot{V}(x(t)) \leqslant -x^t Qx(t) + x^t PBf(x, t) - \frac{1}{(1-\gamma)}x^t PB\{1 + J(x, t)E_2\}\phi_c(x, t) \tag{16}$$

Consider the last term of (16), once again using the bound on $J(x, t)E_2$ given in (4), we have the following inequality

$$\frac{1}{(1-\gamma)}x^t PB(I + J(x, t)E_2)\phi_c(x, t) \geqslant x' PB\phi_c(x, t) \tag{17}$$

This leads to the following result:

$$\dot{V}(x(t)) \leqslant -x^t Q x(t) + x^t PB[f(x,t) - \phi_c(x,t)] \tag{18}$$

Then, utilizing (4), we have

$$\begin{aligned}\dot{V}(x(t)) &\leqslant -x^t(t)Qx(t) + \|B^tPx\rho(x,t)\| - \frac{\|B^tPx\rho(x,t)\|^4}{\|B^tPx\rho(x,t)\|^3 + \varepsilon^{*3}\|x\|^6} \\ &\leqslant -x^t(t)Qx(t) + \frac{\|B^tPx\rho(x,t)\|\varepsilon^{*3}\|x\|^6}{\|B^tPx\rho(x,t)\|^3 + \varepsilon^{*3}\|x\|^6}\end{aligned} \tag{19}$$

Therefore, it follows from (19), (11) and the inequality below

$$0 \leqslant \frac{ab^3}{a^3 + b^3} < b, \ \forall a, b \geqslant 0 \tag{20}$$

that

$$\dot{V}(x(t)) \leqslant -x^t Q x + \varepsilon^* \|x\|^2 \leqslant -\tfrac{1}{2} x^t Q x \tag{21}$$

Therefore, (Σ) is globally exponential stabilized, according to Definition 1. □

Remark 6

We note that Theorem 3.1 is a generalization of some known results. More precisely, when $J(x,t)$ and $f(x,t)$ in (Σ) are set to zero, our result will reduce to a result by Petersen.[18] Also Dawson, Qu and Carroll's result[19] will follow by setting $F(x,t)$ and $J(x,t)$ in (Σ) to be zero.

3.2. *Static output feedback*

In certain applications it is more desirable to use output feedback control rather than state feedback. The output feedback control problem for nonlinear uncertain systems is very difficult to solve in general, because observers are hard to construct. It is, however, simple to extend the state feedback stabilization result in Theorem 3.1 to the static output feedback under some additional conditions.

Assumption 3

There exists a positive scalar Carathéodory function $\rho(y,t)$ satisfying the following conditions:

(1) $\rho(y,t) \geqslant \|f(x,t)\|$, $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$, where $y = Cx$ as in (1).
(2) $\rho(y,t)$ contains terms which are quadratic or higher in y.
(3) Also, $\lim_{t\to\infty} \rho(\cdot,t) < \infty$, $\forall y \in \mathbb{R}^n$.

Now, we state our static output feedback stabilization result.

Corollary 3.2

The system (Σ) satisfying Assumptions 1 and 3 is quadratically stabilizable via a nonlinear and time-invariant static output feedback controller, if there exist $\varepsilon > 0$, positive-definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a constant matrix $H \in \mathbb{R}^{m\times p}$ such that the following algebraic Riccati equation

$$\tfrac{1}{2}\{A^tP + PA + \varepsilon\xi PDD^tP - 2PBB^tP + \tfrac{1}{\varepsilon}E_1^tE_1\} + Q = 0 \tag{22}$$

has a positive-definite symmetric solution P which satisfies the following constraint:

$$B^t P = HC \tag{23}$$

In this case, a suitable stabilizing control law is given by

$$u(t) = -\frac{1}{1-\gamma} Hy(t) - \frac{1}{(1-\gamma)} \phi_c(y, t) \tag{24}$$

where

$$\phi_c(y, t) = \frac{Hy\rho^2(y, t) \| Hy\rho(y, t) \|^2}{\| \mathrm{Hy}\rho(\mathrm{y}, \mathrm{t}) \|^3 + \varepsilon^{*3} \| y \|^6} \tag{25}$$

and $\varepsilon^* \leqslant \frac{1}{2} \lambda_{\min}[Q]$.

3.3. Extension to generalized matching conditions

This section extends the main result in Section 3.1 to a more general class of nonlinear uncertain systems, namely nonlinear systems satisfying the *generalized matching conditions*.

Systems satisfying the generalized matching conditions normally have the following form:

$$\Sigma_{GMC} \begin{cases} \dot{x}_1 = f_1(x_1, t) + g_1(x_1, t) x_2 \\ \dot{x}_2 = f_2(x_1, x_2, t) + g_2(x_1, x_2, t) x_3 \\ \cdots \\ \dot{x}_m = f_m(x_1, \ldots, x_m, t) + g_m(x_1, \ldots, x_m, t) u \end{cases} \tag{26}$$

where, $x_i(t) \in \mathbb{R}^n$, $i = 1, \ldots, m$, represent the state. For all $i = 1, \ldots, m$ the matrix functions $f_i(x_1, \ldots, x_i)$: $\mathbb{R}^{ni} \mapsto \mathbb{R}^n$ and $g_i(\cdot)$: $\mathbb{R}^{i+1} \mapsto \mathbb{R}$ are continuous and satisfying the following assumption.

Assumption 4

$f_i(x_1, \ldots, x_i, t)$ have similar bounds as $f(x, t)$ in Assumption 2, but to ensure asymptotic stability, we assume that the bounded functions $\rho_i(0, \ldots, 0, t) = 0$ for all $i = 1, \ldots, m$. The matrix functions $g_i(x_1, \ldots, x_i, t)$ are continuous and satisfying

$$0 < g_i(x_1, \ldots, x_i, t) \leqslant \gamma_i \ \forall x_1, \ldots, x_i, t \in \mathbb{R}^n \text{ for some constant } \gamma_i \tag{27}$$

In this subsection, we further generalize the *generalized matching conditions* to allow the following type of system:

$$\Sigma_{FGMC} \begin{cases} \dot{x}_1 = (A + \Delta A(x_1, t)) x_1 + (B + \Delta B(x_1, t)) x_2 + Bf(x_1, t) \\ \dot{x}_2 = f_2(x_1, x_2, t) + g_2(x_1, x_2, t) x_2 \\ \cdots \\ \dot{x}_i = f_i(x_1, \ldots, x_i, t) + g_i(x_1, \ldots, x_i, t) x_{i+1} \\ \cdots \\ \dot{x}_m = f_m(x_1, \ldots, x_m, t) + g_m(x_1, \ldots, x_m, t) u \end{cases} \tag{28}$$

where $x_1(t)$ is a vector and $\Delta A(x_1, t)$, $\Delta B(x_1, t)$ and $f(x_1, t)$ are as in Section 3.1, and the rest of the system is same as in system (Σ_{GMC}).

Corollary 3.3

The system (Σ_{FGMC}) satisfying Assumptions 1 and 4 is asymptotically stabilizable if there exists $\varepsilon > 0$ and a positive-definite symmetric matrix $Q \in \mathbb{R}^{n \times n}$ such that the following algebraic Riccati equation

$$\tfrac{1}{2}\{A^t P + PA + \varepsilon\xi PDD^t P - 2PBB^t P + \tfrac{1}{\varepsilon} E_1^t E_1\} + Q = 0 \tag{29}$$

has a positive-definite symmetric matrix solution P.

Refer to the Appendix for proof.

We finally point our that the conditions for the system (Σ_{FGMC}) can be relaxed.[5] For example, x_1 can be allowed to be multi-dimensional to some extent, and weaker conditions on $g_i(\cdot)$ are also allowed.

4. ROBUSTNESS ANALYSIS OF THE CLOSED-LOOP SYSTEM

The purpose of this section is to analyse the robustness of the closed-loop system (Σ) with (9) or (24). We first show, via an example, that the global stability of the closed-loop system is very fragile in the sense that it may be destroyed with the slight additional perturbation in the input matrix. This non-robustness property is not only for the controllers given in Section 3, but also for a large class of stabilizing controllers, due to the presence of nonlinear $f(x, t)$. Based on this observation, we derive a robustness analysis result which gives a relationship between the size of additional uncertainty (in certain sense) and size of the semi-global stability region of the perturbed system.

4.1. Non-robustness of global exponential stability

Consider the following simple example:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - \delta u(t) \\ \dot{x}_2(t) &= x_2^2(t) + x_2^2(t) + u(t) \end{aligned} \tag{30}$$

where δ is a constant to be specified. When $\delta = 0$, the system (30) satisfies the matching conditions (i.e. Assumption 1), and is therefore globally stabilizable. We claim that the system (30) is not globally stabilizable when $\delta \neq 0$. Indeed, define

$$z(t) = x_1(t) + \delta x_2(t) \tag{31}$$

then we have

$$\dot{z}(t) = x_2(t) + \delta x_1^2(t) + \delta x_2^2(t) \tag{32}$$

Without loss of generality, we assume $\delta > 0$. Choose the initial condition $x_1(0)$ and $x_2(0)$ such that

$$z(0) = \delta^{-1}, x_2(0) + \delta x_1^2(0) + \delta x_2^2(0) > 0 \tag{33}$$

Then, we argue that $\dot{z}(t) > 0\ \forall t \geqslant 0$. Indeed, from (32) and (33) it is obvious that $\dot{z}(0) > 0$. Let, on the contrary, $t_0 > 0$ be the least time at which $\dot{z}(t_0) = 0$. Then, (32) implies that

$$-\delta^{-1} < x_2(t_0) < 0$$

and in this case,

$$\dot{z}(t_0) \geqslant -\frac{\delta^{-1}}{4} + \delta x_1^2(t_0)$$

Note that

$$x_1(t_0) = z(t_0) - \delta x_2(t_0)$$
$$\geqslant z(t_0) \geqslant z(0) = \delta^{-1}$$

Hence,

$$\dot{z}(t_0) \geqslant -\frac{\delta^{-1}}{4} + \delta^{-1} = \frac{3\delta^{-1}}{4} > 0$$

contradicting the assumption $\dot{z}(t_0) = 0$. That is, $\dot{z}(t) > 0$, $\forall t \geqslant 0$, implying that the system (30) is not globally stabilizable.

We emphasize that the loss of global stabilizability above holds for arbitrarily sufficiently small $|\delta|$ and this phenomenon actually exists for a large class of systems.

4.2. Estimate of semi-global exponential stability bound

Consider the robust controller (7) for system (Σ) with an additional uncertainty in the input matrix assumed to have following structure.

Assumption 5

$$\Delta B_u(x, t) = D_u F_u(x, t) E_u \text{ and } \| F_u(x, t) \| \leqslant \eta \tag{34}$$

where, D_u and E_u are matrices representing the structure of the additional uncertainty.

The time derivative of the Lyapunov function $V(x) = x^t P x$ along the trajectory or trajectories of the system (Σ) will be given by (see (12))

$$\dot{V}(x(t)) = -\lambda_3 \| x \|^2 + x^t P \Delta B_u(x, t) \left(-Kx - \frac{1}{(1-\gamma)} \phi_c(x(t)) \right) \tag{35}$$

$$\leqslant -\lambda_3 \| x \|^2 + \eta r(x, t) \tag{36}$$

where,

$$r(x, t) = (\| x^t P D_u \| \, \| x^t P B E_u \|) \left(\| Kx \| + \frac{1}{(1-\gamma)} \| \phi_c(x(t)) \| \right) \tag{37}$$

Rewriting it in a more compact form, we get

$$\dot{V}(x(t)) \leqslant -\lambda_3 (1 - \eta \lambda_u(x, t)) \| x(t) \|^2 \tag{38}$$

where,

$$\lambda_u(x, t) = \frac{r(x, t)}{\lambda_3 \| x \|^2} \tag{39}$$

Clearly, from (38), the exponential stability region of the system is determined by the function λ_u(x, t). This is summarized in the following result.

Theorem 4.1

Suppose the system (1), satisfying Assumptions 1 and 2, is globally quadratically stabilized via the stabilizing control law (13)–(15) in Theorem 3.1. Also suppose the systme's input matrix is subject to an additional uncertainty given by (34). Then for any $\xi > 0$, $M = \{x: \|x\| \leqslant \lambda_{\min}[P]/\lambda_{\max}[P]\xi: x \in \mathscr{R}^n\}$ is a region of semi-global quadratic stability of the closed-loop system if

$$\eta < \lambda_u^{-1}(x, t), \forall x \in M, \forall t \geqslant 0$$

Remark 7

The function $\lambda_u(x, t)$ in (39) is unbounded in general, hence, no global exponential stability is guaranteed by (38) for the system (Σ), except for two special cases. The first case is obvious: $\eta = 0$; i.e., the unmodelled uncertainty $\Delta B_u(x, t)$ disappears completely. In this case, (38) recovers (12). The second case is when $\rho(x, t)$ is Lipschtz bounded and η is sufficiently small. In this case, it is straightforward to see from (37) and (39) that $\|\lambda_u(x, t)\|$ is uniformly bounded. Thus, the global exponential stability of the system (Σ) is established by (38) as long as

$$0 < \eta < \min\{\lambda_u^{-1}(x, t): (x, t) \in \mathscr{R}^n \times \mathscr{R}\}$$

5. A NUMERICAL EXAMPLE

Consider the following uncertain nonlinear system

$$\dot{x}(t) = (A + DF(x, t)E)x(t) + Bu(t) + Bf(x, t) \tag{40}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = [0, 1]^t, \quad E = [0, 1] \tag{41}$$

$$f(x, t) = \sin(t) x_1^2 \tag{42}$$

$$F(x, t) = \sin(x_1(t)) \tag{43}$$

Note that with the presence of $f(x, t)$ this system cannot be globally stabilized by any linear controller. However, using the design given in Theorem 3.1, if we choose $Q = I$ and $\varepsilon = 1$, then we have the following global exponential stabilizer

$$u(t) = -Kx(t) - \phi_c(x, t) \tag{44}$$

where

$$\phi_c(x, t) = \frac{K\rho^2(x, t)\|Kx\rho(x, t)\|^2}{\|Kx\rho(x, t)\|^3 + \varepsilon^{*3}\|x\|^6} \tag{45}$$

and $K = [20{\cdot}0995 \quad 6{\cdot}4186]$, $\rho(x, t) = x^2$, and $\varepsilon^* = 0{\cdot}1$. The closed-loop responses of x_1 and x_2 are given in Figure 1 and the control input in Figure 2.

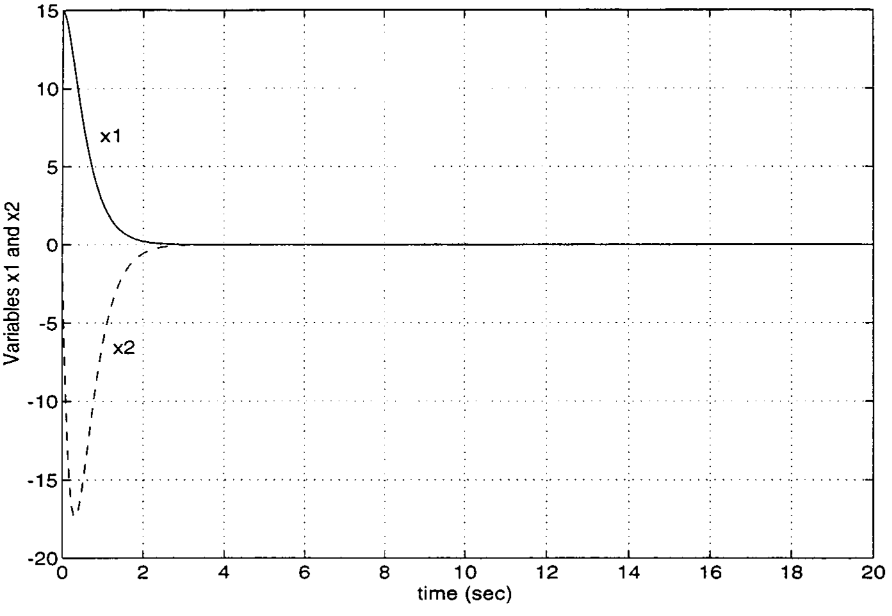

Figure 1. Closed-loop responses of a numerical example

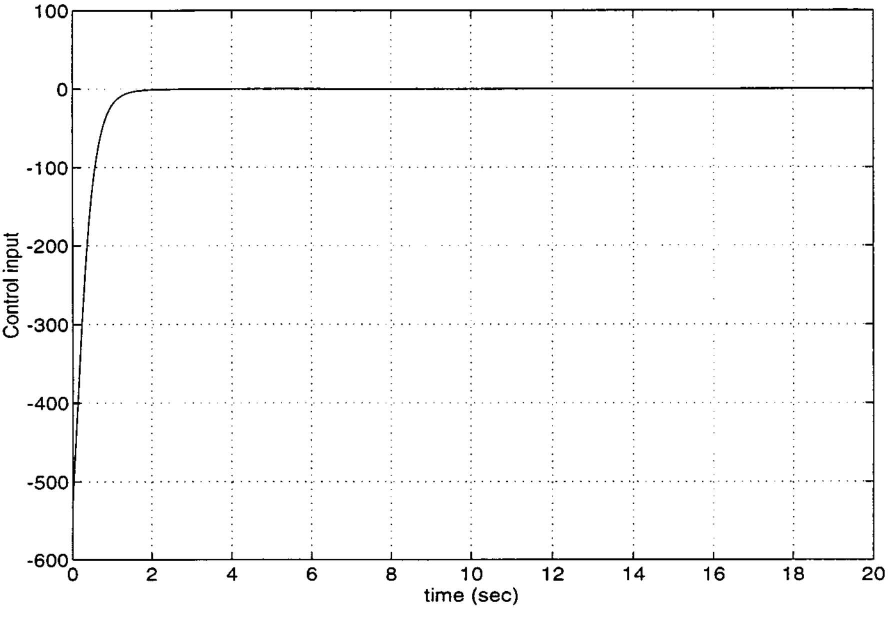

Figure 2. Control input

6. CONCLUSIONS

In this paper, we have solved a robust stabilization problem for systems with both Lipschtz bounded and unbounded uncertainties. We have presented a state feedback controller design technique to globally exponentially stabilize the system. In a special case, the state feedback controller can be implemented via static output feedback. We have further analysed the robustness of the controlled in the presence of some unmodelled uncertainty which causes a possible loss of global exponential stability. Consequently, some estimate of the region of semi-global exponential stability in the state space is provided.

APPENDIX. PROOF OF COROLLARY 3.3

Proof

The design procedure adopted here is very similar to backstepping procedure,[5–7] except that in the first step of design, Theorem 3.1 is applied.

Step 1. Define $z_1 = x_1$ and $z_2 = x_2 - \phi_1(z_1, t)$, where $\phi_1(z_1, t)$ is a smooth function yet to be determined. The first equation of the system (Σ_{FGMC}) can be written as

$$\dot{z}_1 = (A + \Delta A(z_1, t))z_1 + (B + \Delta B(z_1, t))[z_2 + \phi_1(z_1, t)] + Bf(z_1, t) \tag{46}$$

First ignore the term $(B + \Delta B(z_1, t))z_2$ and choose the Lyapunov function $V_1(z_1) = \frac{1}{2} z_1^t P z_1$. Then the time derivative of $V_1(z_1)$ along (46) is

$$\begin{aligned}\dot{V}_1(z_1) = {} & z_1^t P(A + \Delta A(z_1, t))z_1 + z_1^t PBf_1(z_1, t) + z_1^t P(B + \Delta B(z_1, t))\phi_1(z_1, t) \\ & + z_1^t P(B + \Delta B(z_1, t))z_2\end{aligned} \tag{47}$$

Using the design given in Theorem 3.1, we have

$$\phi_1(z_1, t) = -\frac{1}{(1-\gamma)}[B^t P z_1 + \phi_c(z_1, t)] \tag{48}$$

where

$$\phi_c(z_1, t) = \frac{B^t P z_1 \rho^2(z_1, t) \| B^t P z_1 \rho(z_1, t) \|^2}{\| B^t P z_1 \rho(z_1, t) \|^3 + \varepsilon^{*3} \| z_1 \|^6} \tag{49}$$

Using (48), (49) and (29), we will have

$$\dot{V}_1(z_1) \leqslant -(c_1 - \varepsilon^*) \| z_1 \|^2 + z_1^t P(B + \Delta B(z_1, t))z_2 \tag{50}$$

where ε^* is any positive scalar function yet to be chosen. Note that $\phi_1(z_1, t)$ given in (48) does not contain first-order Euclidean norms. Hence its partial derivatives exist.

Step 2. Define $z_3 = x_3 - \phi_2(z_1, z_2, t)$, again $\phi_2(z_1, z_2, t)$ is another smooth function yet to be determined. Using this definition, the second equation of system (Σ_{FGMC}) will become

$$\dot{z}_2 = f_2(z_1, z_2, t) + g_2(z_1, z_2, t)[z_3 + \phi_2(z_1, z_2, t)] + \psi_1(z_1, t) \tag{51}$$

where

$$\begin{aligned}\psi_1(z_1, z_2, t) &= \frac{\mathrm{d}\phi_1(z_1, t)}{\mathrm{d}t} \\ &= \frac{\partial \phi_1(z_1, t)}{\partial z_1}\{(A + \Delta A(z_1, t))z_1 + (B + \Delta B(z_1, t))[z_2 + \phi_1(z_1, t)] + Bf(z_1, t)\} + \frac{\partial \phi_1(z_1, t)}{\partial t}\end{aligned} \tag{52}$$

Penalizing the distance between x_2 and $\phi_1(z_1, t)$, we choose our new Lyapunov function as

$$V_2(z_1, z_2) = V_1(z_1) + \tfrac{1}{2} z_2^t z_2 \tag{53}$$

It can be interpreted as steering x_2 towards the manifold $\phi_1(z_1, t)$. Now computing its time derivative along (46)–(51), we have

$$\begin{aligned}\dot{V}_2(z_1, z_2) &\leqslant -(c_1 - \varepsilon^*)\|z_1\|^2 + z_1^t P(B + \Delta B(z_1, t))z_2 + z_2\{f_2(z_1, z_2, t) \\ &\quad + g_2(z_1, z_2, t)[z_3 + \phi_2(z_1, z_2, t)] + \psi_1(z_1, z_2, t)\} \\ &= -(c_1 - \varepsilon^*)\|z_1\|^2 + z_2 g_2(z_1, z_2, t)z_3 + z_2\{f_2(z_1, z_2, t) \\ &\quad + z_1^t P(B + \Delta B(z_1, t)) + g_2(z_1, z_2, t)\phi_2(z_1, z_2, t) + \psi_1(z_1, z_2, t\} \\ &\leqslant -(c_1 - \varepsilon^*)\|z_1\|^2 + z_2 g_2(z_1, z_2, t)[z_3 + \phi_2(z_1, z_2, t)] + \varphi_2(z_1, z_2, t)\end{aligned} \tag{54}$$

where

$$\varphi_2(z_1, z_2, t) \geqslant \|z_1^t P(B + \Delta B(z_1, t)) + \psi_1(z_1, t) + f_2(z_1, z_2, t)\| \tag{55}$$

is sufficiently smooth function with $\varphi_2(0, 0, t) = 0$. Note that such a bound can always be chosen because of Assumption 4. Similar to **Step 1**, we first forget about the term containing z_3. Following the similar design technique given in Theorem 3.1, we choose

$$\phi_2(z_1, z_2, t) = -\frac{1}{\gamma^2}\left(c_1 z_2 - \frac{z_2\varphi_2^2(z_1, z_2, t)\|z_2\varphi_2(z_1, z_2, t)\|^2}{\|z_2\varphi_2(z_1, z_2, t)\|^3 + \varepsilon^{*3}(\|z_1\| + \|z_2\|)^6}\right) \tag{56}$$

Then (54) will become

$$\begin{aligned}\dot{V}_2(z_1, z_2) \leqslant\ & -(c_1 - \varepsilon^*)\|z_1\|^2 - c_2\|z_2\|^2 + z_2 g_2(z_1, z_2, t)z_3 \\ & + \frac{\|z_2\varphi_2(z_1, z_2, t)\|\varepsilon^{*3}(\|z_1\| + \|z_2\|)^6}{\|z_2\varphi_2(z_1, z_2, t)\|^3 + \varepsilon^{*3}(\|z_1\| + \|z_2\|)^6}\end{aligned} \tag{57}$$

Using the fact given in (20) we will have

$$\dot{V}_2(z_1, z_2) \leqslant\ -(c_1 - \varepsilon^*)(\|z_1\|^2 + \|z_2\|^2) + z_2 g_2(z_1, z_2, t)z_3 \tag{58}$$

Step i $(3 \leqslant i \leqslant n-1)$. We repeat the above procedure, defining $z_{i+1} = x_{i+1} - \phi_i(z_1, \ldots, z_i, t)$, then

$$\dot{z}_i = f_i(z_1, \ldots, z_i, t) + g_i(z_1, \ldots, z_i, t)[z_{i+1} + \phi_i(z_1, \ldots, z_i, t)] + \psi_{i-1}(z_1, \ldots, z_i, t) \tag{59}$$

where

$$\begin{aligned}\psi_{i-1}(z_1, \ldots, z_i, t) = \sum_{k=1}^{i-1} \frac{\partial\phi_{i-1}(z_1, \ldots, z_k, t)}{\partial z_k}\{f_k(z_1, \ldots, z_k, t) + g_k(z_1, \ldots, z_k, t)[z_{k+1} \\ + \phi_k(z_1, \ldots, z_k, t)] + \psi_{k-1}(z_1, \ldots, z_k, t)\} + \frac{\partial\phi_{i-1}(z_1, \ldots, z_{i-1}, t)}{\partial t}\end{aligned} \tag{60}$$

Choose our new Lyapunov function as

$$V_i(z_1, \ldots, z_i) = V_1(z_1) + \sum_{k=2}^{i-1} \tfrac{1}{2} z_k^t z_k \tag{61}$$

and compute its time derivative along (46)–(59) as

$$\begin{aligned}\dot{V}_i(z_1, \ldots, z_i) \leqslant\ & -(c_1 - \varepsilon^*)\sum_{k=1}^{i-1}\|z_k\|^2 + z_{i-1}^t g_i(z_1, \ldots, z_i, t)z_i + z_i\{f_i(z_1, \ldots, z_i, t) \\ & + g_i(z_1, \ldots, z_i, t)[z_{i+1} + \phi_i(z_1, \ldots, z_i, t)] + \psi_{i-1}(z_1, \ldots, z_{i-1}, t)\}\end{aligned} \tag{62}$$

Once again forget about the term containing z_{i+1}. Choose

$$\phi_i(z_1, \ldots, z_i, t) = -\frac{1}{\gamma_i}\left(c_1 z_i - \frac{z_i\varphi_i^2(z_1, \ldots, z_i, t)\|z_i\varphi_i(z_1, \ldots, z_i, t)\|^2}{\|\varphi_i(z_1, \ldots, z_i, t)\|^3 + \varepsilon^{*3}(\|z_1\| + \cdots + \|z_i\|)^6}\right) \tag{63}$$

where

$$\varphi_i(z_1, \ldots, z_i, t) \geqslant \| z_i^t g_i(z_1, \ldots, z_i, t) + \psi_{i-1}(z_1, \ldots, z_i, t) + f_i(z_1, \ldots, z_i, t)\| \tag{64}$$

and $\varphi_i(z_1, \ldots, z_i, t)$ are sufficiently smooth functions with $\varphi_i(0, \ldots, 0, t) = 0$. Following the same analysis as in the $i = 2$ case, we will have

$$\dot{V}_1(z_1, \ldots, z_i, t) \leqslant -(c_1 - \varepsilon^*) \sum_{k=1}^{i} \| z_k \|^2 + z_i^t g_i(z_1, \ldots, z_i, t) z_{i+1} \tag{65}$$

Step n. Choose the Lyapunov function as

$$V_n(z_1, \ldots, z_n) = V(z_1) + \sum_{k=2}^{n} z_k^t z_k \tag{66}$$

and it has the following time derivative along the trajectories of the system (Σ_{FGMC}) with feedback $u(t)$

$$\begin{aligned} \dot{V}_n(z_1, \ldots, z_n) \leqslant & -(c_1 - \varepsilon^*) \sum_{k=1}^{n-1} \| z_k \|^2 + z_{n-1}^t g_{n-1}(z_1, \ldots, z_{n-1}, t) z_n \\ & + z_n \{ f_n(z_1, \ldots, z_n, t) + g_n(z_1, \ldots, z_n, t) u + \psi_{n-1}(z_1, \ldots, z_n, t) \} \end{aligned} \tag{67}$$

Choose

$$u(z_1, \ldots, z_n, t) = -\frac{1}{\gamma^n} \left(c_1 z_n - \frac{z_n \varphi_n(z_1, \ldots, z_n, t) \| z_n \varphi_n(z_1, \ldots, z_n, t) \|^2}{\| z_n \varphi_n(z_1, \ldots, z_n, t) \|^3 + \varepsilon^{*3} (\| z_1 \| + \cdots + \| z_n \|)^6} \right) \tag{68}$$

where

$$\varphi_n(z_1, \ldots, z_n, t) \geqslant \| z_{n-1}^t g_{n-1}(z_1, \ldots, z_{n-1}, t) + \psi_{n-1}(z_1, \ldots, z_n, t) + f_n(z_1, \ldots, z_n, t) \| \tag{69}$$

which does not have to be smooth, and with $\varphi_n(0, \ldots, 0, t) = 0$. Repeating the same analysis as in the $i = 2$ case, we have

$$\dot{V}_n(z_1, \ldots, z_n) \leqslant -(c_1 - \varepsilon^*) \sum_{k=1}^{n} \| z_k \|^2 \tag{70}$$

Choose $\varepsilon^* < c_1$. Then by Definition 1, the variables z_i $\forall i = 1, \ldots, n$ are quadratically stable.

Now we need to show that the system (Σ_{FGMC}) is asymptotically stable. This follows from the fact that the transformation defined between x and z satisfies

$$\begin{aligned} \| x_i \| &= \| z_i + \phi_i(z_1, \ldots, z_{i-1} \| \\ &\leqslant \| z_i \| + \| \phi_i(z_1, \ldots, z_{i-1}) \| \end{aligned} \tag{71}$$

Since $\rho(0, t) = 0$ and $\varphi_i(0, \ldots, 0, t) = 0$ for all $i = 2, \ldots, n$ which implies that $\phi_i(0, \ldots, 0, t) = 0$ $\forall i = 1, \ldots, n$, clearly from (71) we deduced that the system (Σ_{FGMC}) is asymptotically stabilized.

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