Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Guaranteed Cost Control of Uncertain Nonlinear Systems via Polynomial Lyapunov Functions [1]

D. Coutinho[†,◇] A. Trofino[‡] M. Fu[†]
[†]Department of Electrical and Computer Engineering,
The University of Newcastle, Callaghan, NSW 2308, Australia.
[‡] Department of Automation and Systems, UFSC, Brazil.
[◇] On leave from Department of Electrical Engineering, PUC-RS, Brazil.
E-mails: daniel@ee.pucrs.br, trofino@lcmi.ufsc.br, eemf@ee.newcastle.edu.au

Abstract
In this paper, we consider the problem of guaranteed cost control for a class of uncertain nonlinear systems. We derive LMI conditions for the regional robust stability and performance problem based on Lyapunov functions which are polynomial functions of the state and uncertain parameters. Based on the stability results, we discuss the synthesis problem for a class of affine control systems. Numerical examples are presented to illustrate our method.

1 Introduction

The development of robustness and performance analysis as well as design techniques for nonlinear systems is an important field of research. Despite the existence of powerful techniques to cope with these problems in the context of uncertain linear systems, the generalization to the nonlinear case is a difficult task that has motivated many researchers to study these problems. Many control design methods dealing with nonlinear systems use linear control methodologies such as the LQR method, i.e. in some way the nonlinear system is approximated by linear systems [1, 2, 3]. Sometimes, these approaches can be potentially conservative or restrictive. On the other hand, it is well known that the nonlinear optimal control due to difficulties in the solution of the Hamilton-Jacob equation is not a practical approach, [4]. Since the work [5], that showed a solution for rational systems in terms of linear matrix inequalities (LMIs) and based on quadratic Lyapunov functions, some authors have proposed more sofisticated Lyapunov functions to derive less conservative results using the LMI framework [6, 7, 8].

In this paper, we derive LMI conditions for the Guaranteed Cost Control problem for a class of uncertain nonlinear systems in a given polytopic neighbourhood of the equilibrium point. These conditions assure the regional stability of the unforced system, determine a bound on the energy of the output signal for a given set of initial conditions, and are based on Lyapunov functions which are polynomial functions of the state and uncertain parameters. Via an iterative algorithm, this approach is applied to the synthesis problem considering a class of affine control systems. We point out that the system matrices may be rational functions of the state and uncertain parameters, and certain exponential and trigonometric functions are also allowed.

The structure of this paper is as follows. First, we state the problem of concern and derive an upper bound on the 2-norm of the output performance for a set of initial conditions. In the sequel, section 3 presents an application of the derived method to the guaranteed cost control problem. Numerical examples are given in section 4, and some conclusions are drawn in the final section. The notation used in this paper is standard, and the matrix and vector dimensions are omitted whenever they can be determined from the context. Some results were omitted because of space restriction. For more details, the reader is referred to the full version of this paper [9] (available at *www.ee.newcastle.edu.au/reports/reports_index.html*).

2 Performance of Nonlinear Systems

Consider the uncertain nonlinear system

$$\dot{x} = A(x,\delta)x, \;\; z = C(x,\delta)x \tag{1}$$

where $x(0) = x_0$, and $x \in \mathbb{R}^{n_0}$, $\delta \in \mathbb{R}^{n_\delta}$, $z \in \mathbb{R}^{n_z}$ denote respectively the state vector, uncertain parameters and the output performance vector.

With respect to the system (1), we consider the following assumptions:

A1 The uncertain parameters vector, δ, and its time-derivative, $\dot{\delta}$, lie in a given polytope $\mathcal{B}_\delta$, with known vertices, i.e. $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$.

[1]This work was partially supported by 'CAPES', Brazil under grant BEX0784/00-1, by 'CNPq', Brazil under grants 147055/99-7 and 300459/93-9/PQ and by the Australian Research Council.

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A2 The origin, $x = 0$, of the system is an equilibrium point.

A3 The right handside of the differential equation is bounded for all values of $x, \delta, \dot{\delta}$ of interest.

A4 $\mathcal{B}_x$ is a neighbourhood of the equilibrium point of the system.

In this section, the problem of concern is to compute a bound on the 2-norm of the performance output signal for a given set of initial state $x_0 \in \mathcal{B}_x$ and for all values of $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$. To this end, we will introduce some definitions and notations.

We do not assume that $\mathcal{B}_x$ is an invariant set with respect to (1). Hence, there may exist trajectories that reach the boundary of $\mathcal{B}_x$ at an instant T and then leave it. Thus, we define the output energy as follows:

$$\|z\|_2^2 = \begin{cases} \infty & \text{if the trajectory of } x(t) \text{ leaves } \mathcal{B}_x; \\ \lim\limits_{T\to\infty} \int_0^T z'z \, dt & \text{otherwise.} \end{cases}$$

Let us suppose that the system (1) may be decomposed as

$$\begin{array}{l} \dot{x} = \sum_{i=0}^{q} A_i(x,\delta)\pi_i = \mathbf{A}(x,\delta)\pi \\ z = \sum_{i=0}^{q} C_i(x,\delta)\pi_i = \mathbf{C}(x,\delta)\pi \end{array} \qquad (2)$$

where π is such that: $\pi = [\ \pi_0' \ \cdots \ \pi_q'\ ]'$, $\pi_0 = x$, $\pi_1 = \Theta(x,\delta)\pi_0$ and $\Omega(x,\delta)\pi = 0$. In the associated system, the vectors $\pi_i \in \mathbb{R}^{m_i}$ are auxiliary functions of x, δ associated with the decomposition of the system nonlinearities; $\Theta(x,\delta) \in \mathbb{R}^{m_1 \times m_0}$ is an affine matrix, function of (x,δ), used to decompose bilinear terms; $\Omega(x,\delta) \in \mathbb{R}^{m_\Omega \times m}$, with $m = \sum_{i=0}^{q} m_i$, is an affine matrix function of (x,δ) used to decompose polynomial and rational nonlinearities and to represent eventually additional constraints, such as algebraic equations, on the vectors π_i; $A_i(x,\delta) \in \mathbb{R}^{m_0 \times m_i}$, $i = 0, \ldots, q$, are affine matrix functions of (x,δ) associated with the dependence of the system with respect to the π_i functions; and $C_i(x,\delta) \in \mathbb{R}^{n_z \times m_i}$, $i = 0, \ldots, q$, are affine functions of (x,δ) associated with the output structure of the system. The matrices $\mathbf{A}(x,\delta) \in \mathbb{R}^{m_0 \times m}$ and $\mathbf{C}(x,\delta) \in \mathbb{R}^{n_z \times m}$ are defined to represent the equivalent system (2) in a concise form. To simplify the notation, we always use them and Θ, Ω, A_i, C_i, without explicitly specifying their respective dependence on x, δ and t.

Hereafter, we assume that:

A5 the right-hand side of the differential equation (2) is bounded for all values of π of interest and the system representation in terms of the auxiliary variable π is equivalent to the representation (1).

By definition in (2), notice that Θ is an affine matrix function of (x,δ). Then, we can represent it as follows: $\Theta = \sum_{j=1}^{m_0} T_j x_j + \sum_{j=1}^{n_\delta} U_j \delta_j + V$, where x_i, δ_i are the entries of the vectors x and δ respectively, and T_j, U_j, V are constant matrices of structure having the same dimensions of Θ.

With above, define the following matrices $\tilde{\Theta} = \sum_{j=1}^{m_0} T_j x s_j$ and $\hat{\Theta} = \sum_{j=1}^{n_\delta} U_j \dot{\delta}_j$, where s_j is the j-th row of the identity matrix I_{m_0}.

Now, with these definitions, we can state the main result of this work as follows.

Theorem 1 *Consider that system (2) satisfies* **A1**-**A5**. *Let* $\tilde{\Theta}$ *and* $\hat{\Theta}$ *be as above, further define* $N_0 = [I_{m_0+m_1}\ 0] \in \mathbb{R}^{(m_0+m_1)\times m}$; *and*

$$\Psi_a = \begin{bmatrix} M_x & 0 \\ -\Theta & I_{m_1} \end{bmatrix}; \; E = \begin{bmatrix} I_{m_0} & 0 \\ -(\tilde{\Theta}+\Theta) & I_{m_1} \end{bmatrix};$$

$$F = \begin{bmatrix} \mathbf{A} \\ [\ \hat{\Theta} \ \ 0\ ] \end{bmatrix}; \; \Psi_b = \begin{bmatrix} 0 & \Omega \\ 0 & [\ \Psi_a \ \ 0\ ] \\ -E & F \end{bmatrix};$$

$$M_x = \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_{m_0} & -x_{m_0-1} \end{bmatrix}.$$

Suppose that there exist matrices P, L_a, *and* L_b *that solve the following optimization problem, where the LMIs are satisfied at all vertices of the meta-polytope* $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$.

$$\min \ \text{trace} \ \ (P + L_a\Psi_a + \Psi_a' L_a') \ \text{subject to:}$$
$$P + L_a\Psi_a + \Psi_a' L_a' > 0, \ P = P'$$
$$\begin{bmatrix} \left(\begin{bmatrix} 0 & PN_0 \\ N_0' P & 0 \end{bmatrix} + L_b\Psi_b + \Psi_b' L_b' \right) & \begin{bmatrix} 0 \\ \mathbf{C}' \end{bmatrix} \\ [\ 0 \quad \mathbf{C}\ ] & -I_{n_z} \end{bmatrix} < 0$$

Define the following Lyapunov function $v(x,\delta) = x'\mathcal{P}(x,\delta)x$, *with*

$$\mathcal{P}(x,\delta) = \begin{bmatrix} I_{m_0} \\ \Theta \end{bmatrix}' P \begin{bmatrix} I_{m_0} \\ \Theta \end{bmatrix} \qquad (3)$$

Define the sets $\mathcal{R}_c$, *level surfaces of the above Lyapunov function, as* $\mathcal{R}_c = \{x : v(x,\delta) \le c \ , \ 0 < c < \infty\}$ *and* $\mathcal{R}_{c^*}$ *as the largest level surface of* $v(x,\delta)$ *that belongs to* $\mathcal{B}_x$, *where* $c^* = \max \ c$ *such that* $\mathcal{R}_c \subset \mathcal{B}_x$.

Then, $\mathcal{R}_{c^*}$ *is an invariant set and the 2-norm of the output signal satisfies the following for all* $x_0 \in \mathcal{R}_{c^*}$ *and* $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$:

$$\|z\|_2^2 \ \le \ v(x_0, \delta(0)) \ \le \ c^* \qquad (4)$$

Until now, we propose a methodology for robust stability and performance analysis for a class of uncertain nonlinear systems. In the following section, we will consider the synthesis problem for a class of affine control systems.

3 Control

Consider the uncertain nonlinear system

$$\begin{aligned} \dot{x} &= A(x,\delta)x + B(x,\delta)u, \\ z &= C(x,\delta)x + D(x,\delta)u \end{aligned} \tag{5}$$

where $x(0) = x_0$, $u \in \mathbb{R}^{n_u}$ denotes the control input, and $B(x,\delta)$, $D(x,\delta)$ are affine matrix functions of x, δ with appropriate dimensions.

In this section, we are concerned with the problem of determining a control law of the type $u = K(x,\delta)x$ in order to minimize the output energy of system (5), where $K(x,\delta) = \sum_{i=0}^{q} K_i \pi_i$ and the matrices $K_i \in \mathbb{R}^{n_u \times m_i}$ are fixed gains to be determined. The uncertain parameters in the nonlinear control law can represent scheduling parameters or a possible mismatch between the applied and designed control gains.

For simplicity, we assume that:

A6 All states are available for measurement and feedback;

A7 When we implement a gain scheduling control technique, the scheduling parameters are known on-line to the controller.

Note that with an appropriate choice of the matrix gains K_i the proposed control law can implement different design techniques such as robust (static or nonlinear), gain-scheduling or non-fragile controllers. In the non-fragile case, the parameters introduced in the control law via an appropriate choice of the auxiliary vectors π_i belong to a given range representing the possible deviations with respect to the nominal value. For design purposes, these parameters must be viewed as time invariant uncertainties.

The theorem 1 provides the foundation for our synthesis framework. Suppose that the matrix control-gains K_i $(i = 0, \dots, q)$ are given. Thus, we can redefine the matrices $\mathbf{A}(x,\delta)$ and $\mathbf{C}(x,\delta)$ in (2) as follows:

$$\begin{aligned} \mathbf{A}(x,\delta) &= \begin{bmatrix} A_0 + BK_0 & \cdots & A_q + BK_q \end{bmatrix} \\ \mathbf{C}(x,\delta) &= \begin{bmatrix} C_0 + DK_0 & \cdots & C_q + DK_q \end{bmatrix} \end{aligned} \tag{6}$$

and apply the theorem 1 for closed-loop stability analysis. However, we now focus on extending its results to control design feedback with guaranteed cost.

Observe that the matrix inequalities in (1) for the design case are bilinear matrix inequalities (BMIs) [10]. The BMI problems appear commonly in the multiplier theory based robust control design. In order to solve the technical difficulties arising from the numerical solution of such BMIs, we will use the method proposed by [11] where the BMI problem is solved via two LMI sub-problems. In the following, we show this algorithm specialized to our synthesis problem.

Algorithm 1 *Let $\mathcal{B}_x$ and $\mathcal{B}_\delta$ be given polytopes. Consider the theorem 1 with matrices $\mathbf{A}(x,\delta)$ and $\mathbf{C}(x,\delta)$ as defined in (6).*

STEP 1 *Determine a stabilizing controller in the domain $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$;*

STEP 2 *For a given stabilizing controller, solve the optimization problem in theorem 1 taking into account the definition (6), obtaining the matrix L_b;*

STEP 3 *For a given matrix L_b, solve the optimization problem in theorem 1, obtaining the new controller.*

STEP 4 *Iterate over steps 2,3 until convergence or satisfaction of a pre-defined 2-norm of the output signal.*

At each iteration i, note that above algorithm guarantees the regional stability of the closed-loop system and $c^*_{(i)} \le c^*_{(i-1)}$. As a result, the above algorithm will converge on a local minimum. To overcome the problem of finding a stabilizing controller (STEP 1), we may use the classical LQR technique [12] applied to the linearized model of the nominal nonlinear system and check the domain of stability. Observe that the LQR controller may not stabilize the system for given domains $\mathcal{B}_x$ and $\mathcal{B}_\delta$. When this occurs in STEP 1, we suggest the use of the controller proposed in [9], theorem 2, with guaranteed domain of stability.

4 Numerical Results

To illustrate the analysis/synthesis results, we show two numerical examples. The first one is based on the *Van der Pol's* equation for reversed time with a time-invariant uncertain dumping factor. We analyse the local properties using the proposed method for different Lyapunov functions. In the second exemple, from [4], the objective is to design a static state-feedback control law with guaranteed cost for a time invariant two-dimensional nonlinear oscillattor (with no uncertain parameters).

Example 1. Consider the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -1 \\ 1 & \varepsilon(x_1^2-1) \end{bmatrix} x \\ z &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \; x(0) \in \mathcal{B}_x \end{aligned} \tag{7}$$

where the nonlinear dumping factor ε is constant and approximately known, i.e. $\varepsilon \in [\varepsilon_0 - \Delta\varepsilon, \varepsilon_0 + \Delta\varepsilon]$ with $\varepsilon_0 = 0.8$ and $\Delta\varepsilon = 0.2$.

In order to determine an upper bound on the output energy, let us consider that $\mathcal{B}_x$ is defined by the following vertices:

$$\left\{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}, \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}, \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix} \right\}$$

where α is a given scalar.

Now, consider the following partition of the matrix P of $\mathcal{P}(x,\delta)$:

$$P = \begin{bmatrix} P_0 & P_1 \\ P_1' & P_2 \end{bmatrix}$$

With above partition, we can obtain the following Lyapunov matrices: (i) P_0, P_1 and P_2 free, i.e. the matrix $\mathcal{P}(x,\delta)$ is quadratic in (x,δ); (ii) P_0, P_1 free and $P_2 = 0$, i.e. the matrix $\mathcal{P}(x,\delta)$ is affine in (x,δ); and (iii) $P_1 = 0$, $P_2 = 0$ and P_0 free, i.e. the matrix $\mathcal{P}(x,\delta)$ is constant.

The table 1 shows estimated upper-bounds on the 2-norm of the output signal for the proposed approach, using theorem 1 and different Lyapunov matrices. For all solutions, we considered $\alpha = 0.8$.

Upper-bound on $\|z\|_2$	Lyapunov Matrix		
	constant	affine	quadratic
$v(x_0,\delta(0))^{1/2}$	9.2	9.0	1.8

Table 1: Estimated upper-bound on $\|z\|_2$ for system (7).

As expected, the polynomial Lyapunov functions (quadratic Lyapunov matrix) achieves less conservative results, thus justifying the required extra computation. We point out that the piecewise quadratic approach [13], with four linearized state space partitions (a similar computational effort), surprisingly fails in the analysis of stability for this system. This further shows the potential of our approach.

Example 2. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1(\tfrac{\pi}{2} + \arctan(5x_1)) + \\ &\quad -\tfrac{5x_1^2}{2(1+25x_1^2)} + 4x_2 + 3u \end{aligned} \tag{8}$$

with a performance index $z = x_2 + u$.

Note that system (8) has a non-rational nonlinearity, then it is not straight rewritten in the equivalent form (2). To deal with this problem, we define the following auxiliary variables $x_3 = \arctan(5x_1)$ and $\tau = \operatorname{arccot}(5x_1)$. With this auxiliary variables, it is possible to construct: a differential equation $\dot{x}_3 = 5\frac{x_2}{1+25x_1^2}$ and an algebraic one $x_3 + \tau - \frac{\pi}{2} = 0$. Leading to the following augmented-system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\left(\tfrac{\pi}{2} + x_3\right) - \tfrac{5x_1^2}{2(1+25x_1^2)} + 4x_2 + 3u \\ \dot{x}_3 &= \tfrac{5x_2}{1+25x_1^2} \\ 0 &= x_3 + \tau - \tfrac{\pi}{2} \end{aligned} \tag{9}$$

The system (9) has only rational nonlinearities and hence can be rewritten in the equivalent form (2). By construction, the trajectories of the system representation (9) include all trajectories of the original system (8). In particular, suppose that $x_1(0)$, $x_2(0)$ are the initial conditions of the system (8). Then for the initial conditions $x_1(0)$, $x_2(0)$ and $x_3(0) = \arctan(5x_1(0))$ both systems have equal trajectories in the x_1, x_2 sub-space. Also, system (8) is open-loop unstable, then we need a stabilizing controller. To this end, we can use the LMI-LQR techniques from [12] applied to the linearized model (keep in mind that for system (8) there is no uncertainty). Where, we obtain the following control matrix $K_{lin} = [\ 0 \ \ -1.667\ ]$. In the augmented state-space, the stabilizing control matrix is given by $K = [\ 0 \ \ -1.667 \ \ 0\ ]$. This controller (that ensures the stability of the cloosed-loop system) is used as starting point of algorithm 1. As we are dealing with static state feedback of the system (8), we will impose the following structure in the control matrix: $K = [\ k_1 \ k_2 \ 0\ ]$. After 2 iterations, we obtained the following control law $u = [\ -0.03 \ \ -1.73 \ \ 0\ ][\ x_1 \ \ x_2 \ \ \arctan(5x_1)\ ]'$, in which the guaranteed cost satisfies $\|z\|_2^2 \leq 32$. Note that the augmented system (9) allows only the use of a quadratic Lyapunov function in the proposed approach. It means that we obtain a potentially conservative result, but in the other hand we have a lower computational effort.

It should be noted that the optimal solution has a cost of 31 for the initial condition $x(t=0) = [\ 3 \ \ -2\ ]'$ with the optimal Lyapunov function given by $v(x) = x_1^2(\frac{\pi}{2} + \arctan(5x_1)) + x_2^2$, and the Receding Horizon Control with Control Lyapunov Function (RHC+CLF) scheme, proposed in [4], achieves a cost of 36 with a horizon of 1.

The above results show the potential of our approach and when compared with the optimal solution are in some way surprisingly. This fact can be explained due to the behaviour of the Lyapunov function candidate in the x_1, x_2 sub-space. Note that in the x_1, x_2 sub-space this function depends not only on x_1, x_2, but also on

$\arctan(5x_1)$, i.e. the Lyapunov function candidate is more complex than the quadratic one and similar to the optimal solution.

5 Concluding Remarks

In this paper, we showed an LMI based technique to compute a bound on the guaranteed cost for a class of uncertain nonlinear systems given a set of intial conditions represented by the polytope $\mathcal{B}_x$. To ascertain the system stability and performance criterion, we use Lyapunov functions of the type $v(x,\delta) = x'\mathcal{P}(x,\delta)x$, where the matrix $\mathcal{P}(x,\delta)$ is a polynomial function of x and δ. Based on the analysis results, we proposed an iterative algorithm for the synthesis problem. The first example showed as expected that polynomial Lyapunov functions leads to better performance criteria, thus justifying the required extra computation. The other one illustrated how to compute a guaranteed cost control law for an unstable open-loop system with non-rational nonlinearities. Future research will be concentrated on the design problem in order to obtain LMI conditions.

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Proof of Theorem 1

Consider the system (2) and define the following partition of the π vector.

$$\pi_a = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} \in \mathbb{R}^{m_a};\ \pi_b = \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_q \end{bmatrix} \in \mathbb{R}^{m_b}$$
$$\text{and} \quad \pi = \begin{bmatrix} \pi_a \\ \pi_b \end{bmatrix} \in \mathbb{R}^{m} \tag{10}$$

where $m_a = m_0 + m_1$, $m_b = m_2 + \ldots + m_q$ and $m = m_a + m_b$.

Let $N_a \in \mathbb{R}^{m_0 \times m_a}$ be a matrix such that $N_a\pi_a = x$ and define $N_b = [\ 0_{m_0\times m_a} \quad N_a \quad 0_{m_0 \times m_b}\ ]$.

Suppose that the optimization problem (1) has a solution for all vertices of $\mathcal{B}$. Then, by convexity, it is also satisfied $\forall\, x \in \mathcal{B}_x$ and $\forall\,(\delta,\dot{\delta}) \in \mathcal{B}_\delta$.

For convenience, let us represent the first LMI of (1) by $Q_a > 0$. Since this inequality is strict, for some sufficient small positive scalar ϵ_a, it is possible to add the term $-\epsilon_a N_a' N_a$ to Q_a without change its sign, i.e. the condition $Q_a - \epsilon_a N_a' N_a \geq 0$ is still satisfied. Pre- and post-multiplying $Q_a - \epsilon_a N_a' N_a \geq 0$ by π_a' and π_a, respectively, we get:

$$\pi_a' P \pi_a \geq \epsilon_a x' x\ ,\ \ \forall\ (\pi,\delta) \in \mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta \tag{11}$$

since by construction $\Psi_a \pi_a = 0$, i.e. $M_x x = 0$ and $\Theta x - \pi_1 = 0$.

From (2) and (10) notice that

$$\pi_a = \begin{bmatrix} I_{m_0} \\ \Theta \end{bmatrix} x \qquad (12)$$

Then, from (11) and (12) we obtain

$$\begin{aligned} v(x,\delta) &= \pi_a' P \pi_a = x' \begin{bmatrix} I_{m_0} \\ \Theta \end{bmatrix}' P \begin{bmatrix} I_{m_0} \\ \Theta \end{bmatrix} x \\ &= x' \mathcal{P}(x,\delta) x \geq \epsilon_a x' x \ , \quad \forall (x,\delta) \in \mathcal{B} \end{aligned} \qquad (13)$$

where $\mathcal{P}(x,\delta)$ is the state-parameter dependent matrix indicated in (3).

Since x, δ belong to a polytope, the entries of the matrix Ψ_a in Q_a are bounded. Then there exists a sufficient large positive scalar ϵ_{a1} such that $\epsilon_{a1} I_{m_a} \geq Q_a$. Thus, $\epsilon_{a1} \pi_a' \pi_a \geq \pi_a' Q_a \pi_a$ that in turn yields $\epsilon_{a1}(x' x + x' \Theta' \Theta x) \geq x' \mathcal{P}(x,\delta) x$. Keeping in mind that x, δ belong to a polytope there exists a sufficient large positive scalar ϵ_{a2} such that $\epsilon_{a2} I_{m_a} \geq \Theta' \Theta$. Hence:

$$v(x,\delta) = x' \mathcal{P}(x,\delta) x \leq \epsilon_{a1}(1+\epsilon_{a2}) x' x \qquad (14)$$

for all $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$.

Applying the *Schur* complement on the 2nd LMI of (1), we have

$$\begin{bmatrix} 0 & P N_0 \\ N_0' P & \mathbf{C}' \mathbf{C} \end{bmatrix} + L_b \Psi_b + \Psi_b' L_b' < 0 \qquad (15)$$

Let us represent the above matrix inequality by $Q_b < 0$. Since this inequality is strict, for some sufficient small positive scalar ϵ_b, it is possible to add the term $\epsilon_b N_b' N_b$ to this LMI without change the sign, i.e. the condition $Q_b + \epsilon_b N_b' N_b \leq 0$ is still satisfied.

Pre-multiplying by $[\ \dot{\pi}_a' \quad \pi'\ ]$ and post-multiplying by its transpose, we get:

$$\begin{bmatrix} \dot{\pi}_a \\ \pi \end{bmatrix}' \begin{bmatrix} 0 & P N_0 \\ N_0' P & \mathbf{C}' \mathbf{C} \end{bmatrix} \begin{bmatrix} \dot{\pi}_a \\ \pi \end{bmatrix} \leq -\epsilon_b x' x$$
$$\forall (x,\delta,\dot{\delta}) \in \mathcal{B} : \begin{cases} -E \dot{\pi}_a + F \pi = 0 \\ \Psi_a \pi_a = 0 \\ \Omega \pi = 0 \end{cases} \qquad (16)$$

Taking the time derivative of $\pi_1 = \Theta \pi_0$ yields

$$\dot{\pi}_1 = \dot{\Theta} \pi_0 + \Theta \dot{\pi}_0 = \sum_{j=1}^{m_0} T_j \dot{x}_j x + \sum_{j=1}^{n_\delta} U_j \dot{\delta} x + \Theta \dot{x}$$

Since $\dot{x}_j = s_j \dot{x}$ is a scalar it follows that $\dot{x}_j x = x \dot{x}_j = x s_j \dot{x}$. This yields

$$\dot{\pi}_1 = \hat{\Theta} x + \left(\bar{\Theta} + \Theta \right) \dot{x}$$

It is easy to verify that the above equality has the following compact form

$$-E \dot{\pi}_a + F \pi = 0 \qquad (17)$$

Note that, from (2) we define the auxiliary vectors π_i as $\pi_1 - \Theta \pi_0 = 0$ and $\Omega \pi = 0$. These can be written in the following compact form $\Psi_b [\ \dot{\pi}_a' \ \pi'\ ]' = 0$.

Then it is possible to write (16) as follows

$$\dot{\pi}_a' P \pi_a + \pi_a' P \dot{\pi}_a + \pi' \mathbf{C}' \mathbf{C} \pi \leq -\epsilon_b x' x$$

for all $(x,\delta,\dot{\delta}) \in \mathcal{B}$.

Since $z = \mathbf{C}\pi$, the above expression for all $(x,\delta,\dot{\delta}) \in \mathcal{B}$ is equivalent to

$$\begin{aligned} & x' (A(x,\delta)' \mathcal{P}(x,\delta) + \mathcal{P}(x,\delta) A(x,\delta) + \\ & + \dot{\mathcal{P}}(x,\delta)) x + z' z \leq -\epsilon_b x' x \end{aligned} \qquad (18)$$

i.e. $\dot{v}(x,\delta) + z' z \leq -\epsilon_b x' x$.

From (13), (14) and (18), the system (1) is locally exponentially stable and $v(x,\delta) = x' \mathcal{P}(x,\delta) x$ is a Lyapunov function for the origin of the system.

Keep in mind that (13), (14) and (18) are satisfied for all $x \in \mathcal{B}_x$ and $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$. Then, there is a sufficient small constant $c > 0$ such that $\mathcal{R}_c \subset \mathcal{B}_x$. Hence, $\mathcal{R}_{c^*}$ is an invariant set.

Integrating (18) from 0 to T, we have

$$v(x(T),\delta(T)) - v(x(0),\delta(0)) \leq -\int_0^T z' z \, dt$$

for all $T > 0$ and $x_0 \in \mathcal{R}_{c^*}$.

As $T \to \infty$, the above expression leads to

$$\|z\|_2^2 = \lim_{T\to\infty} \int_0^T z' z \, dt \leq v(x_0,\delta(0)) \leq c^*$$

for all $x_0 \in \mathcal{R}_{c^*}$, $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$. □