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On guaranteed cost control of linear systems with input saturation

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Abstract: This work studies the problem of control design for linear systems with input saturation. It is well known that integral quadratic constraints (IQC) can be used to describe input saturation and that the use of IQC in analysis can lead to less conservative performance bound and larger domain of attraction. In this work, it is shown that a class of commonly used IQCs may not help in control synthesis. That is, the use of these IQCs does not enlarge the guaranteed domain of performance for synthesis.

Key words: integral quadratic constraints; input saturation; cost control

0 Introduction

Actuator saturation is common in feedback control systems. Control design often requires us to take such actuation into account, or the design may "wind up" the actuator, possibly resulting in degraded performance or even instability. Common solutions include anti-windup compensations (see, for example, 【1, 2】). This approach is relatively simple to be used, but the resulting performance may be limited. Design methods based on Riccati equations, such as 【3 ~ 6】, tend to improve the performance. More recent results also include those developed using the circle and the Popov criteria within the frame-work of linear matrix inequalities (LMIs) (see, for example, 【7, 8】).

This paper studies the problem of control design for linear systems with input saturation. It is well known that integral quadratic constraints (IQC) can be used to describe input saturation and that the use of IQC in analysis can lead to less conservative performance bound and larger domain of attraction. However, the situation is rather different in control synthesis. It was shown by Iwasaki that the use of circle criterion (a special type of IQC) does not yield a larger guaranteed domain of stabilizability when compared with a linear controller without saturation. It was further shown by Iwasaki and Fu that the use of circle criterion does not lead to a larger domain of attraction for guaranteed cost control either. These negative results are somewhat surprising because they seem to be against the intuition that a saturated controller should achieve better performance or stability.

In this work, we study the use of a more general IQC in synthesis. It is shown that the negative results above also hold for a class of commonly used IQCs. That is, the use of these IQCs does not enlarge the guaranteed domain of performance for synthesis.

1 IQC based analysis

Consider the following linear time-invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Kx(t) \\ e(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

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where $u(t)\in\mathscr{R}^m$ is the control input, $x(t)\in\mathscr{R}^n$ is the state, $z(t)\in\mathscr{R}^m$ is the feedback signal, and $e(t)\in\mathscr{R}^r$ is the output. Suppose the control input u is subjected to the following saturation nonlinearity, i.e.

$$u=\phi(z)\Leftrightarrow u_i=\begin{cases}\alpha_i & \text{if } u_i>\alpha_i\\ z_i & \text{if } |u_i|\le\alpha_i\\ -\alpha_i & \text{if } u_i<-\alpha_i\end{cases}\tag{2}$$

where $\alpha\in\mathscr{R}^m$ is a positive constant vector.

A given set of state vectors X is caled a domain of attraction if any state trajectory starting from a point $x_0\in X$ converges to the origin as the time goes to infinity. The set X is called an invariant set if any state trajectory starting from a point $x_0\in X$ stays in X. The set X is called a domain of performance with level γ if it is a domain of attraction and any output e in response to $x_0\in X$ has its $\mathscr{L}_2$ norm less than or equal to $\sqrt{\gamma}$.

Integral quadratic constraints (IQC) are convenient and powerful for capturing various types of nonlinearities and uncertainties in the system. In this work, The use of IQC to describe the input saturation is considered, i.e., replaceing the input saturation (2) with the following【9】:

$$\int_0^\infty\sigma(x_\pi(t),\ \varepsilon w(t),\ z(t))\mathrm{d}t\ge 0,\ \forall\varepsilon\in[0,1]\tag{3}$$

where $w=z-u=z-\phi(z)$, $\sigma(\cdot)$ is a quadratic form, and x_π is dened by

$$\dot{x}_\pi=A_\pi x_\pi+B_u u+B_z z,\ x_\pi(0)=0\tag{4}$$

with a Hurwitz matrix A_π. The inequality (3) needs to be satisfied for all $z\in\mathbf{Z}\subset\mathscr{L}_2^m[0,\infty)$, where $\mathbf{Z}$ is an admissible set for z which will be specied later.

An alternative representation of the IQC (3)-(4) is given by

$$\int_{-\infty}^{\infty}\begin{bmatrix}z(j\omega)\\ \varepsilon w(j\omega)\end{bmatrix}^*\prod(j\omega)\begin{bmatrix}z(j\omega)\\ \varepsilon w(j\omega)\end{bmatrix}\mathrm{d}\omega\ge 0,\ \forall\varepsilon\in[0,1]\tag{5}$$

where $\prod(j\omega)$ is a rational matrix function of $j\omega$. An IQC can be either static (where $\Phi(j\omega)$ is a constant matrix and hence x_π is void) or dynamic (where $\Phi(j\omega)$ depends on ω).

Examples of IQCs are plenty. Here, three commonly used ones are listed for input saturation.

Example 1 Norm Bound. In this case, x_π is void and we have

$$\sigma(w,z)=\sum_{i=1}^m\tau_i(z_i^2-u_i^2)=\sum_{i=1}^m\tau_i w_i(2z_i-w_i)\ge 0\tag{6}$$

where $\tau_i>0$ are arbitrary constants. In this example, $\mathbf{Z}=\mathscr{L}_2^m$.

Example 2 Sector Bound. Suppose it is known in advance that $|z_i|$ never exceeds $\rho_i\ge\alpha_i$ (This happens, e.g., when the state is confined in a given set). We can take

$$\mathbf{Z}=\{z:z\in\mathscr{L}_2^m,\ |z_i(t)|\le\rho_i,\ \forall t\ge 0,\ i=1,\cdots,m\}\tag{7}$$

Then we can bound u_i by the following:

$$(z_i-u_i)(u_i-s_i z_i)=w_i((1-s_i)z_i-w_i)\ge 0$$

where $s_i=\alpha_i/\rho_i$. Hence, we have

$$\sigma(w,z)=\sum_{i=1}^m\tau_i w_i((1-s_i)z_i-w_i)\ge 0\tag{8}$$

for any $\tau_i>0$. Again, x_π is void in this case. Obviously, this is a tighter constraint than (6). In fact, this is the tightest static IQC in the sense that the $\sigma(w;z)$ in (8) lower-bounds any other $\sigma(w;z)$ over $\mathbf{Z}$.

Example 3 Zames-Falb Bound【10】. Suppose $H_i(s)$ are any stable rational functions with

$$\mathrm{Re}(1+H_i(j\omega))\ge 0,\ \forall\omega\tag{9}$$

Denote by h_i the impulse response of $H_i(s)$. Then, we have

$$(z_i-u_i)(u_i+h_i\circ u_i)=w_i h_i\circ(u_i-w_i)\ge 0,\ \forall i$$

Hence, we have

$$\sigma(x_\pi, w, z) = \sum_{i=1}^{m} \tau_i w_i (h_i \circ (u_i - w_i)) \geq 0 \tag{10}$$

where $\tau_i > 0$ are any constants. In this case, x_π is the state associated with $H(s) = \mathrm{diag}\{H_1(s), \cdots, H_m(s)\}$. In this example, $\mathbf{Z} = \mathscr{L}_2^m$.

In the presence of x_π, we can generalize the feedback signal z to involve x_π as well, i.e., $z = Kx + K_\pi x_\pi$. We may combine (1) and (4) to obtain the following augmented system:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \mathscr{A}\tilde{x}(t) + \mathscr{B}u(t) \\ z(t) &= \mathscr{K}\tilde{x} \\ e(t) &= \mathscr{C}\tilde{x} + \mathscr{D}u(t) \end{aligned} \tag{11}$$

where $\tilde{x} = [x^T \quad x_\pi^T]^T \in \mathscr{R}^{\tilde{n}}$,

$$\mathscr{A} = \begin{bmatrix} A & 0 \\ B_z K & A_\pi + B_z K_\pi \end{bmatrix}; \ \mathscr{B} = \begin{bmatrix} B \\ B_u \end{bmatrix}; \ \mathscr{K} = [K \quad K_\pi]; \ \mathscr{C} = [C \quad 0]; \ \mathscr{D} = D. \tag{12}$$

We can also rewrite the quadratic form $\sigma(\cdot)$ as follows:

$$\sigma(x_\pi, w, z) = \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}^T \Omega \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \tag{13}$$

for some suitable matrix Ω.

System analysis using IQC is based on the following lemma:

Lemma 1 Consider the argumented system (12) and a given constant $\gamma > 0$. Let $V(\tilde{x}) = \tilde{x}^T \mathscr{P} \tilde{x}$ be a candidate Lyapunov function, where $\mathscr{P} = \mathscr{P}^T > 0$. Take

$$\tilde{X} = \{\tilde{x}: V(\tilde{x}) \leq 1\} \tag{14}$$

and the corresponding admissible set for z as

$$\mathbf{Z} = \{\mathscr{K}\tilde{x}: \tilde{x} \in \mathscr{L}_2^{\tilde{n}}, \ \tilde{x}(t) \in \tilde{X}, \ \forall t \geq 0\} \tag{15}$$

Suppose the IQC (5) is satisfied for all $z \in \mathbf{Z}$. Also suppose

$$\begin{bmatrix} \mathscr{A}^T \mathscr{P} + \mathscr{P}\mathscr{A} & \mathscr{P}\mathscr{B} \\ \mathscr{B}^T \mathscr{P} & 0 \end{bmatrix} + \gamma^{-1} \begin{bmatrix} \mathscr{C}^T \\ \mathscr{D}^T \end{bmatrix} [\mathscr{C} \quad \mathscr{D}] + \Omega \leq 0 \tag{16}$$

and

$$r^{-1} \| \mathscr{C}\tilde{x} + \mathscr{D}u \|^2 + \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}^T \Omega \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \geq 0, \ \forall \tilde{x} \in \tilde{X}, \ u = \phi(\mathscr{K}\tilde{x}) \tag{17}$$

Then, $\tilde{X}$ is an invariant set. Further, the set

$$X = \{x: \begin{bmatrix} x \\ 0 \end{bmatrix} \in \tilde{X}\} \tag{18}$$

is a domain of performance with level γ.

Proof Consider any $\tilde{x} \in \mathscr{L}_2^{\tilde{n}}$ with $\tilde{x}(t) \in \tilde{X}$, $\forall t \geq 0$ and $u = \phi(\mathscr{K}\tilde{x})$. From (16), we get

$$2\tilde{x}^T \mathscr{P} \dot{\tilde{x}} + \gamma^{-1} \| \mathscr{C}\tilde{x} + \mathscr{D}u \|^2 + \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}^T \Omega \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \leq 0 \tag{19}$$

Using (17), the above leads to

$$2\tilde{x}^T \mathscr{P} \dot{\tilde{x}} \leq 0$$

which implies that $\tilde{x}^T \mathscr{P} \tilde{x}$ is nonincreasing and hence $\tilde{X}$ is an invariant set. Hence the IQC (5) holds for any trajectory of $\tilde{x}$ starting in $\tilde{X}$. Returning to (19) and integrating it from 0 to ∞, we get

$$\gamma^{-1} \int_0^\infty \| e(t) \|^2 \mathrm{d}t \leq \tilde{x}^T(0) \mathscr{P} \tilde{x}(0) + \int_0^\infty \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}^T \Omega \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \mathrm{d}t$$

Using (5), we conclude that

$$\gamma^{-1} \int_0^\infty \| e(t) \|^2 \mathrm{d}t \leq \tilde{x}^T(0) \mathscr{P} \tilde{x}(0) \leq 1$$

Hence, $\tilde{X}$ is a domain of perfomance with γ, so is X.

2 IQC based synthesis

Now a key technical result is presented.

Theorem 1 Consider the argumented system (12) and a given constant $\gamma > 0$. Let the conditions in **Lemma 1** be satisfied. Suppose there exists a scaling(constant) matrix F such that the following conditions are also satisfied:

(ⅰ) $|F_i z(t)| \leq \sigma_i$ for all $z \in \mathbf{Z}$, where F_i is the ith row of F;

(ⅱ) The IQC condition (5) is replaced by

$$\begin{bmatrix} I \\ \varepsilon(I-F) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ \varepsilon(I-F) \end{bmatrix} \geq 0 \quad \forall \varepsilon \in [0,1], \ \forall \omega \tag{20}$$

(ⅲ) The inequality (17) is replaced by

$$\gamma^{-1}(\mathscr{C}^T + F^T \mathscr{D}^T)(\mathscr{C} + \mathscr{D}F) + \begin{bmatrix} I \\ F \end{bmatrix}^T \Omega \begin{bmatrix} I \\ F \end{bmatrix} \geq 0 \tag{21}$$

Then, $u = Fz$ is an unsaturated linear (dynamic) controller that guarantees the same domain of performance as the (possibly) saturated controller $u(t) = \phi(\tilde{K}\tilde{x}(t))$.

Proof Take $u = Fz$. Condition (ⅰ) ensures that input saturation does not occur. Condition (ⅱ) guarantees that the IQC (5) is satisfied for the corresponding (w, z) pair and condition (ⅲ) implies that (17) is also satisfied for $u = Fz$. Hence, from **Lemma 1**, $u = Fz$ also guarantees the same domain of performance X.

Remark 1 The verification of (20) can often be simplified as far as the ε parameter is concerned. Writing

$$\Phi(j\omega) = \begin{bmatrix} \Phi_{11}(j\omega) & \Phi_{12}(j\omega) \\ \Phi_{21}(j\omega) & \Phi_{22}(j\omega) \end{bmatrix},$$

it is pointed out that most IQCs (including those listed in [9]) have the property that

$$\Phi_{11}(j\omega) \geq 0; \ \Phi_{22}(j\omega) \leq 0, \ \forall \omega \tag{22}$$

In this case, (20) is convex in ε. Thus, it is sufficient to check at $\varepsilon = 1$ (because (20) at $\varepsilon = 0$ is the same as $\Phi_{11}(j\omega) \geq 0$). That is, (20) is simplified to checking

$$\begin{bmatrix} I \\ I-F \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ I-F \end{bmatrix} \geq 0 \quad \forall \omega \tag{23}$$

The same comment applies to (5), i.e., it sufficies to check at $\varepsilon = 1$ when (22) holds.

Next we show several cases where an IQC-based synthesis does not yield a better design as far as a guaranteed domain of performance is concerned.

2.1 Synthesis using static IQCs

We apply **Theorem 1** to synthesis using a static IQC.

Let $\gamma > 0$, K and $P = P^T > 0$ be given. Form $X = \{x : x^T P x \leq 1\}$. Define

$$\rho_i = \max\{\alpha_i, \max_{x \in X} |K_i x|\}$$

and $s_i = \alpha_i / \rho_i$ for $i = 1, \cdots, m$. Then, the sector bound based IQC is given by (8) and the corresponding admissible set for z is given by (7). Note that (17) is satisfied for any $\gamma > 0$.

To ensure that X is a domain of performance, the IQC-based synthesis requires (16) which is equivalent to

$$2x^T P(Ax + Bu) + \gamma^{-1}(Cx + Du)^T(Cx + Du) + \sum_{i=1}^{m} \tau_i (K_i x - u_i)(u_i - s_i K_i x) < 0, \ \forall \begin{bmatrix} x \\ u \end{bmatrix} \neq 0 \tag{24}$$

Now take the new control signals $u_i = s_i K_i x$. We see that $u_i(t)$ are notsaturated in X and that $\sigma(w, z) = 0$. Hence, by **Theorem 1**, this unsaturated controller achieves the same domain of performance X. Indeed, this can be trivially verified from (24). Taking $u_i = s_i K_i x$, (24) reduces to

$$2x^T P(Ax + Bu) + \gamma^{-1}(Cx + Du)^T(Cx + Du) < 0, \ \forall x \neq 0$$

Integrating the above from $t = 0$ to ∞ gives

$$\int_0^\infty \| e(t) \|^2 dt < \gamma x^T(0) P x(0) \le \gamma$$

Recall the remark that (8) is the tightest static IQC (see **Example 2**), it is concluded that any domain of performance guranteed by using a static IQC-based synthesis is also achievable by using a non-saturated controller.

2.2 Synthesis using dynamic IQCs: Negative result

We apply **Theorem 1** further to show that a dynamic IQC based synthesis may not offer improvement either.

Let γ, $\mathscr{K}$, $\mathscr{P}$ and **Z** be similarly defined as in the previous subsection (except for the change of notation due to dimension augmentation of the state) and consider any dynamic IQC in (5). If we choose $z(t)$ to be constant, the dynamic IQC should also be held but reduced to:

$$\begin{bmatrix} z \\ \varepsilon w \end{bmatrix}^T \Phi(0) \begin{bmatrix} z \\ \varepsilon w \end{bmatrix} \ge 0, \ \forall \text{ constant } z \in \mathbf{Z}$$

We can denote the above by

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \Omega_0 \begin{bmatrix} x \\ u \end{bmatrix} \ge 0, \ \forall x \in X \tag{25}$$

for some constant matrix Ω_0. Note that x_π is not involved here because z is constant, implying that u is also constant and $x_\pi(t)$ remains zero at all t. Also note that (25) implies (17).

Let decompose

$$\mathscr{P} = \begin{bmatrix} P & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

multiplying

$$\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}$$

and its transpose to the right and left of (16), respectively, we reduce it to

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \gamma^{-1} \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C \quad D] + \Omega_0 < 0 \tag{26}$$

By now, we have established a static IQC (25) and an equivalent of (16), which is (26). Hence, the result can be applied in the previous subsection and an conclusion can be drawn that there exists a unsaturated controller which gives the same domain of performance X. Moreover, the unsaturated controller can be proportional to z, as in the static IQC case.

Example Synthesis using Zames-Falb and sector bounds. We consider the case of synthesis using both a Zames-Falb bound (10) and a sector bound. We will demonstrate the result above by a more direct proof.

Due to the dynamic nature of the IQC, we need to consider the augmented system (11). The IQC is given as follows:

$$\sigma(x_\pi, w, z) = \sum_{i=1}^m \tau_i (z_i - u_i)(u_i - s_i z_i) + \hat{\tau}_i (z_i - u_i)(u_i + v_i) \tag{27}$$

with any positive τ_i and $\hat{\tau}_i$, where $v_i(j\omega) = H_i(j\omega) u_i(j\omega)$ and $H(j\omega)$ satisfies(9).

Let $\gamma > 0$, $\mathscr{K}$ and $P = P^T > 0$ be given. Define

$$\rho_i = \max\{\alpha_i, \max_{\tilde{x} \in \tilde{X}} | \mathscr{K}_i \tilde{x} |\}$$

and s_i as in Case 1, where $\tilde{X}$ is given in (14). Suppose (16) holds. Then, by **Lemma 1**, X in (18) is a domain of performance with level γ.

Now, we again take $u_i(t) = s_i z_i(t)$. This implies that $u_i(t)$ are not saturated in $\tilde{X}$. We have

$$\sigma(x_\pi, w, z) = \sum_{i=1}^m \hat{\tau}_i (1 - s_i) s_i z_i (z_i + h_i \circ z_i)$$

where h_i denotes the impulse response of $H_i(s)$. It follows that (下转第 126 页)