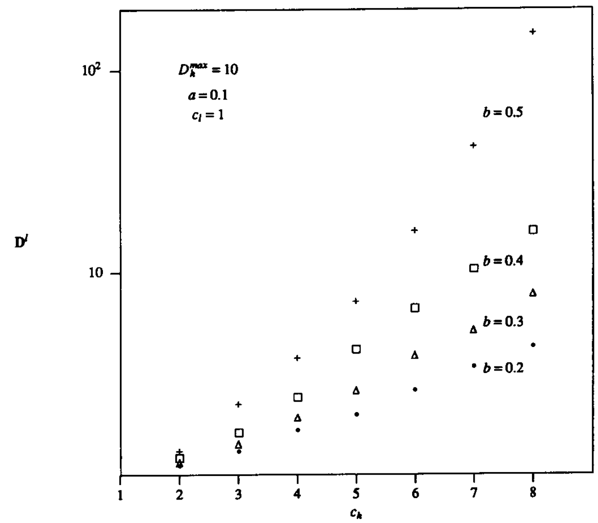

Fig. 4. Expected delays per low priority arrival for π_{MND} and exponential interarrivals (for fixed a).

arrivals are controlled by the acting $D_h^{\max}$ value. The other factor which strongly affects the delays of the low priority traffic is the processing time of the high priority arrivals, especially for relatively high rates of the high priority traffic; indeed, in the presence of relatively high c_h and a values, the high priority traffic dominates the processor resulting in severe delay penalties for the low priority arrivals. In general, the low priority traffic is vulnerable to any changes concerning the high priority traffic while the delays of the latter are always controlled by $D_h^{\max}$, regardless of possible variations in the rate of the low priority arrivals. For relatively low arrival rates of the high priority traffic, the factor which affects the delays of the low priority traffic most is the processing time of the high priority arrivals; the second most effective such factor is the rate of the low priority traffic. In Figs. 2–4, we exhibit the effects of various factors discussed above on the expected delays induced by the π_{MND}; Fig. 2 depicts the effects of $D_h^{\max}$ and c_h on the expected delays of the high priority arrivals, while Figs. 3 and 4 show the effects of the arrival rates and c_h on the expected delays of the low priority traffic.

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H_∞ Analysis and Synthesis of Discrete-Time Systems with Time-Varying Uncertainty

Carlos E. de Souza, Minyue Fu, and Lihua Xie

Abstract—**The problems of H_∞ analysis and synthesis of discrete-time systems with block-diagonal real time-varying uncertainty are considered. We show that these problems can be converted into "scaled" H_∞ analysis and synthesis problems. The problems of quadratic stability analysis and quadratic stabilization of these types of systems are dealt with as a special case. The results on synthesis are established for general linear dynamic output feedback control.**

I. Introduction

This note is aimed at the problems of H_∞ analysis and synthesis of discrete-time systems with real time-varying norm-bounded uncertainty. Similar problems with different settings have been studied elsewhere. In [1] and [2], continuous-time systems with time-invariant complex parameter uncertainty are considered and the so-called μ-analysis and μ-synthesis are developed for analyzing robust H_∞ performance and designing robust H_∞ controllers. In [3]–[5], continuous-time systems with time-varying parameter uncertainty are treated and results similar to the μ-analysis and μ-synthesis are given. In [6], quadratic stability of discrete-time systems with complex and real uncertainties are considered, and certain relationships between quadratic stability and H_∞ analysis are reported. Some results on quadratic stabilization of discrete-time uncertain systems are given in [7] and [8].

The emphasis of this note is as follows: 1) we deal with the problems of H_∞ analysis and synthesis, that is, both quadratic stability and robust H_∞ performance need to be achieved; 2) linear dynamic output feedback control is of our interest; and 3) time-varying parameter uncertainty with a block-diagonal structure is considered. By replacing the parameter uncertainty by some superfluous exogenous disturbance, we show the following two parallel results:

i) The problem of robust H_∞ performance analysis of such an uncertain system can be converted into a "scaled" H_∞ performance analysis problem of a system without uncertainty; and similarly;

ii) The H_∞ control problem of such an uncertain system

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can be converted into a "scaled" H_∞ control problem of a system without uncertainty. General linear dynamic output feedback control is considered and parameter uncertainty is allowed in both state and output equations.

Therefore, techniques for standard H_∞ analysis and synthesis problems can be applied to solve the robust H_∞ analysis and synthesis problems. These results can be viewed as a version of the μ-analysis and μ-synthesis [1], [2] for discrete-time systems with time-varying real parameter uncertainty. As a special case, the problems of quadratic stability analysis and quadratic stabilization of certain type of discrete-time uncertain systems via linear dynamic output feedback are solved by using the same technique.

II. Problem Formulation

We consider discrete-time uncertain systems of the following form:

$$\begin{aligned} x(k+1) &= [A + \Delta A(k)]x(k) + B_1 w(k) \\ &\quad + [B + \Delta B(k)]u(k) \\ z(k) &= C_1 x(k) + D_{12} u(k) \\ y(k) &= [C + \Delta C(k)]x(k) + D_{21} w(k) + [D + \Delta D(k)]u(k) \end{aligned} \tag{1}$$

where $x(k) \in \boldsymbol{R}^n$ is the *state*, $u(k) \in \boldsymbol{R}^m$ is the *control input*, $w(k) \in \boldsymbol{R}^q$ is the *exogenous disturbance* which belongs to $l_2[0,\infty)$, $y(k) \in \boldsymbol{R}^r$ is the *measured output*, $z(k) \in \boldsymbol{R}^p$ is the controlled output, A, B, C, D, B_1, C_1, D_{12}, and D_{21} are known real constant matrices of appropriate dimensions which describe the "nominal" system, and $\Delta A(k), \Delta B(k), \Delta C(k), \Delta D(k)$ represent the *time-varying parameter uncertainty*. The parameter uncertainty is assumed to be of the following structure:

$$\begin{bmatrix} \Delta A(k) & \Delta B(k) \\ \Delta C(k) & \Delta D(k) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(k) \begin{bmatrix} E_1 & E_2 \end{bmatrix} \tag{2}$$

where H_1, H_2, E_1, and E_2 are known real constant matrices which capture the *structure of the uncertainty*, and $F(k) \in \boldsymbol{R}^{\alpha \times \beta}$ is the *uncertainty matrix* in the following block-diagonal form:

$$F(k) = \operatorname{diag}\{F_1(k), F_2(k), \cdots, F_v(k)\}, \qquad F_i^T(k)F_i(k) \le \rho_i^2 I, \quad k = 0,1,2,\cdots \tag{3}$$

for some $\rho_i > 0$, $i = 1,2,\cdots,v$.

When the robust stability or stabilization is the only concern, the system (1) reduces to

$$\begin{aligned} x(k+1) &= [A + \Delta A(k)]x(k) + [B + \Delta B(k)]u(k) \\ y(k) &= [C + \Delta C(k)]x(k) + [D + \Delta D(k)]u(k). \end{aligned} \tag{4}$$

We recall [9] that the uncertain system

$$x(k+1) = [A + \Delta A(k)]x(k) \tag{5}$$

is said to be *quadratically stable* if there exists some symmetric positive definite matrix P such that

$$[A + \Delta A(k)]^T P [A + \Delta A(k)] - P < 0, \qquad k = 0,1,2,\cdots, \tag{6}$$

for all admissible $\Delta A(\cdot)$. An associated Lyapunov function for establishing robust stability is $x^T P x$. The notion of quadratic stability is conservative for robust stability in view of the fact that a constant P matrix is used in the Lyapunov function above. However, this notion is very popular for dealing with time-varying uncertainty (see, e.g., [9], [6] and the references thereof) due to its simplicity and lack of better methods for doing so.

Definition 1: Consider the following time-varying system:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k) \\ z(k) &= C(k)x(k) \end{aligned} \tag{7}$$

where $x(k), w(k), z(k)$ are the same as in (1) and $A(k)$, $B(k)$, and $C(k)$ are time-varying real matrices of appropriate dimensions. The system (7) is said to have *H_∞ disturbance attenuation γ* for some $\gamma > 0$ if it is asymptotically stable with the following property:

$$\|z\|_2 < \gamma \|w\|_2, \quad \text{for all nonzero } w \in l_2[0,\infty) \text{ whenever } x(0) = 0 \tag{8}$$

where $\|\cdot\|_2$ denotes the usual $l_2[0,\infty)$ norm.

III. Robust H_∞ Analysis and Synthesis

Consider the uncertain system (1)–(2) with block-diagonal parameter uncertainty (3). Then, for any constant vector

$$\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_v)^T, \qquad \epsilon_k > 0, \qquad k = 1,2,\cdots,v \tag{9}$$

we define

$$M(\epsilon) = \operatorname{diag}\{\rho_1 \epsilon_1 I_{i_1 \times i_1}, \rho_2 \epsilon_2 I_{i_2 \times i_2}, \cdots, \rho_v \epsilon_v I_{i_v \times i_v}\} \tag{10}$$

$$N(\epsilon) = \operatorname{diag}\{\epsilon_1^{-1} I_{j_1 \times j_1}, \epsilon_2^{-1} I_{j_2 \times j_2}, \cdots, \epsilon_v^{-1} I_{j_v \times j_v}\} \tag{11}$$

$$\overline{H}_l(\epsilon) = H_l M(\epsilon), \quad \overline{E}_l(\epsilon) = N(\epsilon) E_l, \quad l = 1,2 \tag{12}$$

and

$$\overline{F}(k) = M^{-1}(\epsilon) F(k) N^{-1}(\epsilon). \tag{13}$$

Obviously, we have

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(k) \begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} \overline{H}_1(\epsilon) \\ \overline{H}_2(\epsilon) \end{bmatrix} \overline{F}(k) \begin{bmatrix} \overline{E}_1(\epsilon) & \overline{E}_2(\epsilon) \end{bmatrix} \tag{14}$$

and $\overline{F}^T(k)\overline{F}(k) \le I$.

We further define the following "auxiliary" system:

$$\begin{aligned} x(k+1) &= Ax(k) + \begin{bmatrix} \overline{H}_1(\epsilon) & \gamma^{-1} B_1 \end{bmatrix} \tilde{w}(k) + Bu(k) \\ \tilde{z}(k) &= \begin{bmatrix} \overline{E}_1(\epsilon) \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} \overline{E}_2(\epsilon) \\ D_{12} \end{bmatrix} u(k) \\ y(k) &= Cx(k) + \begin{bmatrix} \overline{H}_2(\epsilon) & \gamma^{-1} D_{21} \end{bmatrix} \tilde{w}(k) + Du(k) \end{aligned} \tag{15}$$

where $x(k) \in \boldsymbol{R}^n$ is the *state*, $\tilde{w}(k) \in \boldsymbol{R}^{q+\alpha}$ is the *exogenous disturbance*, $\tilde{z}(k) \in \boldsymbol{R}^{p+\beta}$ is the *controlled output*, $y(k) \in \boldsymbol{R}^r$ is the *measured output*, $\gamma > 0$ represents the desired disturbance attenuation level, ϵ is a scaling vector to be tuned, and the other variables are the same as in (1).

We have the following results.

Theorem 1: Suppose the unforced system of (15) (by setting $u(k) \equiv 0$) has unitary H_∞ disturbance attenuation for some $\gamma > 0$ and $\epsilon_i > 0$, $i = 1,2,\cdots,v$. Then, the unforced uncertain system of (1)–(3) (by setting $u(k) \equiv 0$) is quadratically stable and has H_∞ disturbance attenuation γ for all admissible uncertainty.

See Appendix for proof.

Theorem 2: Consider the uncertain system (1)–(3). Given a linear dynamic strictly proper output controller such that the resulting closed-loop system of (15) has unitary H_∞ disturbance attenuation for some $\gamma > 0$ and $\epsilon_i > 0$, $i = 1, 2, \cdots, v$. Then, the closed-loop system corresponding to (1)–(3) and the same controller is quadratically stable and has H_∞ disturbance attenuation γ for all admissible uncertainty.

Proof: Let the controller be of the following state-space realization:

$$\xi(k+1) = A_c\xi(k) + B_c y(k)$$
$$u(k) = C_c\xi(k) \quad (16)$$

where the dimension of the controller and the matrices A_c, B_c, and C_c are to be chosen. The desired result can be established by applying Theorem 1 to the closed-loop system of (1) with (16) and the closed-loop system of (15) with (16). The detail is lengthy but straightforward, and is therefore omitted. ▽ ▽ ▽

Remark 1: Note that the "scaled" H_∞ synthesis problem in Theorem 2 can be solved by using existing results on discrete-time H_∞ control such as those in [10]. More specifically, the scaled problem can be solved in terms of two algebraic Ricatti equations and the class of controllers corresponding to the scaled problem which guarantee both quadratic stability and robust H_∞ performance can be parameterized.

IV. Quadratic Stability and Stabilization

Similar to the case of robust H_∞ control, we define the following auxiliary system for the uncertain system (4), (2), and the block-diagonal uncertainty (3) as follows:

$$x(k+1) = Ax(k) + \overline{H}_1(\epsilon)\hat{w}(k) + Bu(k)$$
$$\hat{z}(k) = \overline{E}_1(\epsilon)x(k) + \overline{E}_2(\epsilon)u(k)$$
$$y(k) = Cx(k) + \overline{H}_2(\epsilon)\hat{w}(k) + Du(k) \quad (17)$$

where $x(k) \in \mathbf{R}^n$ is the *state*, $\hat{w}(k) \in \mathbf{R}^\alpha$ is the *exogenous disturbance*, $\hat{z}(k) \in \mathbf{R}^\beta$ is the *controlled output*, $y(k) \in \mathbf{R}^r$ is the *measured output*, and all other variables are the same as in (15) except that $\epsilon_v = 1$. The reason for setting ϵ_v to 1 is that only $v - 1$ scaling parameters are needed, adding another one will not contribute any more.

The following results are derived. The proofs can be carried out by the same way as for Theorems 1 and 2, and are therefore omitted.

Theorem 3: The unforced uncertain system of (4), (2), and (3) is quadratically stable if the unforced system of (17) has unitary H_∞ disturbance attenuation for some $\epsilon_i > 0$, $i = 1, 2, \cdots, v-1$ and $\epsilon_v = 1$.

Theorem 4: The uncertain system described by (4), (2), and (3) is quadratically stabilizable via a linear dynamic strictly proper output feedback controller if the closed-loop system of (17) with the same controller has unitary H_∞ disturbance attenuation for some $\epsilon_i > 0$, $i = 1, 2, \cdots, v-1$ and $\epsilon_v = 1$.

Remark 2: The results above are for block-diagonal real time-varying uncertainty. For the case of single-block real time-varying uncertainty, it is claimed in [6] that the result in Theorem 3 is not only sufficient but also necessary. This implies that the result in Theorem 4 is also necessary and sufficient for quadratic stabilization of systems with single-block real time-varying uncertainty. Note that in this case the system (17) does not involve any scaling parameter.

Remark 3: Similar to Remark 1, the "scaled" H_∞ synthesis problem in Theorem 4 can be solved in terms of two algebraic Ricatti equations, and quadratically stabilizing controllers can be parameterized.

Appendix
Proof of Theorem 1

The following lemma is essential for the proof of Theorem 1.

Lemma A: Let $A \in \mathbf{R}^{n\times n}$, $H \in \mathbf{R}^{n\times\alpha}$, $E \in \mathbf{R}^{\beta\times n}$, and $Q = Q^T \in \mathbf{R}^{n\times n}$ be given matrices. Suppose there exists a symmetric positive-definite matrix P such that the following hold:

a) $H^TPH < I$, and
b) $A^TPA + A^TPH[I - H^TPH]^{-1}H^TPA + E^TE + Q < 0$.

Then, we have

$$[A + HF(k)E]^TP[A + HF(k)E] + Q < 0 \quad (18)$$

for all $F(k)$ satisfying $F^T(k)F(k) \le I, k = 0, 1, \cdots$.

Proof: Introducing

$$W(k) = [I - H^TPH]^{-1/2}H^TPA - [I - H^TPH]^{1/2}F(k)E$$

we have

$$W^T(k)W(k) = A^TPH[I - H^TPH]^{-1}H^TPA - E^TF^T(k)H^TPA$$
$$- A^TPHF(k)E + E^TF^T(k)[I - H^TPH]F(k)E.$$

Now, considering a) together with the fact that $F^T(k)F(k) \le I$, we obtain

$$A^TPH[I - H^TPH]^{-1}H^TPA + E^TE \ge E^TF^T(k)H^TPA$$
$$+ A^TPHF(k)E + E^TF^T(k)H^TPHF(k)E.$$

Consequently, (18) follows from a) and b). ▽ ▽ ▽

Now we turn into the proof of Theorem 1.

Proof: Let $\overline{B} = [\overline{H}_1(\epsilon), \gamma^{-1}B_1]$. By Lemma 2.1 in [11], the unforced system of (15) (obtained by setting $u(k) = 0$) is stable with unitary H_∞ disturbance attenuation if and only if there exists a symmetric positive-definite matrix X satisfying $I - \overline{B}^TX\overline{B} > 0$ and

$$A^TXA - X + A^TX\overline{B}(I - \overline{B}^TX\overline{B})^{-1}\overline{B}^TXA$$
$$+ C_1^TC_1 + \overline{E}_1^T(\epsilon)\overline{E}_1(\epsilon) < 0. \quad (19)$$

By using the matrix inversion lemma, we can rewrite (19) as

$$A^T[X^{-1} - \gamma^{-2}B_1B_1^T - \overline{H}_1(\epsilon)\overline{H}_1^T(\epsilon)]^{-1}A - X$$
$$+ C_1^TC_1 + \overline{E}_1^T(\epsilon)E_1(\epsilon) < 0. \quad (20)$$

Define

$$P = [X^{-1} - \gamma^{-2}B_1B_1^T]^{-1} = \left[X^{-1} - \overline{B}\overline{B}^T + \overline{H}_1(\epsilon)\overline{H}_1^T(\epsilon)\right]^{-1}. \quad (21)$$

By using the matrix inversion lemma again, we obtain

$$(X^{-1} - \overline{B}\overline{B}^T)^{-1} = X + X\overline{B}(I - \overline{B}^TX\overline{B})^{-1}\overline{B}^TX > 0. \quad (22)$$

Since $I - \overline{B}^TX\overline{B} > 0$, (21) and (22) give $P > 0$ and

$$P^{-1} - \overline{H}_1(\epsilon)\overline{H}_1^T(\epsilon) > 0. \quad (23)$$

From (23) and further application of the matrix inversion lemma on (20), we obtain the following:

a′) $\overline{H}_1^T(\epsilon)P\overline{H}_1(\epsilon) < I$, and
b′) $A^TPA + A^TP\overline{H}_1(\epsilon)[I - \overline{H}_1^T(\epsilon)P\overline{H}_1(\epsilon)]^{-1}\overline{H}_1^T(\epsilon)PA + \overline{E}_1^T(\epsilon)\overline{E}_1(\epsilon) + Q < 0$

where

$$Q = \gamma^{-2}PB_1[I + \gamma^{-2}B_1^TPB_1]^{-1}B_1^TP - P + C_1^TC_1.$$

Hence, using Lemma A and (14), it follows that

$$[A + H_1F(k)E_1]^TP[A + H_1F(k)E_1] - P$$
$$+ \gamma^{-2}PB_1[I + \gamma^{-2}B_1^TPB_1]^{-1}B_1^TP + C_1^TC_1 < 0 \quad (24)$$

which implies that quadratic stability of the unforced system of (1) because the last two terms in the inequality are positive-semi-definite. In order to establish the H_∞ disturbance attenuation property, we assume $x(0) = 0$ and need to show that

$$J := \sum_{k=0}^{\infty} [z^T(k)z(k) - \gamma^2 w^T(k)w(k)] < 0, \quad \text{whenever } w(k) \not\equiv 0. \tag{25}$$

The existence of the sum in (25) is guaranteed by the boundedness of $w(k)$ and the quadratic stability of the unforced system of (1). It is obvious that (25) holds if $x(k) = 0$ for all $k \geq 0$. Therefore, we assume $x(k) \not\equiv 0$ in the sequel.

Abbreviating $A + \Delta A(k)$ by A_Δ and defining

$$\Gamma = [P^{-1} + \gamma^2 B_1 B_1^T]^{-1} > 0 \tag{26}$$

it is straightforward to show by using the matrix inversion lemma that (26) and (22) imply

$$B_1^T \Gamma B_1 < \gamma^2 I \tag{27}$$

and

$$U(k) := A_\Delta^T \Gamma A_\Delta - \Gamma + \gamma^{-2} A_\Delta^T \Gamma B_1 [I - \gamma^{-2} B_1^T \Gamma B_1^T]^{-1} \cdot B_1^T \Gamma A_\Delta + C_1^T C_1 < 0. \tag{28}$$

Using $x(0) = 0$, we have

$$\sum_{k=0}^{N} [x^T(k+1)\Gamma x(k+1) - x^T(k)\Gamma x(k)] = x^T(N+1)\Gamma x(N+1) \geq 0.$$

Let

$$J(k) := z^T(k)z(k) - \gamma^2 w^T(k)w(k) + x^T(k+1)\Gamma x(k+1) - x^T(k)\Gamma x(k)$$

$$J_N := \sum_{k=0}^{N} [z^T(k)z(k) - \gamma^2 w^T(k)w(k)].$$

Then, we have

$$J_N = \sum_{k=0}^{N} J(k) - x^T(N+1)\Gamma x(N+1).$$

Using (27) and (28), it can be verified that

$$J(k) = x^T(k)U(k)x(k) - \gamma^2 V^T(k)[I - \gamma^{-2} B_1^T \Gamma B_1] V(k) \leq 0$$

where

$$V(k) := w(k) - \gamma^{-2}[I - \gamma^{-2} B_1^T \Gamma B_1]^{-1} B_1^T \Gamma A_\Delta x(k).$$

Since we assumed that $x(k) \not\equiv 0$, we must have $J(k) < 0$ for some $k \geq 0$. Hence, $J_N < 0$ for sufficiently large N, which implies $J < 0$. ▽▽▽

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Minimal Periodic Realizations of Transfer Matrices

Ching-An Lin and Chwan-Wen King

***Abstract*—Periodic controllers designed based on the so-called lifting technique are usually represented by transfer matrices. Real-time operations require that the controllers be implemented as periodic systems. We study the problem of realizing an $Nn_o \times Nn_i$ proper rational transfer matrix as an n_i-input n_o-output N-periodic discrete-time system. We propose an algorithm to obtain periodic realizations which have a minimal number of states. The result can also be used to remove any redundant states that exist in a periodic system.**

I. INTRODUCTION

It is reported in the literature that linear periodic controllers may be superior to the linear time-invariant ones for a large class of control problems [5], [2], [1], [3]. For discrete-time systems, Khargonekar, Poolla and Tannenbaum [5] proposed a framework for the design of linear periodic controllers for linear time-invariant plants. They show that to an n_i-input n_o-output linear N-periodic system there corresponds an Nn_i-input Nn_o-output linear time-invariant system and conversely to an Nn_i-input Nn_o-output linear time-invariant system there corresponds an n_i-input n_o-output linear N-periodic system. It is asserted [5] that from an input–output point of view, this correspondence is isomorphic in that both algebraic and analytic properties of systems are preserved and hence, the design of periodic controllers can be done using various LTI design techniques. However, the n_i-input n_o-output N-periodic controller so designed is "represented" as an Nn_i-input Nn_o-output time-invariant system, e.g., an $Nn_o \times Nn_i$ transfer matrix. Real-time operations require that the controller be realized as a periodic system. There are straightforward realizations of such a transfer matrix as an N-periodic system but usually with a large number of

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