A Linear Matrix Inequality Approach to Robust H_∞ Filtering

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***Abstract*—In this paper, we consider the robust H_∞ filtering problem for a general class of uncertain linear systems described by the so-called integral quadratic constraints (IQC's). This problem is important in many signal processing applications where noises, nonlinearity, quantization errors, time delays, and unmodeled dynamics can be naturally described by IQC's. The main contribution of this paper is to show that the robust H_∞ filtering problem can be solved using linear matrix inequality (LMI) techniques, which are numerically efficient owing to recent advances in convex optimization. The paper deals with both continuous and discrete-time uncertain linear systems.**

***Index Terms*— H_∞ filtering, integral quadratic constraints, linear matrix inequalities, robust filtering.**

I. Introduction

THE H_∞ filtering technique has been widely studied for the benefit of different time and frequency domain properties to the H_2 filtering technique. In the H_∞ setting, the exogenous input signal is assumed to be energy bounded rather than Gaussian. An H_∞ filter is designed such that the H_∞ norm of the system, which reflects the worst-case "gain" of the system, is minimized. The advantage of using an H_∞ filter in comparison with an H_2 filter is twofold. First, no statiscal assumption on the input is needed. Second, the filter tends to be more robust when there exists additional uncertainty in the system, such as quantization errors, delays, and unmodeled dynamics [23]. These features make H_∞ filtering technique useful in certain applications. One such application is reported in [21] for seismic signal deconvolution. Applications of H_∞ filters in multirate signal processing are studied in [4] and [5]. We also note that the well-known equiripple filters are in fact a class of H_∞ filters because the design objective is to minimize the H_∞ norm of the difference between the filter to be designed and a given ideal filter [19], although the term H_∞ is rarely used.

There are three approaches to H_∞ filtering:

1) algebraic Riccati equation (ARE) approach (see, e.g., [1], [15]);
2) polynomial equation approach (see, e.g., [13], [14]);
3) interpolation approach (see, e.g., [8]).

Manuscript received July 5, 1995; revised November 11, 1996. This work was supported by the Australian Research Council. The associate editor coordinating the review of this paper and approving it for publication was Dr. Andreas S. Spanias.

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Publisher Item Identifier S 1053-587X(97)05790-5.

The polynomial approach and the interpolation approach use transfer functions directly. That is, they both are frequency domain approaches. They seem to be most suitable when specific frequency domain information, such as zeros, poles, bandwidth, etc., is available. In addition, frequency weighting on filtering errors and noise signals can be easily performed without worrying about the dimension increase of the weighted system, which may add the computational complexity. In fact, the standard design method for equiripple filters is a kind of interpolation approach. In this case, neither the ideal filter nor the weighting function can be expressed by a finite-order transfer function. The main problem with the frequency approaches is that the formula are quite complicated, especially in the multivariable case. The ARE approach is a state-space approach. It is more popular due to the fact that solutions are expressed in simple formula and that efficient numerical algorithms exist for solving ARE's.

However, the works mentioned above require that the system does not have any uncertainty, apart from the exogenous noise input. Therefore, the robustness of the H_∞ filter deserves consideration. Several results have been obtained on robust H_∞ filtering for continuous-time and discrete-time linear systems; see [6], [9], [24], and [26], for example. These results deal with the so-called norm-bounded uncertainty and are obtained using the ARE approach. The problem of robust H_∞ filtering contains two aspects: H_∞ filter analysis and H_∞ filter synthesis. The analysis aspect is to determine the worst-case H_∞ performance when a filter is given, whereas the synthesis aspect is to design a filter such that the worst-case H_∞ performance is satisfactory. The ARE approach in [9], [24], and [26] involves a conversion of the robust H_∞ filtering problem into a "scaled" H_∞ filtering problem, which does not involve uncertainty. This is done by converting the norm-bounded uncertainty into some scaling parameters. The conversion used there significantly simplifies the robust H_∞ filtering problem and makes it possible to use the standard H_∞ filtering results. However, the introduction of the scaling parameters makes the resulting "scaled" ARE's difficult to solve. Indeed, these scaling parameters enter the ARE's nonlinearly. Further, the norm-bounded uncertainty assumption is somewhat conservative in many applications.

In this paper, we consider a new approach to the robust H_∞ filtering problem for continuous- and discrete-time uncertain systems. Apart from an energy bounded exogenous noise input, the system is allowed to have uncertainties described by the so-called integral quadratic constraints (IQC's), which are more general than the norm bounded structure. Similar to [9],

[24], and [26], the robust H_∞ filtering problem in this paper also involves two aspects, namely, the H_∞ filter analysis and the H_∞ filter synthesis. We apply the so-called $\boldsymbol{S}$-procedure to deal with the IQC's and provide solutions to the robust H_∞ filtering problem in terms of linear matrix inequalities (LMI's). An LMI is a semidefinite inequality that is linear in unknown variables. Due to recent advancement in convex optimization, efficient algorithms exist for solving LMI's; see [3] for a tutorial and [12] for a Matlab Toolbox. It turns out that the analysis problem can be solved using a single LMI. However, the synthesis problem is more complicated as it involves two matrix inequalities that are separately linear but not quite jointly linear. We then discuss two special cases where these two matrix inequalities are jointly linear. Our results reduce to those in [9], [24], and [26] when the norm-bounded uncertainty assumption is enforced.

This paper is organized as follows: Section II gives a brief overview for the H_∞ filtering problem; Section III discusses IQCs; Section IV presents the problem statement and preliminaries; Section V studies the robust H_∞ filter analysis problem; Section VI deals with the robust H_∞ filter synthesis problem; Section VII offers illustrative examples; and some concluding remarks are given in Section VIII.

We use the following notational table throughout this paper.

Notation	Continuous	Discrete
$\delta x(t)$	$\dot{x}(t)$	$x(t+1)$
$\boldsymbol{S}_0^T \vert\vert\xi_i(t)\vert\vert^2$	$\int_0^T \vert\vert\xi(t)\vert\vert^2\, dt$	$\sum_{t=0}^{T} \vert\vert\xi_i(t)\vert\vert^2$

II. Overview of the H_∞ Filtering Problem

In this section, we briefly review the standard H_∞ filtering approaches for linear systems without model uncertainty.

Let T_{zw} denote a stable operator from signal w to signal z, where $w \in L_2[0,\infty)$ in the continuous-time case, and $w \in \ell_2[0,\infty)$ in the discrete-time case. The H_∞ norm of T_{zw} is defined as

$$||T_{zw}||_\infty = \sup_w \frac{||z||_2}{||w||_2}. \tag{1}$$

If T_{zw} is linear time invariant, then

$$||T_{zw}||_\infty = \sup_\omega \overline{\sigma}(T_{zw}(j\omega)) \tag{2}$$

where $\overline{\sigma}$ denotes the largest singular value.

In the following, we will review the results for continuous-time systems only. The discrete-time cases can be addressed similarly.

Consider the following system:

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{3a}$$
$$z(t) = C_1 x(t) \tag{3b}$$
$$y(t) = C_2 x(t) + Dw(t) \tag{3c}$$

where

$x(t) \in \boldsymbol{R}^n$ state;
$w(t) \in \boldsymbol{R}^q$ exogenous noise input in $L_2[0,\infty)$;
$z(t) \in \boldsymbol{R}^r$ output to be estimated;
$y(t) \in \boldsymbol{R}^v$ measured output.

The following assumptions are standard:

A1) (A, B, C_1) is stabilizable and detectable.
A2) $DD' > 0$.

The H_∞ filter structure is

$$\dot{x}_f(t) = Ax_f(t) + K(y(t) - C_2 x_f(t)) \tag{4a}$$
$$z_f(t) = C_1 x_f(t) \tag{4b}$$

where $x_f(t) \in \boldsymbol{R}^n$ and $z_f(t) \in \boldsymbol{R}^r$ are the estimated state and output, respectively, and K is well known as the Kalman gain.

The estimation error is defined by

$$e(t) = z(t) - z_f(t). \tag{5}$$

Therefore, the error dynamics are

$$\dot{x}_e(t) = (A - KC_2)x_e(t) + (B - KD)w(t) \tag{6a}$$
$$e(t) = C_1 x_e(t) \tag{6b}$$

where $x_e(t) = x(t) - x_f(t)$ is the state estimation error.

The H_∞ filtering problem associated with the system (3) is as follows: Given $\gamma > 0$, find a filter of the form (4) such that the corresponding error dynamics (6) is asymptotically stable and satisfies

$$||T_{ew}||_\infty < \gamma, \quad x_e(0) = 0. \tag{7}$$

We now briefly discuss how this problem is tackled using two different approaches.

A. ARE Approach

It is straightforward to verify that the error dynamics (6) are equivalent to the control system model

$$\dot{x}_c(t) = Ax_c(t) + Bw(t) + u(t) \tag{8a}$$
$$e(t) = C_1 x_c(t) \tag{8b}$$
$$y(t) = C_2 x_c(t) + Dw(t) \tag{8c}$$
$$u(t) = -Ky(t). \tag{8d}$$

Using the control system model (8), it is a standard result from [7] that the H_∞ filtering problem is solvable if and only if the algebra Riccati inequality $P = P' > 0$

$$\begin{aligned}(A - BD'(DD')^{-1}C_2)P + P(A - BD'(DD')^{-1}C_2)' \\ + P(\gamma^{-2}C_1'C_1 - C_2'(DD')^{-1}C_2)P \\ + B(I - D'(DD')^{-1}D)B' < 0\end{aligned} \tag{9}$$

has a solution.

If such P exists, then one of the suitable Kalman gains, called the "central solution," is given by

$$K = (PC_2' + BD')(DD')^{-1}. \tag{10}$$

B. Frequency Domain Approach

From the control system model (8), we have

$$e(s) = G_{ew}(s)w(s) + G_{eu}(s)u(s) \tag{11}$$
$$y(s) = G_{yw}(s)w(s) + G_{yu}(s)u(s) \tag{12}$$
$$u(s) = -Ky(s) \tag{13}$$

where

$$G_{ew}(s) = C_1(sI - A)^{-1}B; \quad G_{eu}(s) = C_1(sI - A)^{-1}$$
$$G_{yw}(s) = C_2(sI - A)^{-1}B + D$$
$$G_{yu}(s) = C_2(sI - A)^{-1}.$$

The transfer function from $w(s)$ to $e(s)$ is

$$T_{ew}(s) = G_{ew}(s) - G_{eu}(s)(1 + KG_{yu}(s))^{-1}KG_{yw}(s). \tag{14}$$

Thus, the H_∞ filtering problem is equivalent to

$$\gamma_{\min} = \min_K \{||T_{ew}(s)||_\infty: A - KC_2 \text{ is stable}\} \tag{15}$$

such that $\gamma_{\min} \le \gamma$. Equation (15) is a standard H_∞ optimization problem.

III. Integral Quadratic Constraints

The IQC is a very general tool for describing properties of linear and nonlinear operators. In particular, they can be used conveniently to describe uncertain parameters, noises, time delays, quantization errors, unmodeled dynamics, etc. The notion of IQC is introduced by [28] and [29] for robust stability analysis of feedback systems involving linear and nonlinear parts. An effective method called the $\boldsymbol{S}$ procedure for treating these IQC's is also introduced in [28] and [29]. The purpose of this section is to show that many kinds of uncertainties in signal processing problems can be described by IQC's.

Definiton 1: Consider a stable LTI operator $\mathcal{G}: L_2[0,\infty) \to L_2[0,\infty)$ (or $\mathcal{G}: \ell_2[0,\infty) \to \ell_2[0,\infty)$), which has the state space realization

$$\delta x(t) = Ax(t) + Bu(t), \quad x(0) = 0 \tag{16a}$$
$$y(t) = Cx(t) + Du(t). \tag{16b}$$

Let $q(x,y,u)$ be a quadratic form, i.e.,

$$q(x,y,u) = (x' \quad y' \quad u')Q\begin{bmatrix} x \\ y \\ u \end{bmatrix}$$

for some constant matrix $Q = Q'$. The associated (weak) IQC is given by

$$\lim_{T\to\infty} \boldsymbol{S}_0^T q(x,y,u) \le 0, \quad \forall u \in L_2^e[0,\infty) \text{ (or } \ell_2^e[0,\infty)) \tag{17}$$

and the strong IQC is given by

$$\boldsymbol{S}_0^T q(x,y,u) \le 0,\ \forall T > 0,\ u \in L_2^e[0,\infty) \text{ (or } \ell_2^e[0,\infty)) \tag{18}$$

where $L_2^e[0,\infty)$ (or $\ell_2^e[0,\infty)$) denotes the extended real $L_2[0,\infty)$ (or $\ell_2[0,\infty)$) space. In the rest of the paper, we will use the (weak) IQC only.

Note that in the discrete-time case, an IQC is actually a *sum* quadratic constraint, but we will call it IQC for consistency with the continuous time case.

Obviously, the IQC used to describe the passivity of an operator $\mathcal{G}$ is simply given by

$$\lim_{T\to\infty} \boldsymbol{S}_0^T(-w'(t)v(t)) \le 0, \quad \forall v \in L_2^e[0,\infty) \text{ (or } \ell_2^e[0,\infty))$$
$$w = \mathcal{G}v. \tag{19}$$

If we want to put the above in the form of (17), we simply take $u = [v' \quad w']'$ and note that

$$-w'v = -\tfrac{1}{2}(w'v + v'w) = u'\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}u.$$

For a comprehensive list of uncertain components that can be described by IQC's, refer to [20] and [25]. These components include time delays, uncertain parameters, unmodeled dynamics, and many nonlinear functions.

IV. Problem Statement and Preliminaries

Consider the uncertain linear system

$$\delta x(t) = Ax(t) + Bw(t) + \sum_{i=1}^p H_{1i}\xi_i(t) \tag{20a}$$
$$z(t) = C_1x(t) + D_1w(t) + \sum_{i=1}^p H_{2i}\xi_i(t) \tag{20b}$$
$$y(t) = C_2x(t) + D_2w(t) + \sum_{i=1}^p H_{3i}\xi_i(t) \tag{20c}$$

where

$x(t) \in \boldsymbol{R}^n$	state;
$w(t) \in \boldsymbol{R}^q$	exogenous noise input belonging to $L_2[0,\infty)$ in the continuous-time case and $\ell_2[0,\infty)$ in the discrete-time case;
$z(t) \in \boldsymbol{R}^r$	output to be estimated;
$y(t) \in \boldsymbol{R}^v$	measured output;
$\xi_i(t) \in \boldsymbol{R}^{k_i}$	uncertain variables satisfying the IQC's

$$\boldsymbol{S}_0^T||\xi_i(t)||^2 \le \boldsymbol{S}_0^T||E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)||^2$$
$$\text{as } T \to \infty,\ i = 1,\cdots,p \tag{21}$$

with

$$\xi(t) = [\xi_1'(t) \cdots \xi_p'(t)]'.$$

In addition, $A, B, C_1, C_2, D_1, D_2, H_{1i}, H_{2i}, E_{1i}, E_{2i},$ and E_{3i} are constant matrices of appropriate dimensions. To simplify notation, we define

$$H_1 = [H_{11} \cdots H_{1p}]; \quad H_2 = [H_{21} \cdots H_{2p}]$$
$$H_3 = [H_{31} \cdots H_{3p}] \tag{22}$$
$$E_1' = [E_{11}' \cdots E_{1p}'] \quad E_2' = [E_{21}' \cdots E_{2p}']$$
$$E_3' = [E_{31}' \cdots E_{3p}']. \tag{23}$$

Remark 1: The uncertainty represented by the IQC's (21) is very general. Apart from examples mentioned in the previous section, several special cases of the system (20) have been treated in the literature. For example, [2], [6], [9], [18], [22]–[24], [26], and [27] consider the system

$$\delta x(t) = (A + \Delta A)x(t) + Bw(t) \tag{24}$$
$$z(t) = C_1 x(t) \tag{25}$$
$$y(t) = (C_2 + \Delta C)x(t) + D_2 w(t) \tag{26}$$

with norm-bounded uncertainty

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)E \tag{27}$$

where $F'(t)F(t) \leq I, \forall t \geq 0$. Another widely used system uncertainty description in H_∞ analysis involves the so-called linear fractional uncertainty (see, e.g., [17]), where

$$\delta x(t) = (A + \Delta A)x(t) + Bw(t) \tag{28}$$
$$z(t) = C_1 x(t) \tag{29}$$
$$y(t) = C_2 x(t) + D_2 w(t) \tag{30}$$

is considered with

$$\begin{aligned} \Delta A &= HF(t)(I - E_3 F(t))^{-1} E_1 \\ F'(t)F(t) &\leq I, \quad \forall t \geq 0. \end{aligned} \tag{31}$$

However, to our knowledge, there is no existing H_∞ filtering result available for linear systems with linear fractional uncertainty. It is easy to see that the aforementioned norm-bounded and linear fractional uncertainties are special cases of the IQC's (21) with $p = 1$.

Consider the filter

$$\delta x_f(t) = A_f x_f(t) + B_f y(t) \tag{32a}$$
$$z_f(t) = C_f x_f(t) + D_f y(t) \tag{32b}$$

where

$x_f(t) \in \boldsymbol{R}^{n_f}$	estimated state;
$z_f(t) \in \boldsymbol{R}^r$	estimated output;
$y(t)$	measured output of (20).

A_f, B_f, C_f, and D_f are constant matrices of appropriate dimensions to be chosen.

Define the filtering error as

$$e(t) = z(t) - z_f(t). \tag{33}$$

Then, the filtering error dynamics are given by

$$\delta x_e(t) = A_e x_e(t) + B_e w(t) + H_{1e}\xi(t) \tag{34a}$$
$$e(t) = C_e x_e(t) + D_e w(t) + H_{2e}\xi(t) \tag{34b}$$

where

$$x_e(t) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} \tag{35}$$
$$A_e = \begin{bmatrix} A & 0 \\ B_f C_2 & A_f \end{bmatrix}; \quad B_e = \begin{bmatrix} B \\ B_f D_2 \end{bmatrix} \tag{36}$$
$$C_e = [C_1 - D_f C_2 \quad C_f]; \quad D_e = D_1 - D_f D_2 \tag{37}$$
$$H_{1e} = \begin{bmatrix} H_1 \\ B_f H_3 \end{bmatrix}; \quad H_{2e} = H_2 - D_f H_3. \tag{38}$$

Remark 2: Note that the robust H_∞ state estimation problem is a special case of the above robust H_∞ filtering problem with $D_1 = 0, H_2 = 0, D_f = 0, C_1 = I$, and $C_f = I$.

We further define the notion of bounded-state (BS) stability.

Definition 2: The filtering error dynamics (34) is called BS stable if for all $x_e(0)$, all $w(t) \in L_2[0, \infty)$ in the continuous-time case and $w(t) \in \ell_2[0, \infty)$ in the discrete-time case, there exists $M \geq 0$ such that

$$||x_e(t)|| \leq M, \qquad t \geq 0. \tag{39}$$

We also recall the well-known $\boldsymbol{S}$-procedure [29].

Lemma 1: Let $\mathcal{F}(\cdot), \mathcal{Y}_1(\cdot), \cdots, \mathcal{Y}_k(\cdot)$ be real valued functionals defined on a set $\boldsymbol{\Lambda}$. Define the domain of constraints $\boldsymbol{D}$

$$\boldsymbol{D} = \{\lambda \in \boldsymbol{\Lambda}: \mathcal{Y}_1(\lambda) \geq 0, \cdots, \mathcal{Y}_k(\lambda) \geq 0\}$$

and two conditions

a) $\mathcal{F}(\lambda) \geq 0, \forall \lambda \in \boldsymbol{D}$;
b) $\exists \tau_1 > 0, \cdots, \tau_k > 0$

such that

$$\boldsymbol{S}(\tau, \lambda) = \mathcal{F}(\lambda) - \sum_{j=1}^{k} \tau_j \mathcal{Y}_j(\lambda) \geq 0, \quad \forall \lambda \in \boldsymbol{\Lambda}.$$

Then, **b)** implies **a)**.

Remark 3: The procedure of replacing **a)** by **b)** is called the $\boldsymbol{S}$ procedure. This procedure is a very convenient way of handling inequality constraints and is known to be conservative in general. Despite its conservatism, the simplicity of this procedure has attracted a lot of applications in stability analysis problems and optimization problems; see [3], [20], [28], and [29]. In particular, note that searching for optimal scaling parameters τ_i is often a convex optimization problem, as we will see in the sequel.

V. Analysis of Robust H_∞ Filters

The robust H_∞ filter analysis problem associated with the uncertain system (20) is as follows: Given $\gamma > 0$ and a filter of the form (32), determine if the error dynamics (34) is BS stable and satisfies

$$\begin{aligned} &\boldsymbol{S}_0^T ||e(t)||^2 < \gamma^2 \boldsymbol{S}_0^T ||w(t)||^2, \quad \text{as } T \to \infty, \quad w(t) \neq 0 \\ &x_e(0) = 0 \end{aligned} \tag{40}$$

for all admissible uncertainty satisfying the IQC's (21).

The parameter γ is a tolerance level, which can be regarded as an indication of the quality of the filter. A small tolerance level indicates a small estimation error in the worst case. However, a small tolerance level requires a large filter gain in general, which may cause implementation difficulties in the continuous-time case or a numerical problem in the discrete-time case. Further, the transient behavior of the error dynamics may be of concern if the tolerance level is too small. In general, there is no simple interpretation of the tolerance level in terms of a linear quadratic cost function. However, it is known that for linear time-invariant systems, the H_∞ filter approaches a special Kalman filter when $\gamma \to \infty$, see, e.g., [13], [14], [21], and [23].

Applying the $\boldsymbol{S}$-procedure, we have the following result.

Lemma 2: Given that (34) is BS stable and that (40) holds for all admissible uncertainty satisfying the IQC's (21) if there exists a positive definite matrix $P = P' \in \boldsymbol{R}^{n\times n}$ and scaling parameters $\tau_1, \cdots, \tau_p > 0$ such that the following condition holds:

For the continuous-time case:

$$\begin{aligned} &2x_e'P(A_ex_e + B_ew + H_{1e}\xi) \\ &\quad + \sum_{i=1}^{p} \tau_i(\|E_{1i}x + E_{2i}w + E_{3i}\xi\|^2 - \|\xi_i\|^2) \\ &\quad + \|C_ex_e + D_ew + H_{2e}\xi\|^2 - \gamma^2\|w\|^2 < 0 \\ &\quad \forall (x_e', w', \xi')' \neq 0. \end{aligned} \tag{41}$$

For the discrete-time case:

$$\begin{aligned} &(A_ex_e + B_ew + H_{1e}\xi)'P(A_ex_e + B_ew + H_{1e}\xi) - x_e'Px_e \\ &\quad + \sum_{i=1}^{p} \tau_i(\|E_{1i}x + E_{2i}w + E_{3i}\xi\|^2 - \|\xi_i\|^2) \\ &\quad + \|C_ex_e + D_ew + H_{2e}\xi\|^2 - \gamma^2\|w\|^2 < 0 \\ &\quad \forall (x_e', w', \xi')' \neq 0. \end{aligned} \tag{42}$$

Proof: Integrating or summing up the left-hand side of the inequality in (41) or (42) along any trajectory of the error dynamics (34), for $w \neq 0$, we have

$$\begin{aligned} &x_e'(T)Px_e(T) - x_e'(0)Px_e(0) \\ &\quad + \sum_{i=1}^{p} \tau_i\{\boldsymbol{S}_0^T\|E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)\|^2 \\ &\quad - \boldsymbol{S}_0^T\|\xi_i(t)\|^2\} + \{\boldsymbol{S}_0^T\|e(t)\|^2 - \gamma^2\boldsymbol{S}_0^T\|w(t)\|^2\} < 0. \end{aligned} \tag{43}$$

Using (21), it follows that

$$x_e'(T)Px_e(T) \le x_e'(0)Px_e(0) + \gamma^2\boldsymbol{S}_0^T\|w(t)\|^2 \quad \text{as } T \to \infty.$$

Hence, the BS stability of the error dynamics is implied. Now, take $x_e(0) = 0$. It follows from (43) and (21) again that (40) holds. □

Remark 4: Assume that (34) is zero detectable, i.e., $e(T) \to 0$ as $T \to \infty$ implies $x_e(T) \to 0$ as $T \to \infty$. Then, (41) for the continuous-time case and (42) for the discrete-time case guarantee that the error dynamics (34) with uncertainty (21) is asymptotically stable. To make this point clear, we take $w = 0$. Using (43) and (21), it follows that

$$\boldsymbol{S}_0^T\|e(t)\|^2 \le x_e'(0)Px_e(0), \quad \text{as } T \to \infty.$$

Hence, $e(T) \to 0$ as $T \to \infty$ implies $x_e(T) \to 0$ as $T \to \infty$.

Using Lemma 2 and denoting

$$E_{1e} = [E_1 \quad 0] \tag{44}$$

$$J = \operatorname{diag}\{\tau_1 I_{k_1}, \cdots, \tau_p I_{k_p}\} \tag{45}$$

we obtain our main result of continuous-time systems for this section.

Theorem 3—Continuous Time: The following conditions are equivalent, and they all guarantee the solution to the robust H_∞ filter analysis problem associated with the uncertain system (20) and the filter (32).

- **i)** There exist $P = P' > 0$ and $\tau_1 > 0, \cdots, \tau_p > 0$ such that (41) holds.
- **ii)** There exist $P = P' > 0$ and $J > 0$ in (45) solving the LMI in (46), shown at the bottom of the page.
- **iii)** There exist $P = P' > 0$ and $J > 0$ in (45) solving the LMI

$$\mathcal{L}_2 = \begin{bmatrix} A_e'P + PA_e & PB_e & PH_{1e} & C_e' & E_{1e}'J \\ B_e'P & -\gamma^2 I & 0 & D_e' & E_2'J \\ H_{1e}'P & 0 & -J & H_{2e}' & E_3'J \\ C_e & D_e & H_{2e} & -I & 0 \\ JE_{1e} & JE_2 & JE_3 & 0 & -J \end{bmatrix} < 0. \tag{47}$$

- **iv)** There exists $J > 0$ in (45) such that the auxiliary system in (48) and (49) is asymptotically stable and that the H_∞-norm of the transfer function from $\hat{w}(\cdot)$ to $\hat{e}(\cdot)$ is less than 1:

$$\dot{\hat{x}}(t) = A_e\hat{x}(t) + [\gamma^{-1}B_e H_{1e}J^{-1/2}]\hat{w}(t) \tag{48}$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ J^{1/2}E_{1e} \end{bmatrix}\hat{x}(t) + \begin{bmatrix} \gamma^{-1}D_e & H_{2e}J^{-1/2} \\ \gamma^{-1}J^{1/2}E_2 & J^{1/2}E_3J^{-1/2} \end{bmatrix}\hat{w}(t). \tag{49}$$

Moreover, the set of all J's satisfying **iv)** is convex.

Proof: "**i)** ⇔ **ii)**": The inequality (41) can be rewritten as

$$\begin{aligned} &2x_e'P(A_ex_e + B_ew + H_{1e}\xi) + (x_e'E_{1e}' + w'E_2' + \xi'E_3') \\ &\quad \cdot J(E_{1e}x_e + E_2w + E_3\xi) - \xi'J\xi \\ &\quad + (x_e'C_e' + w'D_e' + \xi'H_{2e}')(C_ex_e + D_ew + H_{2e}\xi) \\ &\quad - \gamma^2 w'w < 0, \quad \forall (x_e', w', \xi')' \neq 0 \end{aligned} \tag{50}$$

which is equivalent to

$$[x_e' \quad w' \quad \xi']\mathcal{L}_1 \begin{bmatrix} x_e \\ w \\ \xi \end{bmatrix} < 0, \quad \forall (x_e', w', \xi')' \neq 0 \tag{51}$$

i.e, $\mathcal{L}_1 < 0$.

$$\mathcal{L}_1 = \begin{bmatrix} A_e'P + PA_e + E_{1e}'JE_{1e} + C_e'C_e & PB_e + E_{1e}'JE_2 + C_e'D_e & PH_{1e} + C_e'H_{2e} + E_{1e}'JE_3 \\ B_e'P + E_2'JE_{1e} + D_e'C_e & -\gamma^2 I + D_e'D_e + E_2'JE_2 & D_e'H_{2e} + E_2'JE_3 \\ H_{1e}'P + H_{2e}'C_e + E_3'JE_{1e} & H_{2e}'D_e + E_3'JE_2 & -J + H_{2e}'H_{2e} + E_3'JE_3 \end{bmatrix} < 0 \tag{46}$$

"**ii)** $\Leftrightarrow$ **iv)**": Denote

$$\hat{B}_e = [\gamma^{-1} B_e H_{1e} J^{-1/2}] \tag{52}$$
$$\hat{C}'_e = [C'_e E'_{1e} J^{1/2}] \tag{53}$$
$$\hat{D}_e = \begin{bmatrix} \gamma^{-1} D_e & H_{2e} J^{-1/2} \\ \gamma^{-1} J^{1/2} E_2 & J^{1/2} E_3 J^{-1/2} \end{bmatrix} \tag{54}$$

and

$$\hat{\mathcal{L}}_1 = \begin{bmatrix} A'_e P + P A_e + \hat{C}'_e \hat{C}_e & P\hat{B}_e + \hat{C}'_e \hat{D}_e \\ \hat{B}'_e P + \hat{D}'_e \hat{C}_e & -I + \hat{D}'_e \hat{D}_e \end{bmatrix}. \tag{55}$$

The auxiliary system in (48) and (49) can be rewritten as

$$\dot{\hat{x}}(t) = A_e \hat{x}(t) + \hat{B}_e \hat{w}(t) \tag{56}$$
$$\hat{e}(t) = \hat{C}_e \hat{x}(t) + \hat{D}_e \hat{w}(t). \tag{57}$$

In addition, the matrix $\mathcal{L}_1$ in (46) can be expressed as

$$\mathcal{L}_1 = \operatorname{diag}\{I_n, \gamma^{-1} I_q, J^{-1/2}\} \hat{\mathcal{L}}_1 \operatorname{diag}\{I_n, \gamma^{-1} I_q, J^{-1/2}\}. \tag{58}$$

That is, $\mathcal{L}_1 < 0$ if and only if $\hat{\mathcal{L}}_1 < 0$. It is well known that matrix A_e is asymptotically stable and $\|\hat{D}_e + \hat{C}_e(sI - A_e)^{-1}\hat{B}_e\|_\infty < 1$ if and only if $\hat{\mathcal{L}}_1 < 0$ for some $P = P' > 0$. Hence, **ii)** is equivalent to **iv)**.

"**(ii)** $\Leftrightarrow$ **(iii)**": Note that $\hat{\mathcal{L}}_1 < 0$ if and only if

$$\hat{\mathcal{L}}_2 = \begin{bmatrix} A'_e P + P A_e & P\hat{B}_e & \hat{C}'_e \\ \hat{B}'_e P & -I & \hat{D}'_e \\ \hat{C}_e & \hat{D}_e & -I \end{bmatrix} < 0 \tag{59}$$

holds. Equation (59) is derived from the well-known Schur complements that

$$\begin{bmatrix} X_1 & X'_2 \\ X_2 & -I \end{bmatrix} < 0 \Leftrightarrow X_1 + X'_2 X_2 < 0. \tag{60}$$

The equivalence between $\hat{\mathcal{L}}_2 < 0$ and $\mathcal{L}_2 < 0$ can be established by similar manipulations used on $\hat{\mathcal{L}}_1$ and $\mathcal{L}_1$. The details are omitted. □

Similarly, we have the following theorem for discrete-time systems:

Theorem 4—Discrete Time: The following conditions are equivalent, and they all guarantee the solution to the robust H_∞ filter analysis problem associated with the uncertain system (20) and the filter (32).

- **i)** There exist $P = P' > 0$ and $\tau_1 > 0, \cdots, \tau_p > 0$ such that (41) holds.
- **ii)** There exist $P = P' > 0$ and $J > 0$ in (45) solving the LMI in (61), shown at the bottom of the page.
- **iii)** There exist $P = P' > 0$ and $J > 0$ in (45) solving the LMI in (62), shown at the bottom of the page.
- **iv)** There exists $J > 0$ in (45) such that the auxiliary system in (64) is asymptotically stable and that the H_∞-norm of the transfer function from $\hat{w}(\cdot)$ to $\hat{e}(\cdot)$ is less than 1:

$$\hat{x}(t+1) = A_e \hat{x}(t) + [\gamma^{-1} B_e H_{1e} J^{-1/2}] \hat{w}(t) \tag{63}$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ J^{1/2} E_{1e} \end{bmatrix} \hat{x}(t) + \begin{bmatrix} \gamma^{-1} D_e & H_{2e} J^{-1/2} \\ \gamma^{-1} J^{1/2} E_2 & J^{1/2} E_3 J^{-1/2} \end{bmatrix} \hat{w}(t) \tag{64}$$

Moreover, the set of all J's satisfying **iv)** *is convex.*

Proof: The proof is similar to the continuous-time case. □

VI. Synthesis of Robust H_∞ Filters

For the synthesis problem, we need the following assumptions:

- **A1)** A is asymptotically stable.
- **A2)** (A, C_2) is detectable.

The H_∞ filter synthesis problem associated with the uncertain system (20) is as follows: Given $\gamma > 0$, find a filter of the form (32) such that the corresponding error dynamics (34) are BS stable and satisfy

$$\boldsymbol{S}_0^T \|e(t)\|^2 < \gamma^2 \boldsymbol{S}_0^T \|w(t)\|^2, \quad \text{as } T \to \infty, \quad w(t) \neq 0$$
$$x_e(0) = 0 \tag{65}$$

for all admissible uncertainty satisfying the IQC's (21).

$$\mathcal{L}_1 = \begin{bmatrix} A'_e P A_e - P + E'_{1e} J E_{1e} + C'_e C_e & A'_e P B_e + E'_{1e} J E_2 + C'_e D_e & A'_e P H_{1e} + C'_e H_{2e} + E'_{1e} J E_3 \\ B'_e P A_e + E'_2 J E_{1e} + D'_e C_e & -\gamma^2 I + B'_e P B_e + D'_e D_e + E'_2 J E_2 & B'_e P H_{1e} + D'_e H_{2e} + E'_2 J E_3 \\ H'_{1e} P A_e + H'_{2e} C_e + E'_3 J E_{1e} & H'_{1e} P B_e + H'_{2e} D_e + E'_3 J E_2 & -J + H'_{1e} P H_{1e} + H'_{2e} H_{2e} + E'_3 J E_3 \end{bmatrix} < 0 \tag{61}$$

$$\mathcal{L}_2 = \begin{bmatrix} A'_e P A_e - P & A'_e P B_e & A'_e P H_{1e} & C'_e & E'_{1e} J \\ B'_e P A_e & -\gamma^2 I + B'_e P B_e & B'_e P H_{1e} & D'_e & E'_2 J \\ H'_{1e} P A_e & H'_{1e} P B_e & -J + H'_{1e} P H_{1e} & H'_{2e} & E'_3 J \\ C_e & D_e & H_{2e} & -I & 0 \\ J E_{1e} & J E_2 & J E_3 & 0 & -J \end{bmatrix} < 0 \tag{62}$$

Before proceeding further, we need the following lemma, which was originally used for the H_∞ control problem [11].

Lemma 5 [11]: Consider

$$\delta x(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{66}$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{67}$$
$$y(t) = C_2 x(t) + D_{21} w(t) \tag{68}$$

where (A, B_2, C_2) is a stabilizable and detectable triple. Let N_R (respectively, N_S) be any matrix whose columns form a basis of the null space of $[B_2' \quad D_{12}']$ (respectively, $[C_2 \quad D_{21}]$). Then, there exists a controller of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{69}$$
$$u(t) = C_c x_c(t) + D_c y(t) \tag{70}$$

such that the closed-loop system has H_∞ norm less than 1 if and only if there exist symmetric matrices R and S satisfying the following LMI's:

For the continuous-time case:

$$\left[\begin{array}{c|c} N_R' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} AR + RA' & RC_1' & B_1 \\ C_1 R & -I & D_{11} \\ \hline B_1' & D_{11}' & -I \end{array}\right] \cdot \left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{71}$$

$$\left[\begin{array}{c|c} N_S' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} A'S + AS & SB_1 & C_1' \\ B_1' S & -I & D_{11}' \\ \hline C_1 & D_{11} & -I \end{array}\right] \cdot \left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{72}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0. \tag{73}$$

For the discrete-time case:

$$\left[\begin{array}{c|c} N_R' & \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} ARA' - R & ARC_1' & B_1 \\ C_1 RA' & -I + C_1 RC_1' & D_{11} \\ \hline B_1' & D_{11}' & -I \end{array}\right] \cdot \left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{74}$$

$$\left[\begin{array}{c|c} N_S' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} A'SA - S & SB_1 A' & C_1' \\ B_1' SA & -I + B_1' SB_1 & D_{11}' \\ \hline C_1 & D_{11} & -I \end{array}\right] \cdot \left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{75}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0. \tag{76}$$

It is straightforward to see that the auxiliary system (48) and (49) for continuous-time or (63) and (64) for discrete-time is the closed-loop system of the open-loop auxiliary system

$$\delta \tilde{x}(t) = A\tilde{x}(t) + [\gamma^{-1} B H_1 J^{-1/2}]\hat{w}(t) \tag{77}$$

$$\hat{e}(t) = \begin{bmatrix} C_1 \\ J^{1/2} E_1 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} \gamma^{-1} D_1 & H_2 J^{-1/2} \\ \gamma^{-1} J^{1/2} E_2 & J^{1/2} E_3 J^{-1/2} \end{bmatrix} \cdot \hat{w}(t) + \begin{bmatrix} -I \\ 0 \end{bmatrix} u(t) \tag{78}$$

$$y(t) = C_2 \tilde{x}(t) + [\gamma^{-1} D_2 \quad H_3 J^{-1/2}]\hat{w}(t) \tag{79}$$

with the controller

$$\delta x_f(t) = A_f x_f(t) + B_f y(t) \tag{80}$$
$$u(t) = C_f x_f(t) + D_f y(t). \tag{81}$$

Note that the robust H_∞ filter synthesis problem of the original system (20) with the filter (32) is converted into the standard H_∞ control synthesis problem for the open-loop auxiliary system in (77)–(79) with the controller in (80) and (81). It is interesting to see that the auxiliary controller in (80) and (81) is exactly the same form as the filter in (32), which is to be designed. This observation leads to the following theorem, which is the main result for the synthesis problem:

Theorem 6: Given $\gamma > 0$, consider the robust H_∞ filter synthesis problem as described above. Denote by $\mathcal{N}_S$ any matrix whose columns form a basis of the null space of $[C_2 \quad D_2 \quad H_3]$. Then, the robust H_∞ filtering synthesis problem is solvable if there exist symmetric matrices $R, S \in \boldsymbol{R}^{n \times n}$ and $J > 0$ in (45) such that the following LMI's hold:

For the continuous-time case:

$$\left[\begin{array}{cc|cc} AR + RA' & RE_1' & B & H_1 J^{-1} \\ E_1 R & -J^{-1} & E_2 & E_3 J^{-1} \\ \hline B' & E_2' & -\gamma^2 I & 0 \\ J^{-1} H_1' & J^{-1} E_3' & 0 & -J^{-1} \end{array}\right] < 0 \tag{82}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|cc} A'S + SA & SB & SH_1 & C_1' & E_1' J \\ B'S & -\gamma^2 I & 0 & D_1' & E_2' J \\ H_1' S & 0 & -J & H_2' & E_3' J \\ \hline C_1 & D_1 & H_2 & -I & 0 \\ JE_1 & JE_2 & JE_3 & 0 & -J \end{array}\right] \cdot \left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{83}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0. \tag{84}$$

For the discrete-time case:

$$\left[\begin{array}{cc|cc} ARA' - R & ARE_1' & B & H_1 J^{-1} \\ E_1 RA' & -J^{-1} + E_1 RE_1' & E_2 & E_3 J^{-1} \\ \hline B' & E_2' & -\gamma^2 I & 0 \\ J^{-1} H_1' & J^{-1} E_3' & 0 & -J^{-1} \end{array}\right] < 0 \tag{85}$$

and (86), shown at the bottom of the next page, as well as

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0. \tag{87}$$

Proof: Note that the columns of diag $\{I, \gamma I, J^{1/2}\}\mathcal{N}_S$ form a basis of the null space of $[C_2 \quad [\gamma^{-1}D_2 \quad H_3 J^{-1/2}]]$. Comparing (77)–(79) with (66)–(68), we find that $[B_2' \quad D_{12}'] = [0 \quad -I \quad 0]$. The corresponding matrix that forms the basis of the null space of $[B_2' \quad D_{12}']$ is given by

$$\mathcal{N}_R = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}.$$

Apply Lemma 5 to the auxiliary system (77)–(79), it is tedious but straightforward to verify that the LMI (83) is the version of (72) for the continuous-time auxiliary system, but left and right multiplied by diag $\{I, I, I, I, J^{1/2}\}$. In addition, the LMI (82) corresponds to (71). The discrete-time part can be proved similarly.

Remark 5: The LMI (82) is jointly linear in R, J^{-1} and γ^2, whereas the LMI (83) is jointly linear in S, J, and γ^2. Therefore, the LMI's (82)–(84) are not jointly linear in J. A similar problem occurs for the discrete-time counterpart (85)–(87). To overcome this difficulty, two methods can be used to find J. The first one is a simple gridding method. To do this, we first rescale τ_i by defining $\lambda_i = \tau_i/(1+\tau_i)$. Obviously, $\tau_i > 0$ if and only if $\lambda_i \in (0,1)$. Then, we assign a uniform grid on each λ_i. For each grid point of $(\lambda_1, \cdots, \lambda_p)$, we can search for a solution for (82)–(84) or (85)–(87), which is a simple LMI problem. The second method is an iterative procedure. With this method, we add a ρI term to the left-hand side of each inequality in (82) and (83) or (85) and (86), where ρ is a scalar variable. An initial J is guessed. Then, R and S are searched for so that ρ is maximized. This is an LMI problem. Then, let R and S be fixed, and find J to further maximize ρ, which, again, is an LMI problem. If the maximum ρ is negative, repeat the above procedure until either ρ becomes nonnegative, in which case, a feasible solution is found, or a prescribed number of iterations is reached in which the iterative method fails.

The tradeoff of the two algorithms above lies in the fact that the first algorithm can guarantee a near optimal solution, provided the grid size is sufficiently small, whereas the second algorithm is more numerically efficient but with the possibility of missing a global feasible solution.

Remark 6: In the following, we show two special cases for which (82) and (85) can be reformulated so that the resulting matrix inequalities will be jointly linear in J.

Case 1: Suppose $p = 1$, i.e., $J = \tau_1 I$. This is called the "single IQC" case. In this case, we left and right multiply (82) by

$$\text{diag}\{J^{1/2}, J^{1/2}, I, I\}.$$

Then, the LMI (82) is equivalent to

$$\left[\begin{array}{cc|cc} ARJ + RJA' & RJE_1' & J^{1/2}B & H_1 J^{-1/2} \\ E_1 RJ & -I & J^{1/2}E_2 & E_3 J^{-1/2} \\ \hline J^{1/2}B' & J^{1/2}E_2' & -\gamma^2 I & 0 \\ J^{-1/2}H_1' & J^{-1/2}E_3' & 0 & -J^{-1} \end{array}\right] < 0. \tag{88}$$

Following the well-known Schur complement

$$\begin{bmatrix} X_1 & X_2' \\ X_2 & X_4 \end{bmatrix} < 0 \Leftrightarrow X_4 < 0 \quad \text{and} \quad X_1 - X_2' X_4^{-1} X_2 < 0 \tag{89}$$

the LMI (82) is further equivalent to (90), shown at the bottom of the page, where $\tilde{R} = RJ$. Similarly, the LMI (85) is equivalent to (91), also shown at the bottom of the page.

Then, LMI's (90) and (83) and (84) for continuous-time systems as well as LMI's (91) and (86) and (87) for discrete-time systems are jointly linear in $J, \tilde{R}$, and S. Note that γ is a given tolerance level. It is always possible to find a suboptimal $\gamma_{\min}$ using simple iterative procedures.

Case 2: Assume $E_1 = 0$. That is, the uncertain variables $\xi_i(t)$ are independent of the state variables. Left and right multiplying (82) by

$$\text{diag}\{I, J, I, J\}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|cc} A'SA - S & A'SB & A'SH_1 & C_1' & E_1'J \\ B'SA & -\gamma^2 I + B'SB & B'SH_1 & D_1' & E_2'J \\ H_1'SA & H_1'SB & -J + H_1'SH_1 & H_2' & E_3'J \\ \hline C_1 & D_1 & H_2 & -I & 0 \\ JE_1 & JE_2 & JE_3 & 0 & -J \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{86}$$

$$\begin{bmatrix} A\tilde{R} + \tilde{R}A' + \gamma^{-2}BB'J + H_1 H_1' & \tilde{R}E_1' + \gamma^{-2}BE_2'J + H_1 E_3' \\ E_1\tilde{R} + \gamma^{-2}E_2 B'J + E_3 H_1' & -I + \gamma^{-2}E_2 E_2'J + E_3 E_3' \end{bmatrix} < 0 \tag{90}$$

$$\begin{bmatrix} A\tilde{R}A' - \tilde{R} + \gamma^{-2}BB'J + H_1 H_1' & A\tilde{R}E_1' + \gamma^{-2}BE_2'J + H_1 E_3' \\ E_1'\tilde{R}A + \gamma^{-2}E_2 B'J + E_3 H_1' & -I + \gamma^{-2}E_2 E_2'J + E_3 E_3' \end{bmatrix} < 0. \tag{91}$$

the LMI (82) is equivalent to

$$\left[\begin{array}{cc|cc} AR+RA' & 0 & B & H_1 \\ 0 & -J & JE_2 & E_3 \\ \hline B' & E_2'J & -\gamma^2 I & 0 \\ H_1' & E_3'J & 0 & -J \end{array}\right] < 0. \tag{92}$$

We claim that (92) and (84) are equivalent to

$$\left[\begin{array}{c|cc} -J & JE_2 & JE_3 \\ \hline E_2'J & -\gamma^2 I & 0 \\ E_3'J & 0 & -J \end{array}\right] < 0. \tag{93}$$

Obviously, (92) implies (93). To see that (93) implies (92) and (84), we recall the assumption that A is asymptotically stable and that $S>0$. Therefore, there exists $R=R'>0$ such that $AR+RA'<0$. Simply by scaling up R large enough will ensure both (92) and (84).

Now, the LMI's (83) and (93) for continuous-time systems are jointly linear in J and S. Similarly, the LMI's (86) and (93) can be used for discrete-time systems.

Remark 7: Assume $p=1, E_2=0, E_3=0, H_2=0, D_1=0, D_f=0$, LMI's (82)–(84) for continuous-time systems, and LMI's (85)–(87) for discrete-time systems naturally reduce to ARE's that are the same as in [9] and [26]; see [9] and [26] for details about the corresponding ARE's.

Remark 8: We note that Theorem 6 does not offer an explicit formula for constructing robust H_∞ filters. The following is a design procedure cited from [10]:

Step 1) Solve the LMI's (82)–(84) for continuous-time case and LMI's (85)–(87) for discrete-time case. If the LMI's are feasible, we will obtain a feasible solution R, S, and J.

Step 2) Decompose the matrix $I-RS$ into

$$MN' = I - RS$$

where M and N are matrices with full column ranks. This can be easily done via singular value decomposition.

Step 3) Solve P from

$$P\begin{bmatrix} R & I \\ M' & 0 \end{bmatrix} = \begin{bmatrix} I & S \\ 0 & N' \end{bmatrix}.$$

It is guaranteed [10] that $P=P'>0$, that the solution P is unique, and that this solution is also a suitable P for (47) for continuous-time systems or (62) for discrete-time systems.

Step 4) Since P and J are computed, solving (47) or (62) for A_f, B_f, C_f, and D_f is once again an LMI problem. The solution is guaranteed to exist, as mentioned in the previous step. The resulting A_f, B_f, C_f, and D_f will form the desired robust H_∞ filter.

We note that the robust H_∞ filter is not unique because many feasible solutions R, S, and J for (82)–(84) or (85)–(87) exist. It is even possible to characterize the family of the solutions and therefore the family of robust H_∞ filters corresponding to a given γ. However, we do not dive into the details here. The interested reader is referred to [10].

VII. EXAMPLES

Example 1: Consider the continuous-time linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) + H_1\xi(t) \\ z(t) &= C_1 x(t) \\ y(t) &= C_2 x(t) + D_2 w(t) + H_2 \xi(t) \\ \xi(t) &= f(t)(E_1 x(t) + E_2 w(t) + E_3 \xi(t)) \\ f(t) &\in [0,1], \forall t \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -8.8 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_1 &= [20.8]; \quad C_2 = [-1 \quad 0.6]; \quad D_2 = 1.1 \\ H_1 &= \begin{bmatrix} 0 \\ 1.1 \end{bmatrix}; \quad H_2 = 0.2 \\ E_1 &= [0.6 \quad 0.4]; \quad E_2 = 0.35; \quad E_3 = 0.4. \end{aligned}$$

The aim is to design a robust H_∞ filter (32). We consider two design methods for comparison purposes: The first one is the nominal H_∞ filter design based on the nominal system with $f(t)=0$, and the second one is our robust H_∞ filter design that takes the uncertainty into account. Given the disturbance attenuation level $\gamma = 0.62$, a nominal H_∞ filter is obtained with the filter matrices

$$\begin{aligned} A_f &= \begin{bmatrix} -0.0203 & 1.0122 \\ -8.6696 & -2.0783 \end{bmatrix}; \quad B_f = \begin{bmatrix} -0.0203 \\ 0.1304 \end{bmatrix} \\ C_f &= [2 \quad 0.8]; \quad D_f = 0. \end{aligned}$$

Next, we use the IQC approach and take

$$\int_0^T \|\xi(t)\|^2\,dt \le \int_0^T \|E_1 x(t) + E_2 w(t) + E_3 \xi(t)\|^2\,dt \quad \forall T > 0.$$

With the design method proposed in this paper, the filter matrices of the robust H_∞ filter are obtained using the `Matlab LMI Toolbox` [12] as follows:

$$\begin{aligned} A_f &= \begin{bmatrix} -0.1828 & -4.9978 \\ 1.4976 & -2.4775 \end{bmatrix}; \quad B_f = \begin{bmatrix} -0.0135 \\ 0.0256 \end{bmatrix} \\ C_f &= [-15.0497 \quad 22.5981]; \quad D_f = 0 \end{aligned}$$

and the matrices R, S, and the scaling parameter J are given by

$$\begin{aligned} R &= \begin{bmatrix} 6.3095 & -2.5066 \\ -2.5066 & 37.5045 \end{bmatrix}; \quad S = \begin{bmatrix} 25.0040 & 2.4267 \\ 2.4267 & 3.2236 \end{bmatrix} \\ J &= 0.2200. \end{aligned}$$

The error dynamics $T_{ew}(s)$ using both the nominal H_∞ filter and the robust H_∞ filter are compared. In Fig. 1, we plot the maximum singular value for $T_{ew}(j\omega)$ versus frequency ω for different constant values of $f(t)$. An 0.1 increment for $f(t)$ is used, and the effect of $f(t)$ is observed in Fig. 1. It is clear that the robust filter is less sensitive to the variations in $f(t)$.

Fig. 2 shows the singular value comparison for $f(t) = 0, 0.3, 0.6, 0.9$, again taken to be constant. We observe from Fig. 2 that in the nominal case $(f(t)=0)$, $\|T_{ew}(s)\|_\infty$ for the nominal H_∞ filter is less than that for the robust H_∞ filter.

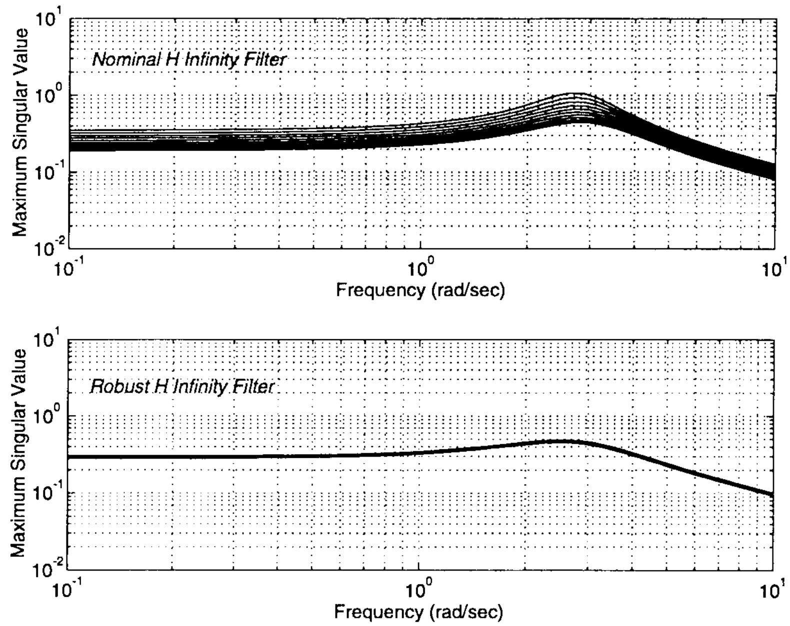

Fig. 1. Maximum singular value plots.

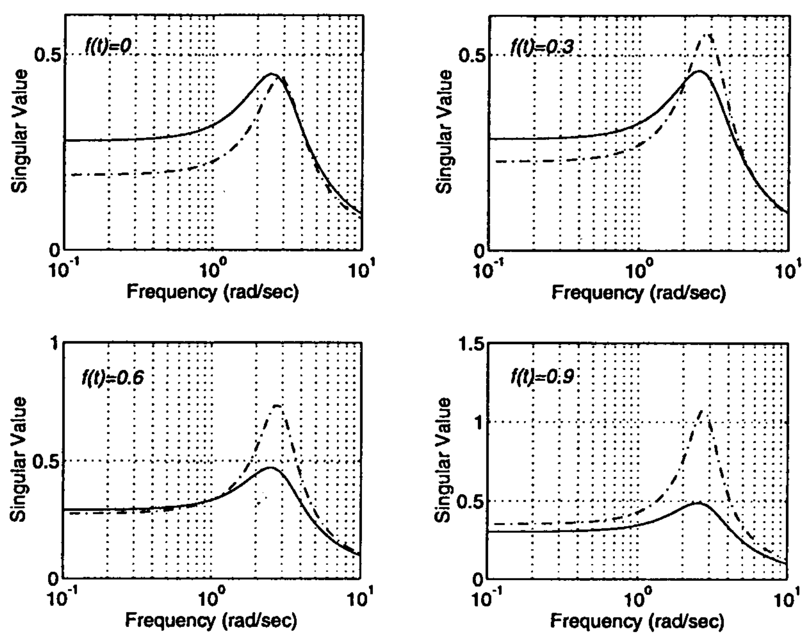

Fig. 2. Singular value comparison. —: robust H_∞ filter; - · -: nominal H_∞ filter.

However, for $f(t) = 0.3, 0.6, 0.9$, the nominal H_∞ filter gives a larger $||T_{ew}(s)||_\infty$ than the robust one. The tradeoff is clear.

Example 2: Our second example takes a communication channel that has the transfer function

$$\frac{Y(s)}{W(s)} = G_0(s) + e^{-\tau s} G_1(s) \tag{94}$$

where $G_0(s)$ and $G_1(s)$ are known. The delay τ is unknown but bounded by some upper bound $\overline{\tau}$. The term $e^{-\tau s} G_1(s)$ represents an echo in the communication channel. The problem is to design a filter such that the input signal $w(t)$ is recovered from the output signal $y(t)$ in the sense that the H_∞ norm of the corresponding error dynamics is minimized. If the delay $\tau = 0$, the problem is relatively simple since the optimal filter transfer function is $(G_0(s) + G_1(s))^{-1}$, provided that the nominal system $G_0(s) + G_1(s)$ is proper and bistable, or an approximate inverse can be found that gives a small H_∞ tolerance level γ when $G_0(s) + G_1(s)$ is strictly proper. When $(G_0(s) + G_1(s))^{-1}$ is used as the filter, the estimation error will be $(e^{-\tau s} - 1)G_1(s)W(s)$ when there exists a delay.

Note that $e^{-\tau s} G_1(s)W(s)$ is an echo signal of $G_0(s)W(s)$. Therefore, we assume that $G_0(s)$ and $G_1(s)$ have the same denominator for simplicity. Denote

$$Y_1(s) = (G_0(s) + G_1(s))W(s) \tag{95}$$

$$Y_2(s) = G_1(s)W(s). \tag{96}$$

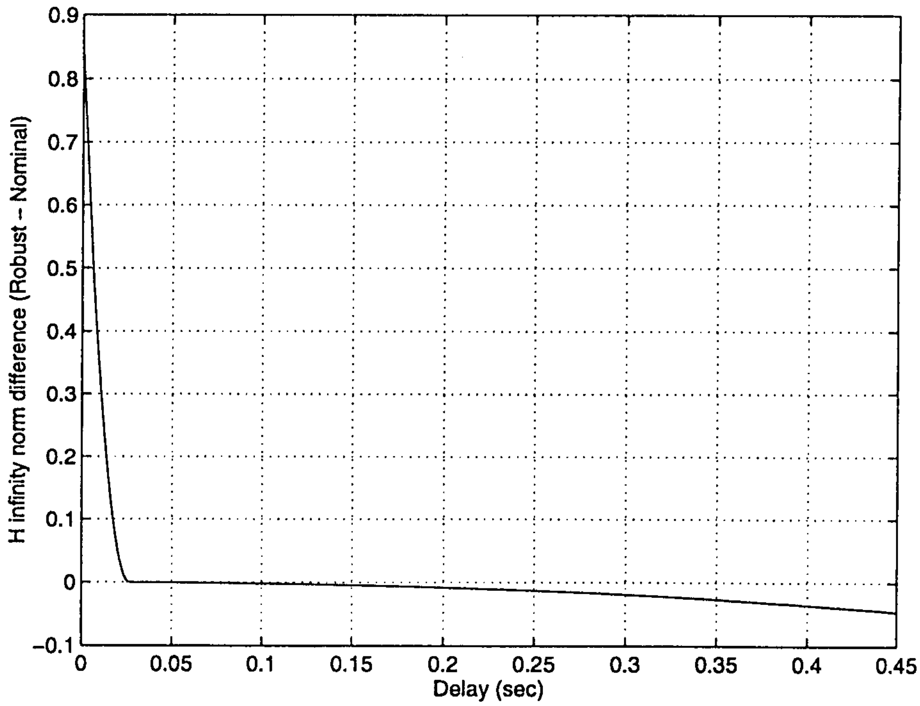

Fig. 3. $||T_{ew}(s)||_\infty - ||T_{ne}(s)||_\infty$ versus τ.

Suppose a state-space realization for (95) and (96) are $\{A, B, C_2, D_2\}$ and $\{A, B, E_1, E_2\}$, respectively. Then, we can rewrite (94) in state-space form as

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{97a}$$
$$z(t) = w(t) \tag{97b}$$
$$y(t) = C_2x(t) + D_2w(t) + \xi(t) \tag{97c}$$
$$y_2(t) = E_1x(t) + E_2w(t) \tag{97d}$$

where $\xi(t)$ and $y_2(t)$ have

$$\Xi(s) = (e^{-\tau s} - 1)Y_2(s) \tag{98}$$

in the frequency domain. It is obvious that

$$\int_{-\infty}^{\infty} \Xi^*(j\omega)\Xi(j\omega)\, d\omega = \int_{-\infty}^{\infty} Y_2^*(j\omega)(e^{j\omega\tau} - 1)(e^{-j\omega\tau} - 1)Y_2(j\omega)\, d\omega. \tag{99}$$

Using $e^{-j\omega\tau} - 1 = -2je^{-j\omega\tau/2}\sin(\omega\tau/2)$, we obtain the IQC

$$\int_{-\infty}^{\infty} \Xi^*(j\omega)\Xi(j\omega)\, d\omega \le \int_{-\infty}^{\infty} (Y_2^*(j\omega))(\max_{|\tau|\le\overline{\tau}} (2\sin(\omega\tau/2))^2)(Y_2(j\omega))\, d\omega. \tag{100}$$

Choose an overbounding filter $f(s)$ with a state-space realization $\{A_c, B_c, C_c, D_c\}$ such that

$$|f(j\omega)| \ge |2\sin(\omega)|, \quad \forall\omega \ge 0.$$

One of the overbounding filter $f(s)$ takes the state-space system matrices

$$A_c = \begin{bmatrix} -1.56 & -0.6084 \\ 1 & 0 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{101a}$$
$$C_c = [-0.66 \quad -1.0988]; \quad D_c = 2. \tag{101b}$$

Note that it is always possible to use higher order filter to get a tighter overbound.

The augmented system (97) and (101) is given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}w(t) \tag{102a}$$
$$z(t) = w(t) \tag{102b}$$
$$y(t) = \tilde{C}_2\tilde{x}(t) + D_2w(t) + \xi(t) \tag{102c}$$
$$\tilde{y}_2(t) = \tilde{E}_1\tilde{x}(t) + \tilde{E}_2w(t) \tag{102d}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \frac{2}{\tau}B_cE_1 & \frac{2}{\tau}A_c \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} B \\ \frac{2}{\tau}B_cE_2 \end{bmatrix}$$
$$\tilde{C}_2 = [C_2 \quad 0]; \quad \tilde{E}_1 = [D_cE_1 \quad C_c]; \quad \tilde{E}_2 = D_cE_2.$$

and the IQC (100) is replaced with the IQC

$$\int_0^T ||\xi(t)||^2\, dt \le \int_0^T ||\tilde{y}_2(t)||\, dt, \quad \text{as } T \to \infty. \tag{103}$$

For our example, we take

$$G_0(s) = \frac{1.5s^2 + 0.5s + 0.05}{2s^2 + 4s + 2}$$
$$G_1(s) = \frac{1.5s^2 + 5.5s + 6.8}{2s^2 + 4s + 2}$$

and $\overline{\tau} = 0.45$. Then

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C_2 = [0 \quad 1.925]$$
$$D_2 = 1.5; \quad E_1 = [1.25 \quad 2.65]; \quad E_2 = 0.75.$$

Using Theorem 6 and the `LMI Toolbox` [12], we find that the robust H_∞ filtering problem is solvable for $\gamma = 1.1$ (although a slightly smaller γ can be achieved). A corresponding solution is given by $J = 0.1089$ and

$$R = \begin{bmatrix} 171.2643 & -65.2981 & 64.9193 & 66.3846 \\ -65.2981 & 37.0121 & -34.4426 & 35.6283 \\ 64.9193 & -34.4426 & 318.9277 & -251.5274 \\ 66.3846 & 35.6283 & -251.5274 & 535.9718 \end{bmatrix}$$
$$S = \begin{bmatrix} 38.1222 & 81.2393 & -11.1980 & -15.1932 \\ 81.2393 & 185.5393 & -23.0484 & -34.2142 \\ -11.1980 & -23.048 & 3.4702 & 4.3528 \\ -15.1932 & -34.2142 & 4.3528 & 6.3717 \end{bmatrix}.$$

We then use the design procedure described in Remark 8. The robust H_∞ filter

$$G_f(s) = \frac{0.1122s^4 + 10.2976s^3 + 38.1741s^2 + 21.7468s^2 + 1.0953}{s^4 + 85.8304s^3 + 353.7919s^2 + 265.6439s + 22.4681}.$$

is then obtained.

In comparison, we observe that the optimal filter for $\tau = 0$ is given by

$$G_{nf}(s) = (G_0(s) + G_1(s))^{-1} = \frac{2s^2 + 4s + 2}{3s^2 + 6s + 6.85}.$$

For $\tau \leq \overline{\tau}$, we can define the error dynamics corresponding to the nominal filter $G_{nf}(s)$

$$T_{ne}(s) = G_{nf}(s)(G_0(s) + e^{-\tau s}G_1(s)) - 1$$

and that to the robust filter $G_f(s)$

$$T_{ew}(s) = G_f(s)(G_0(s) + e^{-\tau s}G_1(s)) - 1.$$

A fine grid of frequency is used to approximate $||T_{ew}(s)||_\infty$ and $||T_{ne}(s)||_\infty$. Fig. 3 shows the curve of $||T_{ew}(s)||_\infty - ||T_{ne}(s)||_\infty$ as a function of τ. It is seen from Fig. 3 that the nominal filter is better than the robust H_∞ filter for $\tau \in [0, 0.027)$, but the situation is reversed for $\tau \in [0.027, 0.45]$. As in Example 1, the tradeoff between a nominal H_∞ filter and a robust H_∞ filter is clearly shown.

VIII. Conclusion

In this paper, we have provided an LMI approach to the robust H_∞ filtering problem for linear systems with uncertainty described by IQC's. Our approach has several features. First, the IQC description of uncertainty is very general and is suitable for many signal processing applications. The use of IQC's is demonstrated in Remark 1 in Section III and the examples in Section VII. We also refer to [20] and [25] for more examples of IQC's. Second, the LMI approach is computationally efficient owing to recent advances in convex optimization [16]. The robust H_∞ filter analysis problem is solved in terms of LMI's. The synthesis problem, however, is solved in terms of matrix inequalities that are linear except for some scaling parameters. Methods have been proposed to overcome this difficulty. Further, we have shown that in two special cases, these matrix inequalities can be converted into LMI's. The first special case, which requires the number of IQC's to be limited to a single one, is of particular interest because this is the case in many applications.

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