Robust Nonlinear Forwarding Design

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Abstract. This paper considers the problem of robust stabilization for a class of nonlinear up-augmented systems with uncertain parameters in a very general form. This system involves a base system with a control input and a forwarding structure. Several robust control design methods are discussed for the cases in which these parts of the system are in SISO and MIMO forms, respectively. The main assumptions required for these methods are quadratic stabilizability for the local linearized model of the system, global asymptotic stabilizability for the base system and some mild conditions on the up-augmentation and the nonlinearity of the system.

Keywords: Nonlinear control; Forwarding; Robust control; Quadratic stabilization.

1 Introduction

In this paper, we address the robust stabilization problem for a class of cascaded nonlinear systems with the so-called *forwarding* or *feedforwarding* structure. More precisely, these systems have the following model:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, q) + g_1(x_1, x_2, u, q) \\ \dot{x}_2 &= f_2(x_2, q) + g_2(x_2, u, q) \end{aligned} \tag{1}$$

where $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ are state variables, $u \in \mathbf{R}^m$ is a control input, q is an uncertain parameter vector belonging to a compact set $Q \subset \mathcal{R}^p$, nonlinear vector functions $f_1(x_1, x_2, q)$, $f_2(x_2, q)$, $g_1(x_1, x_2, q)$ and $g_2(x_2, q)$ are smooth in x_1, x_2 and u and continuous in q, $f_1(0, 0, q) \equiv 0$, $f_2(0, q) \equiv 0$, $g_1(x_1, x_2, 0, q) \equiv 0$ and $g_2(x_2, 0, q) \equiv 0$ for $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$ and $\forall q \in Q$.

Many similar cases of (1) have been studied; see, e.g., [3, 4, 5, 6, 8]. The work in these papers have led to several design control methods. In [8], a saturation function and input-output method are used to design a global asymptotic stabilizer for an upper-triangular system. In this case, the state variable x_1 in (1) is a scalar variable. In [3, 4], cascaded systems similar to that in (1) are studied and Lyapunov function based design methods are proposed. However, the aforementioned design methods rely on very accurate knowledge of the system. The only exception is [8] where uncertainties are allowed in high order (nonlinear) terms.

In [5] and [6], a robust control design approach is developed for an upper-triangular nonlinear system with large size uncertainties in both local linearizied model and global nonlinear model of the system. Apart from providing a robust control design method, the results in [5] and [6] also bridge a gap between linear robust control theory and nonlinear robust control theory in the sense that the design method coincides with the seminal work of Wei [9] on quadratic stabilization of linear systems.

In this paper, we will review the existing results for the robust stabilization problem of (1). Then the robust design method in [5] and [6] is generalized to the system (1). The main advantage of this method is that the key condition for robust stabilization of (1) boils down to quadratic stabilizibility of the locally linearized model. Since there are many well-established methods, such as linear quadratic stabilization theory [2], H_∞ control [1], Wei's method [9], etc. to solve quadratic stabilization problem for a wide class of linear uncertain systems, the system satisified this assumption is much more general than the systems studied in existing literature. The other advantage of this method is that a simpler design method for robust controllers and a simpler proof of the stabilizability of the system are provided. Our design method considers two notions of stability: global asymptotic stability and local quadratic stability. The latter requires the existence of a locally quadratic Lyapunov function for the system (1) for all admissible uncertain parameters $q \in Q$. The stabilizing controller to be proposed involves two steps: The first controller brings the state of the base system, x_2, to a small neighbourhood of the origin, and the second controller is used to bring both x_1 and x_2 to the origin. A non-quadratic Lyapunov function is used to design this robust controller.

2 Forwarding Control Design via Lyapunov Functions

In this section, we review design methods given in [3, 4]. Consider the following system (which is a special case of the system (1)) rewritten as

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2)u\end{aligned} \tag{2}$$

Since $f_1(0, 0) = 0$ and $f_1(x_1, x_2)$ are smooth in x_1, x_2, we can decompose $f_1(x_1, x_2)$ as follows:

$$f_1(x_1, x_2) = \bar{f}_1(x_1) + f_{12}(x_1, x_2)x_2.$$

where $\bar{f}_1(x_1) = f_1(x_1, 0)$ and $f_{12}(x_1, x_2)$ is a smooth function. Denote $h(x_1, x_2) = f_{12}(x_1, x_2)x_2$.

Suppose the system (2) satisfies assumptions below:

Assumption 2.1 Suppose $W_1(x_1)$ is a Lyapunov function of the system

$$\dot{x}_1 = \bar{f}_1(x_1) \tag{3}$$

and $L_{\bar{f}} W_1(x_1) \le 0$. Moveover, there holds

$$\left\| \frac{\partial W_1}{\partial x_1} \right\| \|x_1\| \le k W_1(x_1); \quad \forall \|x\| \ge c \tag{4}$$

for some constants $k > 0$ and $c > 0$.

Assumption 2.2 The system

$$\dot{x}_2 = \bar{f}_2(x_2) \tag{5}$$

is asymptotically stable and $W_2(x_2)$ is a Lyapunov function of the system (5).

Assumption 2.3 There exist $\mathcal{K}$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$\|h(x_1, x_2)\| \le \gamma_1(x_2) + \|x_1\|\gamma_2(x_2) \tag{6}$$

where γ_1 and γ_2 are differentable at the origin.

In [3], under Assumptions 2.1-2.3 and some slightly stronger assumption in the local structure of the system (3) (see the assumption A3 in [3]), a Lyapunov function

$$V(x_1, x_2) = W_1(x_1) + W_2(x_2) + \Psi(x_1, x_2). \tag{7}$$

is constructed by a design method, called *cross term technique*, for the system (2). The cross term $\Psi(x_1, x_2)$ is in the form as below

$$\Psi(x_1, x_2) = \int_0^\infty L_h W_1(\bar{x}_1(\tau, x_1, x_2), \bar{x}_2(\tau, x_2))d\tau \tag{8}$$

where $\bar{x}_1(\tau, x_1, x_2)$, $\bar{x}_2(\tau, x_2)$ are solution of (2) with initial state (x_1, x_2) and $u \equiv 0$.

Applying the Lyapunov function (7), it has been proven in [3] that, under Assumptions 2.1-2.3 and some local stabilizibility of the system (see the assumption A4 in [3]), the system (2) is globally asymptotically stabilizable.

In [4], Mazenc and Praly obtained a Lyapunov function in form

$$V(x_1, x_2) = l(W_1(x_1)) + k(W_2(x_2)) \tag{9}$$

where the functions $l(\cdot)$ and $k(\cdot)$ are nonlinear weight functions to be choosen.

Under assumptions similar to Assumptions 2.1-2.3 and some local stabilibility of the system, a global stabilizier is designed in [4] by applying the Lyapunov function (9) and selecting appropereiate functions $l(\cdot)$ and $k(\cdot)$. A drawback in the methods in [3, 4] is that the Lyapunov functions (7) and (9) are dependent on the precise knowledge of the model of the system (2).

3 Forwarding Design via Saturation Functions

In this section, we will discuss another forwarding design method in [8] which uses saturation functions. Compared with the results in the last section, the forwarding control via saturation function needs more structure information of the system (1). Suppose the system (1) without uncertain parameters q satisfies the following assumptions:

Assumption 3.1 The state x_1 is a scalar state variable and $f_1(x_1, x_2) = x_{21} + \bar{f}_{11}(x_2)$. The functions $\bar{f}_{11}(x_2)$ and $g_1(x_2, u)$ involve only quadratic and higher order terms of $x_{21}, \cdots, x_{2n_2}$ and u, where $x_2 = (x_{21}, \cdots, x_{2n_2})^T$.

Assumption 3.2 Denote $f_2(x_2) = (f_{21}, \cdots, f_{2n_2})^T$ and $g_2(x_2, u) = (g_{21}, \cdots, g_{2n_2})^T$. Then, for $i = 1, \cdots, n_2 - 1$,

$$\begin{aligned}f_{2i} &= x_{2i} + \bar{f}_{2i}(x_{2(i+1)}, \cdots, x_{2n_2}), \\ g_{2i} &= x_{2i} + \bar{g}_{2i}(x_{2(i+1)}, \cdots, x_{2n_2}, u)\end{aligned}$$

where $\bar{f}_{2i}$ and $\bar{g}_{2i}$ only have quadratic and higher order terms of $x_{2(i+1)}, \cdots, x_{2n_2}$ and u only. Further,

$$f_{2n_2} = 0; \qquad g_{2n_2} = u + \bar{g}_{2n_2}(u).$$

where $\bar{g}_{2n_2}(u)$ have quadratic and higher order terms of u only.

Under Assumptions 3.1-3.2, there exists a linear nonsingular coordinate transformation $(x, x_{21}, \cdots, x_{2n_2}) \to (\zeta_1, \cdots, \zeta_{n_2+1})$ such that, in the ζ-coordinate the system (1) can be written as

$$\dot{\zeta} = A\zeta + Bu + \Phi \tag{10}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \tag{11}$$

and

$$\Phi^T = \begin{bmatrix} \phi_1(\zeta_2, \cdots, \zeta_{n_2+1}, u) & \cdots & \phi_{n_2+1}(u) \end{bmatrix} \tag{12}$$

where ϕ_i, $i = 1, \cdots, n_2+1$ contains only quadratic and higher order terms of $\zeta_{i+1}, \cdots, \zeta_{n_2+1}$ and u (see [8] for details).

Denote $\sigma_i(\tau)$, $i = 1, \cdots, n_2+1$ are continuous, nondecreasing saturation functions such that

1. $\sigma_i(\tau) = \tau; \quad |\tau| < \delta_i;$
2. $\sigma_i(\tau)| < \varepsilon_i$ for all $\tau \in \mathbf{R}$

where $\varepsilon_i > \delta_i$.

Select $1 >> \varepsilon_{n_2+1} >> \varepsilon_{n_2} >> \cdots >> \varepsilon_1$ and let controller u such that

$$u = -\sigma_{n_2+1}(\zeta_{n_2+1} + \sigma_{n_2}(\zeta_{n_2} + \cdots + \sigma_1(\zeta_1)\cdots)) \quad (13)$$

Applying the controller (13) into (10), the state variable ζ_{n_2+1} is driven into the region $[-\delta_{n_2+1}, \delta_{n_2+1}]$ after a finite time t_{n_2+1} and the states ζ_{n_2} and ζ_{n_2+1} satisfy following equations

$$\begin{aligned} \dot{\zeta}_{n_2} &= -\sigma_{n_2}(\zeta_{n_2} + \cdots + \sigma_1(\zeta_1)\cdots) \\ \dot{\zeta}_{n_2+1} &= -\zeta_{n_2+1} - \sigma_{n_2}(\zeta_{n_2} + \cdots + \sigma_1(\zeta_1)\cdots) \end{aligned} \quad (14)$$

In general, the controller (13) is a multi-step controller. In every step, a new state of the system is driven into a given samll region by the controller while the previous states are kept in a given small region by the controller. Since these states are very small so that this part of the system can be modelled as a linear system. Further it is stabilizied by the linear part of the controller (13). In the n_2+1 step, there is a finite time t_1 such that, after the finite time t, it holds

$$\begin{aligned} \dot{\zeta}_1 &= -\zeta_1 \\ &\vdots \\ \dot{\zeta}_{n_2} &= -\zeta_1 - \cdots - \zeta_{n_2} \\ \dot{\zeta}_{n_2+1} &= -\zeta_1 - \cdots - \zeta_{n_2+1} \end{aligned} \quad (15)$$

Obviously, the system (15) is asymptotically stable. Thus, under Assumption 3.1-3.2, the system (1) is globally asymptotically stabilizable if the system is without uncertain parameters.

It should also be noted that the forwarding design via saturation function allows some uncertainties in the system but the precise locally linearized model of the system is necessary.

4 Robust Forwarding Deisgn

As we have seen in Section 3, using a very special structure assumption and the precise local linearizied model of the system (1), this system can be stabilizied by a multistep controller. On the other hand, applying a quadratic Lyapunov function, a robust controller can be designed for a much more general uncertain linear system (see, [9]). So there is a big gap between linear and nonlinear systems in terms of robust stabilization techniques. In this section we will merge this gap by using a local nonquadratic Lyapunov function and a multi-step controller [5, 6].

Suppose the system (1) satisfies following assumptions:

Assumption 4.1 The state x_1 is a scalar state variable. The function $f_1(x_1, x_2, q)$ can be decomposed as $f_1(x_1, x_2, q) = \theta(q)x_{21} + \bar{f}_{11}(x_2, u, q)$ where $\theta(q) > 0$, $\forall q \in Q$ and $\bar{f}_{11}(x_2, u, q)$ has quadratic or higher order terms of x_{21} only.

Denote $x_{2,n_2+1} = u$ is a new state variable and v is a new control input. And also denote $\bar{x}_2 = (x_2 \; x_{2,n_2+1})^T$. Then the system (1) can be written as

$$\begin{aligned} \dot{x}_1 &= \theta(q)x_{21} + \tilde{f}_{11}(\bar{x}_2, q) \\ \dot{\bar{x}}_2 &= \bar{f}_2(\bar{x}_2, q) + bv \end{aligned} \quad (16)$$

where $\tilde{f}_{11}(\bar{x}_2, q) = \bar{f}_{11}(\bar{x}_2, q) + g_1(\bar{x}_2, q)$ and

$$\bar{f}_2(\bar{x}_2, q) = \begin{bmatrix} f_2(\bar{x}_2, q) & g_2(\bar{x}_2, q) \\ 1 & 0 \end{bmatrix}; \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assumption 4.2 The system

$$\dot{\bar{x}}_2 = \bar{f}_2(\bar{x}_2, q) + bv \quad (17)$$

is globally asymptotically stable and locally quadratically stable. And

$$V_2(\bar{x}_2) = \bar{x}_2^T P_2 \bar{x}_2 \quad (18)$$

is a local quadratic Lyapunov function of the system (17) in the region

$$\Omega = \{ \; \bar{x}_2 \mid V_2(\bar{x}_2) < \mu \; \} \quad (19)$$

where μ is a given positive constant.

Assumption 4.3 Denote $A_2(q)\bar{x}_2$ is the local linearization of $\bar{f}_2(\bar{x}_2, q)$. Then matrix $A_2(q)$ has a structure as below:

$$A_2(q) = \begin{bmatrix} 0 & A_2^-(q) \\ \star & \star \end{bmatrix}$$

where $A_2^-(q)$ is some uncertain matrix and $\star$ can be some uncertain scalar or vector elements.

In fact, the structure in Assumption 4.3 includes the upper-triangular structure in Assumption 3.2. Thus, compared with Assumption 3.2, Assumption 4.3 is weeker. Further, Assumption 4.1-4.3 allow uncertain parameter q to enter the system, so the controller to be designed in this section will have more robustness.

Assumptions 4.2 leads to that the $\bar{x}_2$ converges into Ω in finite time while keeping x_1 bounded. Hence, we assume in the sequel that $\bar{x}_2(0) \in \Omega$, where $\bar{x}_2(0)$ is the initial value of $\bar{x}_2(t)$. Choose

$$V(x_1, \bar{x}_2) = (x_1 - (\gamma \;\; 0)P\bar{x}_2)^2 + \int_0^{V_2(\bar{x}_2)} s(w)dw \tag{20}$$

as a local Lyapunov function for the system (1), where γ is a negative constant to be choosen and $s(w)$ is a positive, smooth, and monotonically non-decreasing function for $w \in [0, \mu)$, with

$$\int_0^{V_2} s(w)dw \to \infty; \;\; as \;\; V_2 \to \mu. \tag{21}$$

Remark 4.1 A particular choice of $s(\cdot)$ is given by

$$s(w) = \frac{\mu}{\mu - w}.$$

In general, the Lyapunov function (20) is non-quadratic. However, as $x \to 0$, $V(x)$ becomes quadratic in x because $s(0) > 0$. We also note that the function $\int_0^{V_2(x_2)} s(w)dw$ resembles a "potential barrier" and the Lyapunov function (20) is valid only for $x_2 \in \Omega$, i.e.,

$$V(x_1, x_2) \to \infty \quad as \quad V_2(x_2) \to \mu. \tag{22}$$

This implies that future $x_2 \in \Omega$ as long as that $V(x_1, x_2)$ remains bounded. □

Applying the Lyapunov function (20), a robust controller is designed for the system (1) such that the system is robustly globally asymptotically stable and locally quadratically stable (see [5, 6]).

5 Extension of Robust Forwaring

We new extend the robust forwarding method discussed in last section to a more general form.

Denote the local linearized model of the system (1) as

$$\dot{x} = A(q)x + B(q)u \tag{23}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \; A(q) = \begin{bmatrix} A_1(q) & A_{12}(q) \\ 0 & A_2(q) \end{bmatrix}; \; B(q) = \begin{bmatrix} B_1(q) \\ B_2(q) \end{bmatrix}.$$

Without loss of generlity, we assume $B_1(q) \equiv 0$, $\forall q \in Q$. Otherwise, we can extend x_2 to a new state $\bar{x}_2$ by introducing the control input u as a part of the new state. As we showed in the last section, the control input matrix corresponding to x_1 will be equal to zero in the new system.

Suppose the system (1) satisfies assumptions below:

Assumption 5.1 (Local Quadratic Stabilizibility): There exist a linear state feedback matrix

$$K = [K_1 \;\, K_2]$$

and a symmetric and positive-definite matrix P_0 such that

$$P_0[A(q) + B(q)K] + [A(q) + B(q)K]^T P_0 < 0, \;\; \forall q \in Q \tag{24}$$

Without loss of generality, we let

$$P_0 = \begin{bmatrix} P_1 & -P_1 W \\ -W^T P_1 & P_2 + W^T P_1 W \end{bmatrix}. \tag{25}$$

for some $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ and W. Equivalently, the quadratic Lyapunov function is given by

$$V_0(x_1, x_2) = (x_1 - Wx_2)^T P_1 (x_1 - Wx_2) + x_2^T P_2 x_2 \tag{26}$$

Assumption 5.2 (Global Asymptotic Stabilizability of the Base System): There exists a locally smooth controller $u_0(x_2)$ such that the system below

$$\dot{x_2} = f_2(x_2, q) + B_2(q)u_0(x_2) \tag{27}$$

is globally asymptotically stable.

Assumption 5.3 (Local Boundedness of the Forwarding State): The matrix P_1 in Assumption 2.1 is such that

$$P_1 A_1(q) + A_1^T(q) P_1 \le 0; \quad \forall q \in Q. \tag{28}$$

Assumption 5.4 (Smoothness Conditions): We require

$$f_1(x_1, x_2, q) = A_1(q)x_1 + A_{12}(q)x_2 + F_{12}(x, q)x_2 \tag{29}$$

$$f_2(x_2, q) = A_2(q)x_2 + F_2(x_2, q)x_2 \tag{30}$$

for some $F_2(x_2, q)$ which is continuous in q and smooth in x_2 with $F_2(0, q) = 0$, $\forall q \in Q$, and $F_{12}(x, q)$ which is continuous in q and smooth in x and satisfies

$$\max_{q \in Q} \|F_{12}(x, q)\| \le \gamma_1(x_2)\|x_1\| + \gamma_2(x_2) \tag{31}$$

with some smooth functions $\gamma_i(x_2), i = 1, 2$.

Remark 5.1: It is obvious that Assumption 5.1 is necessary for local quadratic stabilization. Since the quadratic stabilization theory for linear uncertain system is well established, we will not discuss methods of solving the linear quadratic control $u = Kx$ and quadratic Lyapunov function $V_0(x_1, x_2)$.

However, the condition in (28) is required for technical reasons (in the proof of Theorem 5.1). To justify this condition, we note several points: 1) When the forwarding state x_1 is a scalar, this condition merely requires $A_1(q)$ to be non-positive. In fact, $A_1(q) = 0$ in the upper-triangular structure. 2) When x_1 is not a scalar, a condition similar to (28) is often used; see [4, 3]. 3) To show that stabilizability may be impossible without (28), we consider the following example:

$$\begin{array}{rcl} \dot{x}_1 & = & \epsilon x_1 + 2x_2 + x_2^2, \quad \epsilon > 0 \\ \dot{x}_2 & = & u \end{array} \tag{32}$$

It is easy to verify that its local linearized model is stabilizable. However, $2x_2 + x_2^2 \geq -1$, implying that $x_1(t)$ will diverge if $x_1(0) > 1/\epsilon$ regardless what control is used.

Finally, we point out that Assumption 2.3, (29) and (31) guarantee that x_1 is bounded for bounded x_2 (see [3]). □

Now, we will pay attention to design a robust controller for the system (1) under Assumptions 5.1-5.4. The closed-loop system is required to have both global asymptotic stability and local quadratic stability.

From Assumption 5.1, we choose

$$V_2(x_2) = x_2^T P_2 x_2 \tag{33}$$

as a local quadratic Lyapunov function for the base system and define a local region Ω as:

$$\Omega = \{x_2 \,| V_2(x_2) < \mu\} \tag{34}$$

where $\mu > 0$ is to be specified. Denote

$$A(x,q) = \begin{bmatrix} A_1(q) & A_{12}(q) + F_{12}(x,q) \\ 0 & A_2(q) + F_2(x_2,q) \end{bmatrix}. \tag{35}$$

From Assumptions 5.1 and 5.4, we know that the following holds for sufficiently small $\mu > 0$ and $\varepsilon > 0$:

$$P_0[A(x,q) + B(q)K] + [A(x,q) + B(q)K]^T P_0 < -\varepsilon I; \tag{36}$$

for all $x_2 \in \Omega$.

Assumptions 5.2 and 5.4 guarantees that the state x_2 can be driven into the region Ω in finite time by the controller $u_0(x)$ while keeping x_1 bounded. Hence, we assume that $x_2(0) \in \Omega$, where $x_2(0)$ is the initial value of $x_2(t)$. Choose

$$V(x_1,x_2) = (x_1 - Wx_2)^T P_1 (x_1 - Wx_2) + \int_0^{V_2(x_2)} s(w)dw \tag{37}$$

as a local Lyapunov function for the system (1), where $s(w)$ is same as the function $s(w)$ in (20).

For $x_2 \in \Omega$, the derivative of $V(x_1,x_2)$ along the trajectory of the system (1) is given by

$$\dot{V}(x_1,x_2) = x^T[PA(x,q) + A^T(x,q)P]x + 2x^T PBu \tag{38}$$

where

$$P = \begin{bmatrix} P_1 & -P_1 W \\ -W^T P_1 & s(V_2)P_2 + W^T P_1 W \end{bmatrix} \tag{39}$$

with its inverse given by

$$S = \begin{bmatrix} P_1^{-1} + s^{-1}(V_2) W P_2^{-1} W^T & s^{-1}(V_2) W P_2^{-1} \\ s^{-1}(V_2) P_2^{-1} W^T & s^{-1}(V_2) P_2^{-1} \end{bmatrix}. \tag{40}$$

Consider the following local controller:

$$\begin{array}{rcl} u_l(x_1,x_2) & = & s^{-1}(V_2(x_2))K_1 x_1 \\ & & +[K_2 + (1 - s^{-1}(V_2(x_2)))K_1 W]x_2 \end{array} \tag{41}$$

Then we have the following main result:

Theorem 5.1: Suppose the system (1) satisfies Assumptions 5.1-5.4. Then the closed-loop system controlled by the following controller

$$u = \begin{cases} u_0(x_2), & x_2 \notin \Omega \\ u_l(x_1,x_2), & x_2 \in \Omega \end{cases} \tag{42}$$

is robustly globally asymptotically stable and locally quadratically stable.

Proof: As discussed before, we only need to consider the case when $x_2(0) \in \Omega$ and $u_l(x_1,x_2)$ is applied. Using the nonlinear coordinate transformation

$$z = [z_1^T \; z_2^T]^T = Px, \tag{43}$$

the equation (38) becomes

$$\dot{V}(x_1,x_2) = z^T[A(x,q)S + SA^T(x,q)]z + 2z^T B(q)u. \tag{44}$$

Denote $s^{-1}(V_2(x_2))$ by s^{-1} and let

$$u = s^{-1}K_1 x + K_2 x_2 + v. \tag{45}$$

Then,

$$\begin{array}{l} \dot{V}(x_1,x_2) = 2z^T(A(x,q) + B(q)K)Sz \\ \qquad -2(1 - s^{-1})z^T B(q)[K_1 \; 0]Sz + 2z^T B(q)v \end{array} \tag{46}$$

Rewrite (36) as

$$\begin{array}{r} [A(x,q) + B(q)K]S_0 + S_0[A(x,q) + B(q)K]^T < -\epsilon S_0^2, \\ \forall x \in \mathbf{R}^{n_1} \times \Omega; \;\; \forall q \in Q \end{array} \tag{47}$$

where $S_0 = P_0^{-1}$. Rewriting S as

$$S = s^{-1}S_0 + (1 - s^{-1}) \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

we simplify (46) to

$$\begin{array}{rl} & \dot{V}(x_1,x_2) \\ = & 2s^{-1}z^T(A(x,q) + B(q)K)S_0 z \\ & +2(1 - s^{-1})z_1^T[A_1(q)P_1^{-1}]z_1 \\ & -2(1 - s^{-1})s^{-1}z_1^T B(q)K_1 W x_2 + 2z^T B(q)v \end{array}$$

Applying Assumption 2.3 and choosing

$$v = (1 - s^{-1})K_1 W x_2, \tag{48}$$

we have

$$\dot{V}(x_1,x_2) = 2s^{-1}z^T(A(x,q)+B(q)K)S_0 z$$
$$= 2s^{-1}x^T P_0(A(x,q)+B(q)K)x \quad (49)$$

From Assumption 2.1, we can see that

$$\dot{V}(x_1,x_2) < 0; \ \ \forall x \in \mathbf{R}^{n_1} \times \Omega - \{0,0\}. \quad (50)$$

This implies that

$$V(x_1(t),x_2(t)) \leq V(x_1(0),x_2(0)), \ \forall t \geq 0.$$

Using (37) and monotonicity of $s(\cdot)$, we have

$$V_2(x_2(t)) \leq V(x_1(0),x_2(0))/s(0) =: \rho$$

Hence, (49) leads to

$$\dot{V}(x_1,x_2) \leq -\varepsilon s^{-1}(\rho)x^T x. \quad (51)$$

Therefore, the system (1) is robustly globally asymptotically stabilizible. Finally, the robust local quadratic stability property follows from (51) and the fact that $V(x_1,x_2)$ becomes quadratic as $x_2 \to 0$. $\nabla\nabla\nabla$

6 Conclusions

In this paper, we have studied the robust stabilization problem for a class of uncertain nonlinear systems in a forwarding structure. Several results on this problem have been reviewed. Then a new design method generalized from [5, 6] is introduced. This method guarantees robust global asymptotic stability and local quadratic stability. Compared with existing results, this method enables us to simplify the design process and the required assumptions and provides better robustness. It should be noted that this method can be combined with the backstepping design method to give a recursive design for robust controllers for a much larger class of uncertain nonlinear systems involving both forwarding and backstepping structures; see [7] for details.

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