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Connections between Quantized Feedback Control and Quantized Estimation

Minyue Fu
School of EECS
University of Newcastle
Australia
minyue.fu@newcastle.edu.au

Lihua Xie
School of EEE
Nanyang Technological University
Singapore
Email: elhxie@ntu.edu.sg

Weizhou Su
College of Automation Science and Engineering
South China University of Technology
China
Email: wzhsu@scut.edu.cn

***Abstract*—Quantized feedback control and quantized estimation have attracted a lot of attention in recent years with many results available on both research topics. In this paper, we investigate connections between quantized feedback control and quantized estimation and try to establish a possible separation principle for quantized output feedback which would allow the control design and state estimation to become independent in a networked control environment. We also consider the use of a variable rate finite-level logarithmic quantizer and show that this may approach the minimum averaged bit rate required for quantized feedback stabilization.**

***Index Terms*—Quantized feedback control, networked control, quantized estimation, quantization.**

I. Introduction

Quantized feedback control problems have attracted a lot of attention in recent years. Samples of works in this area include [1]-[16]. These problems arise in a digital network based control environment where feedback signals must be quantized before transmission. Since communication channels typically have limited bandwidths or bit rates, quantizers and controllers need to be jointly designed in order to minimize the bit rate required for achieving a given control task or to optimize the control performance for a given bit rate constraint.

There are two types of quantizers studied for quantized feedback control. The first type is the so-called memoryless (or static) quantizers [11], [12], [13], [16]. This type of quantizers are easy to use but theoretically require an infinite number of quantization levels (or infinite bit rate). In practice, however, these quantizers are truncated to yield a finite number of quantization levels but extra modifications such as dynamic scaling [14] is needed. In many cases [11], [12], [13], [16], logarithmic quantization is preferred for this type of quantizers. A major advantage of logarithmic quantizers is quantization errors can be readily modeled by multiplicative uncertainties which can be easily handled under the robust control framework [13]. The second type of quantizers used in quantized feedback control are dynamic quantizers; see, e.g., [6], [7], [10], [14]. The main advantage of this type of quantizers is that a finite number of quantization levels is typically needed. However, signals to be quantized typically need to be dynamically scaled, which complicates the design of quantizers and controllers and in addition, they require perfect transmission, or additional measures to handle transmission errors. For the problem of stabilization of a SISO LTI system (in some stochastic sense), it is shown by [6] that it can be achieved using only a finite number of quantization levels, and the minimum number of averaged quantization levels (also known as the minimum *averaged feedback information rate*) is explicitly related to the unstable poles of the system.

In this paper, we consider the problem of quantized feedback stabilization. Several results are to be offered. The first one is a simple separation principle which states that quantized output feedback stabilization can be done by separately designing a state feedback stabilizer and a state estimator. Using this result, we then consider the use of an infinite-level logarithmic quantizer and show that the coarsest quantization density required for the state feedback stabilization can be achieved using output feedback. Our last result considers the use of a variable rate finite-level logarithmic quantizer. For a first order system, we show that the minimum average bit rate for quantized feedback stabilization using this type of quantizers is the same as the minimum feedback rate given by [6]. This is the first time an explicit link is established between the coarsest quantization density for quantized feedback stabilization using logarithmic quantizers [11] and the minimum feedback rate using arbitrary quantizers by [6].

II. Problem Formulation

The problem we study in this paper is a standard quantized feedback control problem where feedback signals must be transmitted over a communication channel.

The controlled system we consider is a discrete-time model given by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $y_t \in \mathbb{R}$ is the measured output which is scalar-valued, $w_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}$ are independent and identically distributed (i.i.d.) sequences of Gaussian random variables with zero mean and covariances Σ_w and Σ_v, respectively, and the initial state x_0 is also assumed to be an independent zero-mean Gaussian random variable with covariance Σ_0. In some cases in the paper, the conditions on the noises will be relaxed or modified. We assume that (A, B) is stabilizable and (C, A) is detectable.

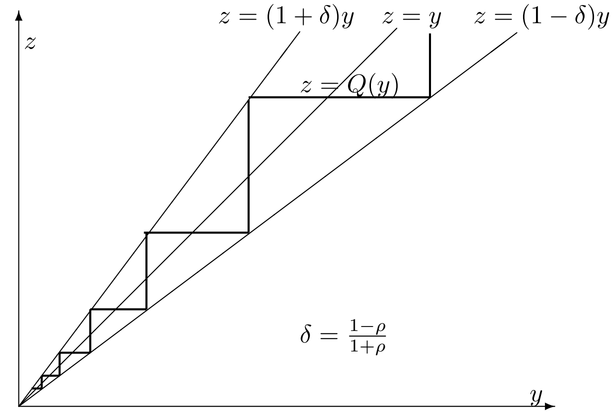

Fig. 1. Logarithmic Quantizer

The communication channel we consider in this paper is assumed to be a memoryless and error-free channel with a fixed transmission rate of R bits per (discrete-time) sample. The encoder is required to be causal mapping from the measured signal y_t, and similarly, the decoder is required to be a causal mapping from the received quantized signal. Under the error-free channel assumption, the encoder and decoder can be lumped together and considered as a quantizer:

$$z_t = Q(y^t, z^{t-1}) \tag{2}$$

where $Q(\cdot)$ takes value in a finite alphabet set with size of 2^R. In the above, we have used the notation $a^t = \{a_0, a_1, \ldots, a_t\}$. The controller takes the form

$$u_t = D[z^t] \tag{3}$$

The quantized output feedback stabilization problem is to design the quantizer (2) and a feedback control law (3) such that the closed-loop system is stable, subject to certain constraints on the quantizer.

In [13], we considered the use of a static logarithmic quantizer (where $R = \infty$). In this case,

$$z_t = Q(y_t) \tag{4}$$

where

$$Q(y)=\begin{cases} \rho^i \mu_0, & \text{if } \frac{1}{1+\delta}\rho^i\mu_0 < y \le \frac{1}{1-\delta}\rho^i\mu_0, \\ 0, & \text{if } y = 0, \\ -Q(-y), & \text{if } y < 0, \end{cases} \tag{5}$$

where $\rho \in (0,\ 1)$, μ_0 is a scaling constant, $i = 0, \pm 1, \pm 2, \cdots$, and

$$\delta = \frac{1-\rho}{1+\rho}. \tag{6}$$

Note that the parameter ρ can be regarded as the quantization density. A small ρ corresponds to a coarse quantizer. Logarithmic quantizers are depicted in Figure 1.

The feedback control law used in [13] is of the form

$$\begin{aligned} \hat{x}_{t+1} &= A_c \hat{x}_t + B_c z_t, \quad \hat{x}_0 = 0, \\ u_t &= C_c \hat{x}_t + D_c z_t, \end{aligned} \tag{7}$$

with $\hat{x}_t \in \mathbb{R}^n$.

For a logarithmic quantizer, the quantization error $y_t - Q(y_t)$ can be described by [13]

$$y_t - Q(y_t) = \Delta(y_t) y_t, \quad |\Delta(y_t)| \le \delta \tag{8}$$

It is straightforward to verify that the closed-loop system is given by

$$\bar{x}_{t+1} = \mathcal{A}(\Delta(y_t))\bar{x}_t \tag{9}$$

where $\bar{x} = [x^T x_c^T]^T$, $\Delta(\cdot)$ is given in (8),

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$
$$\hat{I} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{10}$$

and

$$\mathcal{A}(\Delta) = \bar{A} + \bar{B}\bar{K}(\bar{C} + \hat{I}\Delta\hat{C}) \tag{11}$$

The problem of concern is to find the coarsest quantizer for quadratic stabilization of the closed-loop system.

For the case where state feedback is available, i.e., $y = x$ and $u = Q(Kx)$ for some K to be designed, it is well known [11] that the coarsest quantization density allowed for quadratic stabilization is given by

$$\rho_{\inf} = \frac{1-\delta_{\sup}}{1+\delta_{\sup}}, \quad \delta_{\sup} = \frac{1}{\prod_i |\lambda_i^{(u)}(A)|} \tag{12}$$

where $\lambda_i^{(u)}(A)$ denotes the ith unstable eigenvalue of A.

For the output feedback case, a result is given below [13].

Theorem 2.1: Consider the system (1). For a given quantization density $\rho > 0$, the system is quadratically stabilizable via a quantized controller (7) if and only if the following auxiliary system:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ q_t &= (1+\Delta_t)Cx_t, \quad |\Delta_t| \le \delta \end{aligned} \tag{13}$$

is quadratically stabilizable via (7) with z_t replaced with q_t, where δ and ρ are related by (6).

The largest sector bound $\delta_{\sup}$ (which gives $\rho_{\inf}$) is given by

$$\delta_{\sup} = \frac{1}{\inf_{\bar{K}} \|\bar{G}_c(z)\|_\infty} \tag{14}$$

where $\bar{K}$ is defined in (10) and

$$\bar{G}_c(z) = (1 - H(z)G(z))^{-1} H(z)G(z) \tag{15}$$

where

$$G(z) = C(zI - A)^{-1}B \tag{16}$$

and

$$H(z) = D_c + C_c(zI - A_c)^{-1}B_c \tag{17}$$

Further, if $G(z)$ has relative degree equal to 1 and no unstable zeros, then the coarsest quantization density for quantized state feedback can be reached via quantized output feedback.

III. Separation Principle For Quantized Output Feedback Stabilization

In general, output feedback stabilization using quantized output directly requires a denser logarithmic quantizer than the state feedback case; see [13] for an example. This is due to the reason that in the former case, the output signal is quantized before being used for controller design, which tends to create more information loss than the case of state feedback where quantization happens to the control signal.

A question we ask is whether it is possible to design a quantized output feedback controller with the same coarsest quantization density as in the state feedback case. This question has been looked at in [13] under the assumption that the system does not have any noise. It is shown in [13] that the same coarsest quantization density can be achieved if the output signal is first applied to a deadbeat observer. Under the no noise assumption, perfect state reconstruction is achieved by the deadbeat observer in a finite number of steps. The reconstructed state can then be used to design a quantized state feedback controller.

The idea above does not work in general when the system is subject to noises. Also the use of deadbeat observers is not usually preferred in practice because it may have a poor transient behavior. In this section, we consider using a quantized state estimator and establish a simple separation principle for quantized output feedback stabilization.

Quantized state estimators we consider in this paper are of the following form:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + L_t Q(y_t - \hat{y}_t),\ \hat{x}_0 = 0 \\ \hat{y}_t &= C\hat{x}_t \end{aligned} \tag{18}$$

where L_t is the estimator gain which is bounded. That is, instead of quantizing y_t directly, we quantize the "innovation" signal $y_t - \hat{y}_t$.

Note that the quantized state estimator needs to be implemented on both the transmitter and receiver sides: It is needed on the transmitter side for constructing the innovation signal $y_t - \hat{y}_t$; It is also needed on the receiver side to construct the state estimate $\hat{x}_t$. Under the assumption of error-free transmission, the quantized innovation is available on both sides, thus there is no problem for implementing the estimator.

It is straightforward to verify that the estimation error $e_t = x_t - \hat{x}_t$ admits the following dynamics:

$$\hat{e}_{t+1} = Ae_t - L_t Q(Ce_t + v_t) + w_t,\ e_0 = x_0 \tag{19}$$

We have the following simple separation principle:

Theorem 3.1: Consider the system (1) and any quantizer $Q(\cdot)$. Let $u_t = Kx_t$ be any state feedback stabilizer in the sense that the closed-loop system under this feedback has a bounded state covariance. Then, $u_t = K\hat{x}_t$ with $\hat{x}_t$ given by (18) is also a stabilizer if and only if the estimation error dynamics has a bounded error state covariance.

Proof: Rewriting $K\hat{x}_t = Kx_t - Ke_t$ and using the fact that the error dynamics (19) is independent of x_t, it is clear that the closed-loop system of (1), (18) and $u_t = K\hat{x}_t$ has a bounded covariance for the joint state x_t and $\hat{x}_t$ if and only if (19) has a bounded covariance for the error state e_t. □

Using the separation principle above, we have the following result on quantized output feedback stabilization:

Theorem 3.2: Consider the system (1). Let $\rho_{\inf}$ be the coarsest logarithmic quantizer for quanitzed state feedback quadratic stabilizer $u_t = Q(Kx_t)$. Then, the same quantization density can be achieved using estimated state feedback quadratic stabilizer $u_t = K\hat{x}_t$ with the quantized state estimate (18).

Proof: Using Theorem 3.1, it suffices to show that the coarsest quantization density for $Q(\cdot)$ to make the error state in (19) bounded is the same as for quantized state feedback stabilization. Since a logarithmic quantizer is used,

$$Q(s) = (1 + \Delta(s))s$$

with $|\Delta(s)| \leq \delta = (1-\rho)/(1+\rho)$, where $0 < \rho < 1$ is the quantization density. Denoting $s_t = Ce_t + v_t$, (19) can be rewritten as

$$e_{t+1} = Ae_t - L_t(1+\Delta(s_t))Ce_t + \eta_t = Ae_t - L_t Q(Ce_t) + \eta_t$$

where

$$\eta_t = -L_t(1+\Delta(s_t))v_t + w_t$$

which has a bounded covariance (because $\Delta(s_t)$ and L_t are bounded). It follows that the covariance of e_t is bounded if and only if

$$e_{t+1} = Ae_t - L_t Q(Ce_t)$$

is stable for L_t. This is known [11] to be a dual to the quantized state feedback stabilization problem:

$$x_{t+1} = Ax_t + BQ(Kx_t)$$

and the two require the same quantization density for quadratic stabilization. □

Remark: A natural consequence of the separation principle is that the estimation error e_t should be "minimized" in order to reduce its effect on the state feedback.

IV. Variable Rate Logarithmic Quantization

We now consider the system (1) without noises w_t and v_t. It is well known that quantized feedback stabilization of (1) can be achieved using a finite-level quantizer. Indeed, the minimum average number of bits required for the quantizer to achieve stabilization is given by [6]

$$R_{\inf} = \log_2(\prod_i \lambda_i^{(u)}(A)) \tag{20}$$

Note that this is an average bit rate. In other words, a variable rate quantizer needs to be used.

It is also known in [14] that quantized feedback stabilization can be achieved by using a finite-level logarithmic quantizer. The specific scheme given in [14] gives a fixed bit rate required for stabilization, but this rate is in general greater than $R_{\inf}$ in (20).

Also note the striking fact that the term $\prod_i \lambda_i^{(u)}(A)$ determines both the minimum bit rate $R_{\inf}$ and the coarsest logarithmic quantization density $\rho_{\inf}$.

The question we ask in this section is whether the use of a logarithmic quantizer requires a higher bit rate for stabilization if a variable rate quantizer is used. It turns out that the answer is negative. That is, the use of a variable rate logarithmic quantizer does not require any more bit rate than $R_{\inf}$.

In this paper, we restrict ourselves to a first-order unstable system without noises. The result can be generalized to high order systems and systems with noises. We consider the following system:

$$\begin{aligned} x_{t+1} &= ax_t + u_t \\ y_t &= x_t \end{aligned} \tag{21}$$

Without loss of generality, we let $a > 1$.

We first assume that the initial condition x_0 is known to be within the bound $\Omega_0 = \{x : |x| \le 1\}$ and we consider the use of a fixed-rate R-bit logarithmic quantizer and quantized feedback control to stabilize the system in the sense that the state $x_t \in \Omega_0$ for all t. Denote the quantization density by ρ and $\delta = (1-\rho)/(1+\rho)$ and denote $N = 2^{(R-1)}$. We use the following finite-level logarithmic quantizer:

$$Q(x_t) = \begin{cases} \rho^i/(1-\delta), & \text{if } \rho^{i+1} < y \le \rho^i, 0 \le i \le N-2 \\ 0, & \text{if } 0 \le y \le \rho^{N-1} \\ \text{undefined} & \text{if } y > 1 \\ -Q(-y), & \text{if } y < 0, \end{cases} \tag{22}$$

In comparison with (5), we have chosen $\mu_0 = 1/(1-\rho)$. The quantizer above is well defined for inputs in Ω_0 and it has $2N-1$ or $2^R - 1$ levels. The control law is chosen to be

$$u_t = -aQ(y_t) = -aQ(x_t) \tag{23}$$

We let $x_t \in \Omega_0$ and consider two cases, $Q(x_t) = 0$ and $Q(x_t) \neq 0$. In the former case, $|x_t| \le \rho^{N-1}$ and therefore,

$$|x_{t+1}| = a|x_t| \le a\rho^{N-1}$$

which is still in Ω_0 if

$$a\rho^{N-1} \le 1 \tag{24}$$

In the latter case, it is known [13] that

$$Q(x_t) = (1+\Delta_t)x_t, \quad |\Delta_t| \le \delta$$

Therefore,

$$|x_{t+1}| = a|\Delta_t x_t| \le a\delta$$

Let $\delta = a^{-1}$ or $\rho = (a-1)/(a+1)$ as in (12). Then, the above x_{t+1} remains in Ω_0. Returning to (24), a sufficient condition for x_{t+1} to remain in Ω_0 in both cases is

$$N \ge 1 + \frac{\ln(a)}{\ln(a-1)/(a+1)} \tag{25}$$

Now we consider the following variable rate logarithmic quantizer:

$$u_t = \begin{cases} Q(x_t) & \text{if } t \bmod m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where m is a positive integer number and $Q(\cdot)$ is given in (22). That is, the bit rate is zero for all t except when $t \bmod m = 0$. Since the control input is applied periodically, we define $\bar{x}_t = x_{mt}$ and an equivalent system:

$$\bar{x}_t = a^n \bar{x}_t + \bar{u}_t \tag{27}$$

where $\bar{u}_t = Q(\bar{x}_t)$. It is clear that x_t is bounded (as $t \to \infty$) if and only if $\bar{x}_t$ is bounded. As before, we assume $\bar{x}_0 = x_0 \in \Omega_0$. Using (25), $\bar{x}_t$ remains Ω_0 for all t if

$$N \ge 1 + \frac{\ln(a^m)}{\ln(a^m-1)/(a^m+1)} \tag{28}$$

Let $m \to \infty$, $\ln(a^m-1)/(a^m+1) \to 2a^{-m}$, which implies

$$N \ge 1 + \frac{ma^m}{2}\ln(a)$$

The required minimum average bit rate, as $m \to \infty$, is therefore given by

$$R = \lim_{m\to\infty} \frac{1}{m}\log_2(2N-1) = \log_2(a) \tag{29}$$

which is exactly the same as the minimum average bit rate $R_{\inf}$ in (20).

It remains to remove the assumption that $x_0 \in \Omega_0$. To do so, we need to scale the input to the quantizer by

$$\bar{u}_t = \alpha_t^{-1} Q(\alpha_t \bar{x}_t) \tag{30}$$

where $\alpha_t > 0$ is a scaling parameter to be defined but with initial value α_0, say, equal to 1. Recall that the quantizer $Q(\cdot)$ in (22) has $2N-1$ levels. We may use two more levels to indicate whether the scaled input $\alpha_t\bar{x}_t \in \Omega_0$. That is, if $\alpha_t\bar{x}_t \in \Omega_0$, $Q(\alpha_t\bar{x}_t)$ is given as in (22), otherwise,

$$Q(\alpha_t\bar{x}_t) = \frac{\text{sign}(x_t)}{\rho(1-\delta)}, \tag{31}$$

The scaling parameter can then be updated as follows:

$$\alpha_{t+1} = \begin{cases} \alpha_t/\gamma_1 & \text{if } |Q(\alpha_t\bar{x}_t)| > 1 \\ \alpha_t & \text{otherwise} \end{cases} \tag{32}$$

where γ_1 is chosen to be any constant greater than a^m.

What happens is that when (initially) $|\bar{x}_t|$ is too large, α_t will decrease at a rate such that the scaled state $\alpha_t\bar{x}_t$ decreases exponentially. Eventually, the scaled state will be bounded within Ω_0 and the previous quantized feedback law will keep $\alpha_t\bar{x}_t$ in Ω and α_t stops decreasing for all future t. This way, $\bar{x}_t$ (and thus x_t) will remain bounded.

Note that the scaling above increases only by 2 levels for the quantizer, which does not affect the averaged bit rate in (29).

Finally, we point out that if the given average bit rate $R > R_\infty$, we can modify the scaling factor α_t to drive $\bar{x}_t$ (hence x_t) to zero asymptotically. Indeed, if $R > R_\infty$ and the scaled state $\alpha_t\bar{x}_t \in \Omega_0$, it can be easily shown the next scaled state $\alpha_{t+1}\bar{x}_{t+1} = \alpha_t\bar{x}_{t+1}$ is strictly inside Ω_0. This means that we can modify (33) to the following:

$$\alpha_{t+1} = \begin{cases} \alpha_t/\gamma_1 & \text{if } |Q(\alpha_t\bar{x}_t)| > 1 \\ \alpha_t\gamma_2 & \text{otherwise} \end{cases} \tag{33}$$

where $\gamma_2 > 1$ is such that, if $\alpha_t \bar{x}_t \in \Omega_0$, $\alpha_{t+1}\bar{x}_{t+1} = \alpha_t \gamma_2 \bar{x}_{t+1} \in \Omega_0$. It follows that, as $t \to \infty$, $\alpha_t \to \infty$ and $\alpha_t \bar{x}_t \in \Omega_0$. Hence, $\bar{x}_t \to 0$ as $t \to \infty$.

The analysis above is summarized below:

Theorem 4.1: Given the first order system (21), asymptotic stabilization can be achieved using a variable rate finite-level logarithmic quantizer if and only if the average bit rate exceeds $R_{\inf} = \log_2(a)$.

Proof: The sufficiency is given in the analysis above. The necessity follows from [6]. □

We note that the scaling method in (33) has been used in [14] for fixed rate finite-level quantizers. The tradeoff between a fixed rate quantizer and variable rate quantizer is that the former gives better transient performance whereas the latter requires a smaller averaged bit rate.

V. Conclusion

In this paper, we have studied a number of quantized feedback control problems. Our main contributions include the following. We have given a simple separate principle for quantized feedback stabilization. We have shown that the minimum quantization density for output feedback stabilization using a (static) logarithmic quantizer is the same as the minimum quantization density for bounded state estimation. We also consider the use of variable-rate finite-level logarithmic quantizers and show, in the first order case, that this type of quantizers can reach the minimum averaged bit rate for quantized feedback stabilization.

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