A ROBUST INTERPOLATION ALGORITHM FOR SPECTRAL ESTIMATION

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ABSTRACT

We propose a *robust* interpolation algorithm for model-based spectral estimation. The interpolation data represent information about the half spectrum function associated with a given signal and are computed from an input-to-state filter. Our algorithm allows a large number of noisy interpolation data to be used to optimally fit a half spectrum function of a fixed order. The algorithm involves solving a set of linear matrix inequalities and is thus numerically efficient.

1. INTRODUCTION

This paper is concerned with the following standard spectral estimation problem: Given a discrete-time signal $u(t), t = 0, 1, 2, \ldots$, find an n-th order auto-regressive moving average (ARMA) model $g(z)$ such that the spectrum $s(e^{i\omega})$ of $u(t)$ is *best* approximated by $g(e^{i\omega})g(e^{-i\omega})$ in some appropriate measure. To make this problem more tractable and more meaningful technically, the given signal $u(t)$ is often assumed to be generated by an unknown ARMA model $g(z)$ but possibly corrupted by noises or a relatively small number of samples of $u(t)$ is available, and the aim is to find an algorithm that can best approximate the parameters of $g(z)$.

Since the ARMA model $g(z)$ has only $2n+1$ free parameters, it is natural that the samples of $u(t)$ are used to compute a small set of statistical parameters which are then used to estimate the parameters of $g(z)$. For example, if $g(z)$ is known to be an n-th order AR model, it is common to estimate the first $n+1$ autocorrelation coefficients of $u(t)$ and use them to estimate $g(z)$. If $G(z)$ is an ARMA model, spectral estimation becomes more difficult [1].

In this paper, we consider an interpolation approach to spectral estimation initially proposed in [2]. The basic idea is as follows. Instead of estimating $s(z)$ or $g(z)$ directly, we estimate the so-called half spectrum $f(z)$ defined via spectral factorization:

$$s(z) = f(z) + f(z^{-1}). \tag{1}$$

By definition, $f(z)$ is a positive real, n-th order rational function. The interpolation approach starts with estimating the values of $f(z)$ and possibly its derivatives at a set of user-specified locations $z = \xi_i, i = 0, 1, \ldots, m$. This can be done using the so-called input-to-state filters [3, 4]. These values are then used to estimate the parameters of $f(z)$ by solving an interpolation problem.

The aim of this paper is to propose a robust interpolation algorithm for spectral estimation. This is motivated by two facts. Firstly, the standard interpolation approaches [2, 5] require precise interpolation data. Because these data come from statistical estimation and thus are possibly corrupted by noises, standard interpolation approaches may fail. Secondly, standard interpolation approaches are unable to handle the case where the number of interpolation points exceeds the number of free parameters in the interpolation function. In the proposed robust interpolation algorithm, we allow a large set of noisy interpolation data to estimate a reliable half spectrum. This is done by solving some semidefinite programming problems. In addition, we can allow interpolation conditions involving first or higher order derivatives of $f(z)$. This is an important feature because interpolation conditions involving higher order derivatives of $f(z)$ is useful for estimating the model order, although this topic is not studied in this paper. Some features of our algorithm are similar to the covariance extension approaches proposed in [6, 7], but we will highlight some interesting differences.

Due to the page limit, the technical proofs of the results presented in the paper are not included.

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2. INPUT-TO-STATE FILTERING

Motivated by [3, 4], we use an input-to-state to estimate the values of the half spectrum $f(z)$ and possibly its derivatives at a set of user-specified points $\xi_i, i = 0, 1, 2, \ldots, m$, outside of the unit disc. The filter has the following form:

$$x(t+1) = \mathbf{A}x(t) + \mathbf{b}u(t), \tag{2}$$

where $x(t) \in \mathbb{R}^\nu$ is the state, $\mathbf{A} \in \mathbb{R}^{\nu\times\nu}$ and $\mathbf{b} \in \mathbb{R}^\nu$ are such that $(\mathbf{A}, \mathbf{b})$ is a controllable pair and $\mathbf{A}$ has an eigenvalue of multiplicity $\nu_i \geq 1$ at ξ_i^{-1}, $i = 1, 2, \ldots, m$, and $\nu = \sum_{i=1}^m \nu_i$. The next two lemmas explain how to compute the values of $f(z)$ and its derivatives from $x(t)$.

Lemma 1 *Let R be the covariance matrix of $x(t)$ and S be the unique positive definite solution of the Lyapunov equation*

$$S = \mathbf{A}S\mathbf{A}^\top + \mathbf{b}\mathbf{b}^\top. \tag{3}$$

Then,

$$R = WS + SW^\top, \quad (4)$$

where

$$W = f(\infty)I + \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} f(z^{-1})(zI - \mathbf{A})^{-1}\mathbf{A}\frac{\mathrm{d}z}{z}. \quad (5)$$

■

By definition (5), W admits a unique representation

$$W = w_0 I + w_1 \mathbf{A} + \ldots + w_{n-1}\mathbf{A}^{n-1}, \quad (6)$$

where the coefficients $\{w_k\}_{k=0}^{n=1}$ can be shown to be independent of the co-ordinates of $\mathbf{A}$. We define

$$w(z) = w_0 + w_1 z + \ldots + w_{n-1} z^{n-1}. \quad (7)$$

Lemma 2 *Let λ be an eigenvalue of $\mathbf{A}$ in (2) with multiplicity ρ and $w(z)$ be defined as in (6). Then,*

$$\left.\frac{\mathrm{d}^r w(z)}{\mathrm{d}z^r}\right|_{z=1/\lambda} = \left.\frac{\mathrm{d}^r}{\mathrm{d}z^r} f\left\{\frac{1}{z}\right\}\right|_{z=1/\lambda}, \quad r = 0, 1, \ldots, \rho - 1.$$

■

From the above, it is clear that the value of $f(z)$ and its derivatives at ξ_i can be computed by estimating the covariance matrix R of $v(t)$ from (2), solving S and W in (3)-(5), forming $w(z)$ and evaluating it and its derivatives at ξ_i^{-1}.

3. INTERPOLATION CONDITIONS

The interpolation problem we have is a modified Nevanlinna-Pick interpolation problem as stated below: *Let $\xi_0, \xi_1, \ldots, \xi_m$ be a given set of distinct, real or self-conjugate complex numbers such that $\xi_0 = \infty$ and $|\xi_i| > 1$ for $i = 1, 2, \ldots, m$, and for every such i, the complex numbers $f_i^{(0)}, f_i^{(1)}, \ldots, f_i^{(\nu_k - 1)}$ are also given with $\nu_0 = 1$ and other $\nu_i \geq 1$. Then find a rational function $f(z)$ of degree n, if possible, such that*

$$\left.\frac{1}{r!}\frac{d^r}{dz^r}\{f(z)\}\right|_{z=\xi_i} = f_i^{(r)}, \quad \begin{array}{l} r = 0, 1, \ldots, \nu_k - 1 \\ i = 0, 1, \ldots, m, \end{array} \quad (8)$$

and that $f(z)$ is strictly positive real (SPR), i.e., $f(z)$ is analytic in $|z| \leq 1$ and that

$$f(z) + f(z^{-1}) > 0, \ \ \forall |z| = 1 \quad (9)$$

In the above, the choice of $\xi_0 = \infty$ is for convenience. In the standard Nevanlinna-Pick interpolation problem, $n = \sum_{i=0}^m \nu_i$. In general, this does not uniquely determine the interpolation function. In order to have a reliable spectral estimate, we require $\sum_{i=0}^m \nu_i > 2n + 1$. The technical difficulty with this setting is that on one hand we tend to have an over-determined problem because $f(z)$ has only $2n + 1$ free parameters, and on the other hand we may not necessarily have a solution that satisfies all the constraints. These two problems are studied next.

We start with a parameterization of $f(z)$ which satisfies n interpolation conditions in (8), but not necessarily other interpolation conditions and the SPR condition. The n interpolation points used above are chosen such that they are meant to be most reliable based on some prior knowledge. For simplicity, we assume in the sequel that $n = \sum_{i=1}^{m_1} \nu_i$ for some $0 < m_1 < m$.

In the following we assume that the set $\{\xi_i\}_{i=1}^{m_1}$ is self-conjugate. This allows us to work with real-valued matrices. Define $X_i \in \mathbb{C}^{\nu_i \times \nu_i}$ to be the Jordan block with the eigenvalue ξ_i and

$$\begin{aligned} c_i &= \left[\begin{array}{cccc} f_i^{(\nu_i-1)} & f_i^{(\nu_i-2)} & \ldots & f_i^{(0)} \end{array}\right]^\top \in \mathbb{C}^{\nu_i}, \\ w_i &= \left[\begin{array}{cccc} 0 & \ldots & 0 & 1 \end{array}\right]^\top \in \mathbb{R}^{\nu_i}. \end{aligned}$$

Form $X_\star = \mathrm{diag}\{X_1, \ldots, X_{m_1}\}, w = \left[\begin{array}{ccc} w_1^\top & \ldots & w_{m_1}^\top \end{array}\right]^\top$ and $\bar{c} = \left[\begin{array}{ccc} c_1^\top & \ldots & c_{m_1}^\top \end{array}\right]^\top$. Also, define $d = f_0^{(0)}$. Then we have the following generalization of a result in [8].

Lemma 3 *All n-th order rational function $f(z)$ satisfying the interpolation conditions in (8) for $i = 0, \ldots, m_1$ are parameterized by*

$$f(z) = \frac{d + b[zI_n - X]^{-1}c}{1 + b[zI_n - X]^{-1}e}, \quad b \in \mathbb{R}^{1\times n}, \quad (10)$$

where X, c and e are real-valued matrix and vectors defined using $C = \left[\begin{array}{cccc} w & X_\star w & \ldots & X_\star^{n-1}w \end{array}\right]$ as follows:

$$\begin{aligned} &X = C^{-1}X_\star C, \ \ c = C^{-1}\bar{c} \\ &e = C^{-1}w = \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \end{array}\right]^\top. \end{aligned}$$

The SPR constraint for $f(z)$ can be handled by applying the well-known KYP Lemma [9], as given below.

Lemma 4 Define

$$\Theta = \left[\begin{array}{cc} ce^\top + ec^\top & c + ed \\ de^\top + c^\top & 2d \end{array}\right]. \quad (11)$$

Then, $f(z)$ in (10) is SPR if and only if

$$\Theta + \left[\begin{array}{cc} Q - XQX^\top & -XQb^\top \\ -bQX^\top & -bQb^\top \end{array}\right] > 0, \quad (12)$$

holds for some $Q = Q^\top \in \mathbb{R}^{n\times n}$.

■

The matrix inequality (12) is nonlinear in Q and b, and thus difficult to use. To get around this problem, we reparameterize b by

$$\beta = bQ. \quad (13)$$

Applying Schur complement, (12) is equivalent to

$$\left[\begin{array}{ccc} ce^\top + ec^\top + Q - XQX^\top & c + ed - X\beta^\top & 0 \\ c^\top + de^\top - \beta X^\top & 2d & \beta \\ 0 & \beta^\top & Q \end{array}\right] > 0 \quad (14)$$

which is now linear in Q and β.

In order to ensure the uniqueness of the computed model at least n additional interpolation points are generally required for computing b. Let us define

$$\Gamma(z) = d + b(zI - X)^{-1}c,\ \Delta(z) = 1 + b(zI - X)^{-1}e \quad (15)$$

and $\Omega(z) = [zI - X]^{-1}$. From the definition of $\Gamma(z)$ and $\Delta(z)$ in (15) we have for $r > 0$ that

$$\frac{1}{r!}\frac{\mathrm{d}^r\Gamma(z)}{\mathrm{d}z^r} = -b\Omega^{r+1}(z)c, \quad \frac{1}{r!}\frac{\mathrm{d}^r\Delta(z)}{\mathrm{d}z^r} = -b\Omega^{r+1}(z)e,$$

Making use of the relationship

$$\frac{1}{r!}\frac{\mathrm{d}^r\Gamma(z)}{\mathrm{d}z^r} = \sum_{\ell=0}^{r}\frac{1}{\ell!}\frac{\mathrm{d}^\ell\Delta(z)}{\mathrm{d}z^\ell}\frac{1}{(r-\ell)!}\frac{\mathrm{d}^{r-\ell}f(z)}{\mathrm{d}z^{r-\ell}},$$

we have

$$\begin{aligned} h_r(z) &:= \left[\begin{array}{cccc} f(z)-d & \frac{\mathrm{d}f(z)}{\mathrm{d}z} & \ldots & \frac{1}{(r-1)!}\frac{\mathrm{d}^{r-1}f(z)}{\mathrm{d}z^{r-1}} \end{array}\right] \\ &= b(C_1^{(r)}(z) - C_2^{(r)}(z)G_r(z)), \end{aligned} \quad (16)$$

where

$$\begin{aligned} C_1^{(r)}(z) &= \left[\begin{array}{cccc} \Omega(z)c & \Omega^2(z)c & \ldots & \Omega^r(z)c \end{array}\right] \\ C_2^{(r)}(z) &= \left[\begin{array}{cccc} \Omega(z)e & \Omega^2(z)e & \ldots & \Omega^r(z)e \end{array}\right] \\ G_r(z) &= \left[\begin{array}{cccc} f(z) & \frac{\mathrm{d}f(z)}{\mathrm{d}z} & \cdots & \frac{1}{(r-1)!}\frac{\mathrm{d}^r f(z)}{\mathrm{d}z^r} \\ 0 & f(z) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \frac{\mathrm{d}f(z)}{\mathrm{d}z} \\ 0 & 0 & \ldots & f(z) \end{array}\right]. \end{aligned}$$

If we are given the values of $f(z)$ and its derivatives at ξ_k, $k = m_1+1, m_1+2, \ldots, m$, then b satisfies

$$\boldsymbol{h} = bM, \quad (17)$$

where

$$\begin{aligned} \boldsymbol{h} &= \left[\begin{array}{ccc} h_{\nu_{m_1+1}}(\xi_{m_1+1}) & \ldots & h_{\nu_m}(\xi_m) \end{array}\right] \\ M &= [M_{m_1+1}\ M_{m_1+2}\ \ldots\ M_m] \\ M_i &= C_1^{(\nu_i)}(\xi_i) - C_2^{(\nu_i)}(\xi_i)G_{\nu_i}(\xi_i),\ m_1+1 \le i \le m. \end{aligned}$$

4. ROBUST INTERPOLATION ALGORITHM

Since the given interpolation data can be noisy, we may face two potential problems: 1) There may not be any n-th order $f(z)$ SPR function to satisfy the first n interpolation conditions; 2) The remaining interpolation data may not give a unique solution to b, or even if the solution set for b is non-empty, the resulting $f(z)$ may not be SPR. These problems are addressed by our proposed robust estimation algorithm.

To address the first problem is the same as asking under what conditions the set of β satisfying (14) is non-empty. This is answered by the following result:

Lemma 5 *There exists $b \in \mathbb{R}^n$ such that $f(z)$ in (10) is an n-th order SPR function if and only if the* generalized Pick matrix

$$P := \left[\begin{array}{cc} XEX^\top & c+ed \\ c^\top + de^\top & 2d \end{array}\right] > 0, \quad (18)$$

where $E = E^\top > 0$ is the unique solution to the Lypunov equation

$$XEX^\top - E = ce^\top + ec^\top. \quad (19)$$

Moreover, (18)-(19) are equivalent to solving

$$ce^\top + ec^\top > XQX^\top - Q, \quad (20)$$

$$\left[\begin{array}{cc} ce^\top + ec^\top + Q & c+ed \\ c^\top + de^\top & 2d \end{array}\right] > 0 \quad (21)$$

for some $Q = Q^T > 0$.

Let $\hat{c}$ and $\hat{d}$ be the estimates of c and d, respectively. Denote $y = [c^\top\ \ d]^\top$ and $\hat{y} = [\hat{c}^\top\ \ \hat{d}]^\top$. The first step in robust estimation is to check if (18)-(19) are satisfied or not when c and d are replaced by their estimates. If so, this step is done. Otherwise, we need to modify $\hat{c}$ and $\hat{d}$. One simple modification is to use the following:

$$[\check{c},\ \check{d}] = \arg\min_{c,d}\ (y-\hat{y})^\top W^{-1}(y-\hat{y})$$

subject to the linear inequality constraints (20) and (21). The matrix $W = W^\top > 0$ is chosen as the covariance matrix of $\hat{y}$. This is equivalent to the semidefinte programming problem:

$$[\check{c},\ \check{d}] = \arg\min_{c,d}\ \ell$$

subject to

$$\left[\begin{array}{cc} \ell & (y-\hat{y})^\top \\ (y-\hat{y}) & W \end{array}\right] > 0,$$

$$ce^\top + ec^\top > XQX^\top - Q,$$

$$\left[\begin{array}{cc} ce^\top + ec^\top + Q & c+ed \\ c^\top + de^\top & 2d \end{array}\right] > 0.$$

The second step in robust estimate is to solve the parameter vector b, assuming that the estimates $\check{c}$ and $\check{d}$ for c and d are given. Denote by $\hat{M}$ and $\hat{\boldsymbol{h}}$ the estimate of M and $\boldsymbol{h}$, respectively, computed using the given interpolation data, $\check{c}$ and $\check{d}$. It is assumed that $\hat{M}$ has full column rank. We can estimate β and Q by solving the following optimization problem:

$$\min_{\beta,Q}(\beta - \hat{\boldsymbol{h}}\hat{M}^\dagger Q)\Omega^{-1}(\beta - \hat{\boldsymbol{h}}\hat{M}^\dagger Q)$$

subject to (14) with c and d replaced with their estimates, where $\hat{M}^\dagger$ is the pseudo-inverse of $\hat{M}$ and $\Omega = \Omega^\top > 0$ is a weighting matrix chosen by the user. This is equivalent to the following semidefinite programming problem:

$$[\hat{\beta},\ \hat{Q}] = \arg\min_{\beta,Q}\ \gamma$$

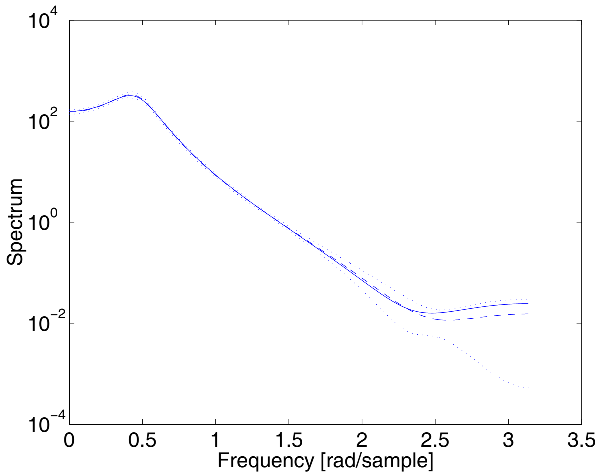

Fig. 1. Estimated Spectrum

subject to

$$\begin{bmatrix} \gamma & (\beta - \hat{\boldsymbol{h}}\hat{M}^{\dagger}Q) \\ (\beta - \hat{\boldsymbol{h}}\hat{M}^{\dagger}Q)^{\top} & \Omega \end{bmatrix} > 0,$$

$$\begin{bmatrix} \check{c}e^{\top} + e\check{c}^{\top} + Q - XQX^{\top} & \check{c} + e\check{d} - X\beta^{\top} & 0 \\ \check{c}^{\top} + \check{d}e^{\top} - \beta X^{\top} & 2\check{d} & \beta \\ 0 & \beta^{\top} & Q \end{bmatrix} > 0.$$

Using (13), we then obtain the estimate of b as

$$\hat{b} = \hat{\beta}\hat{Q}^{-1}.$$

It follows that the estimate of $f(z)$ is given by

$$\hat{f}(z) = \frac{\check{d} + \hat{b}(zI - X)^{-1}\check{c}}{1 + \hat{b}(zI - X)^{-1}e}.$$

5. ILLUSTRATIVE EXAMPLE

To show how the proposed algorithm works, we test it on the following ARMA process:

$$g(z) = \frac{z^2 + z + 0.5}{z^2 - 1.5z + 0.7} \tag{22}$$

The signal $u(t)$ is generated by filtering a Gaussian white noise through $g(z)$, and 2000 points of $u(t)$ are used. 32 interpolation points are chosen by taking $\xi_k = \rho e^{j\theta}$ with $\rho \in \{2,\ 3.5\}$ and $\theta \in \{\pi p/8 : p = 0, 1, \ldots, 15\}$. Simulation results based on 100 runs are shown in Figure 1 which compares the true spectrum (solid) with the mean (dashed) $\pm$ the standard deviation (dotted) of the estimated spectrum.

6. CONCLUSIONS

In this paper we have proposed a robust interpolation algorithm for spectral estimation. The problem considered here can be seen as a generalization of the classical Nevanlinna-Pick interpolation problem. Our algorithm has two main features. Firstly, we allow noise-corrupted interpolation data. Secondly, we can incorporate a large number of interpolation data points. The algorithm employs semidefinite programming and is thus computationally efficient.

The robust spectral estimation approach is generalized easily for multivariable processes. It is also possible to estimate the model order. This requires the availability of sufficient number of derivatives of $f(z)$ at a given interpolation point. The free parameter b determines the poles of the ARMA representation. Note that the stability of the estimated model is guaranteed automatically, since an SPR model is always stable. Unlike the covariance extension algorithm in [7], we do not need any additional regularization step to ensure a stable solution. In addition, a solution is always guaranteed. A similar robust algorithm for covariance extension can also be derived.

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