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Cooperative Localization of a Cascading Quadrilateral Network

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Abstract— **In this paper, we introduce a set of sensor networks, called the cascading quadrilateral network, and study how to compute the positions of its nodes in a cooperative way. We investigate the condition for determining whether all the sensor nodes are localizable. If not, we provide a method to detect the un-localizable nodes for the whole network. The necessary and sufficient conditions for the *network localizability* and *node localizability* are given from the view of algebraic property, respectively. Specifically, we provide algorithms to show how to detect un-localizable nodes from a partially localizable network. Numerical simulation is provided to show the effectiveness of the developed method of computing positions.**

I. INTRODUCTION

Accurately computing the positions of sensor nodes is a key issue for providing localization based service (LBS) using sensor networks in both civil and military applications. The common way of computing locations is inherited from the GPS localization algorithm, which aims to solve an optimization problem with a non-convex cost function. During the past decade, many localization algorithms to solve this optimization problem in a distributed way were proposed, such as the Semi-Definite Programming (SDP) based method [1], sub-gradient algorithm [2], Gaussian iteration [3] and so on. These works focus on finding a way to compute the positions by using distance, angle or any other measurements that can be easily obtained at sensor nodes. The success of these methods mainly depends on finding the solution of an optimization problem. But the cost functions defined by the distance measurements and the estimated positions are not convex, so these optimization based methods cannot guarantee convergence to true positions of sensor nodes.

From the perspective of systems and control, the localization problem can be solved by two categories of methods. The first one is based on local embedding and eigen decomposition. Ravazzi *et al.* [4] provide a randomized algorithm to iteratively estimate the relative positions of sensor nodes by reconstructing the position vector from local measurements of the differences between pairwise neighbors. The convergence of such an algorithm is proved by the randomized PageRank dynamics, as shown in [5].

The second category of method is to solve the localization problem based on consensus algorithms and formation control formulations. From the view of consensus algorithms, the localization problem is strongly related to the distance based formation control [6], which has been well studied in [7], [8], [9], [10], [11], or formation control based on the measurement of relative position in [12], [13], [14], [15]. The success of this method depends on the synchronization of the local coordinate framework held by each single node. In contrast, the demanded measurement in this paper is only the distance measurement, which could be obtained at each node in a fully distributed way.

Before bringing such measurements of distances or angles into the positioning step, we need a 'filtering' process to filter out certain measurements and even some nodes, which would cause ambiguities into the computation of positions. Such a 'filtering' process is called *localizability test*, during which the *un-localizable* nodes are filtered out, whereas the left ones are detected to be *localizable*. Such a localizability test is fundamental for the LBS since the service provided by the un-localizable nodes will be meaningless in practice [16].

From the graphical view, the localizability test is equivalent to determine the uniqueness of a graph-embeddability problem [17], i.e., whether there exists a unique embedding for the graph given a set of distance measurements. For a randomly deployed network, this graph-embeddability problem has been proved to be NP-hard. Thus, people turn to study which kind of graphs have the unique and "generic" embedding [18]. In [19], Moore *et al.* introduce a concept of *robust quadrilateral* and the object network is divided into clusters of quadrilaterals. In this paper, we also studies the network formed by clusters of quadrilaterals, but we give up the stitching strategy, i.e., the network topology is with less constraints than that of [19], which leads to a more generalized scope of networks.

The criterion of the localizability test is called the localizability condition. Most existing conditions are based on the graph rigidity theory, which models the network to be an abstracted distance graph and analyzes the graphical properties of the network topology [20].

The concept of *network localizability* has been proposed to answer whether or not a network is localizable given a set of pairwise distance constraints. In contrast, the concept of *node localizability* focuses on the location-uniqueness of every single node [16]. In practice, a randomly deployed network is usually *partially localizable*, which means the positions of only a part of the nodes can be uniquely determined given

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certain constraints of distance measurements.

Another challenging issue for the localization problem is on how to compute the positions using only the local measurements and communication. In [14], [21], [4], the measurements of relative positions require different nodes to share a unified reference frame, i.e., the coordinate frame at each node should agree with that of others. By the term of local measurements, we refer to the measurements of distances and bearing angles between pairwise neighboring nodes, which can be obtained by each single node. In this paper, we consider only the distance measurements among pairwise neighboring nodes in cascading quadrilateral networks. This mechanism is defined to be a *fully distributed* or *cooperative* way.

In this paper, for a set of networks called the cascading quadrilateral network, we provide a necessary and sufficient condition for its localizability. Then, we show how to detect the un-localizable nodes through analyzing the structure of the eigenvectors of a specific weight matrix. Moreover, the eigenvectors are estimated by each node in a cooperative way. While detecting un-localizable nodes, a necessary and sufficient condition for *node localizability* is also obtained. For the best of our knowledge, this is the first time that a necessary and sufficient condition is proposed for node localizability. Based on the proposed condition in this paper, every un-localizable node can be detected by using only local measurements and estimations.

Notations: $\mathbb{R}$ denotes the set of real numbers. $\iota = \sqrt{-1}$ denotes the imaginary unit. $\mathbf{1}_n$ represents the n-dimensional vector of ones and I_n denotes the identity matrix of order n. $\overline{EIG}(\bullet)$ and $\underline{EIG}(\bullet)$ denote the eigenvalues of a certain matrix with the largest and smallest module, respectively. $\mathbb{E}^d$ denotes the d-dimensional Euclidean space. $[A]_{i,j}$ denotes the j-th entry of the i-th row in matrix A. $\emptyset$ denotes the empty set. We also use $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

II. PRELIMINARIES

A. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph of a network with N sensor nodes, where $\mathcal{V} = \{1, \cdots, N\}$ and $\mathcal{E}$ denote a node set and an edge set, respectively. For $\forall i, j \in \{1, \cdots, N\}$, the edge $\{i, j\}$ satisfies $\{i, j\} \in \mathcal{E}$ if and only if i and j can measure the distance between each other. In this way, the graph $\mathcal{G}$ here is usually called to be a *distance graph*.

Let $\mathcal{H}_i$ be a neighboring set of i, $i \in \mathcal{V}$. For $\forall j \in \mathcal{V}$ and $j \neq i$, it satisfies $j \in \mathcal{H}_i$ if and only if $\{i, j\} \in \mathcal{E}$. The distance graph is undirected, i.e., for all $i, j \in \mathcal{V}$, if the edge $\{i, j\} \in \mathcal{E}$, then $\{j, i\} \in \mathcal{E}$.

A framework is a pair $(\mathcal{G}, \mathbf{p})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes a distance graph and $\mathbf{p} = [p_1, \cdots, p_N]^T$ denotes Euclidean coordinates of the vertices in $\mathcal{V}$, where $N = |\mathcal{V}|$ and $p_i \in \mathbb{R}^2$ for all $i \in \mathcal{V}$. The framework $(\mathcal{G}, \mathbf{p})$ is said to be *generic* if $\mathbf{p}$ does not satisfy any nontrivial algebraic equation with rational coefficients [22]. In this paper we make an implicit assumption that frameworks are generic.

Definition 1 ([20]): A framework is said to be *globally rigid* if any equivalent framework is also congruent. It is said to be *generically globally rigid* if almost all of its representations are globally rigid.

B. Network and Node Localizability

Consider a network of N sensor nodes in $\mathbb{R}^2$, with Euclidean coordinates $\mathbf{p} = [p_1, \cdots, p_N]$. Define the nodes, whose ground truth coordinates are clear to themselves, to be *anchor nodes*, and the remaining ones to be *normal sensor nodes*.

Considering a network associated with a distance graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and given distance measurements $\|p_i - p_j\|$, for all i and j satisfying $\{i, j\} \in \mathcal{E}$, the question arises as to whether or not the whole network is localizable. This is called *network localizability* [16].

The following lemma from [23] states a relationship between the localizability of a network and the rigidity of its underlying graph.

Lemma 1 ([23]): The positions of all the nodes in a 2-dimensional generic framework $(\mathcal{G}, \mathbf{p})$ are uniquely determined if and only if the following three conditions are satisfied simultaneously.

1) Positions of at least three (anchor) nodes are known.
2) The distance measurements $\|p_a - p_b\|$, for all $\{a, b\} \in \mathcal{E}$, are known.
3) The distance graph $\mathcal{G}$ is globally rigid.

If the network is partially localizable, it becomes challenging to determine whether a single node is localizable. This is because it does not exist a necessary and sufficient condition for the localizability of each single node. This problem is called the *node localizability*. As pointed out in [16], network localizability is actually a special case of node localizability.

III. PROBLEM STATEMENT

In this paper, we start by representing the position of each node into a linear combination of its neighbors. Then, a cooperative and recursive method for computing the positions of sensor nodes is proposed, which uses only local information. Its convergence is guaranteed by introducing certain scaling coefficient. Finally, we propose a necessary and sufficient localizability condition for each node.

To represent the position of each node, we use the concept of barycentric coordinates. Their key characteristics are that 1) they represent each node's position as a linear combination of its neighbors, and 2) they can be computed using only distance measurements between pairwise neighboring nodes. On how to compute barycentric coordinates by using distance measurements, readers may refer to [24].

In this paper, we consider a set of networks called *cascading quadrilateral network*. Given four vertices in $\mathbb{E}^2$, connecting each pair of these vertices by an edge forms a fully connected quadrilateral. For a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, it is a cascading quadrilateral network as long as, for all $i \in \mathcal{V}$, there exists at least one quadrilateral containing i. One example of the cascading quadrilateral network is shown in Fig. III.

Remark 1: The requirement of being a cascading quadrilateral network does not imply that the its distance graph

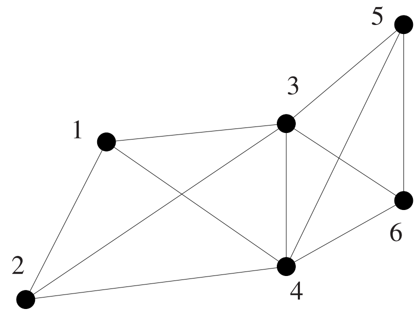


Fig. 1. An example of cascading quadrilateral network.

is complete, in which case the localization problem would become trivial. The example of the cascading quadrilateral network shown in Fig. III is obviously not a complete graph.

According to the definition of barycentric coordinate, the position of Node i in a cascading quadrilateral network can be written into a linear combination of its neighbors' positions and the combination coefficients can be uniquely determined by using the barycentric coordinate of i with respect to its neighbors, say j, k and l. Specifically, for all $i \notin \mathcal{V}_a$, where $\mathcal{V}_a$ denotes the set of anchor nodes, we have

$$\begin{array}{rcl} p_i & = & a_{ij}p_j + a_{ik}p_k + a_{il}p_l \\ 1 & = & a_{ij} + a_{ik} + a_{il} \end{array} \tag{1}$$

Here, p_i denotes the Euclidean coordinates of Node i, and $\{a_{ij}, a_{ik}, a_{il}\}$ are barycentric coordinates of i with respect to its three neighbors $j, k, l \in \mathcal{H}_i$.

When $|\mathcal{H}_i| = m > 3$, for $i \in \mathcal{V}$, there will be $\binom{m}{3}$ possible combinations of writing p_i as a weighted sum of its three neighbors. Let $\mathcal{C}_\tau$, $\tau = 1, \cdots, \binom{m}{3}$, be one of such combinations. Denote $\mathcal{S}_j = \{\mathcal{C}_\tau | j \in \mathcal{C}_\tau, \tau = 1, \cdots, \binom{m}{3}\}$, for each $j \in \mathcal{H}_i$. Then, the number of the appearance of p_j in all $\binom{m}{3}$ combinations should be $\frac{|\mathcal{S}_j|}{3}$. Denote $a_{ij}^{(v)}$, $v \in \{1, \cdots, \frac{|\mathcal{S}_j|}{3}\}$, as one possible barycentric coordinate of Node i with respect to Node j in the set $\mathcal{C}_\tau$. Then,

$$p_i = \sum_{j \in \mathcal{H}_i} \tilde{a}_{ij} p_j, \tag{2}$$

where $\tilde{a}_{ij} = \frac{1}{\binom{m}{3}} \sum_{v=1}^{\frac{|\mathcal{S}_j|}{3}} a_{ij}^{(v)}$, and p_i and p_j are Euclidean coordinates of nodes i and j, respectively.

Given a network with N nodes, each node's position can be written as (2). Writing the positions of all nodes in matrix form, we have

$$\mathbf{p} = (A \otimes I_2)\mathbf{p}, \tag{3}$$

where the vector $\mathbf{p} \in \mathbb{R}^{2N}$ contains the coordinates of the sensor nodes and $A \in \mathbb{R}^{(N \times N)}$ with $[A]_{ij} = \tilde{a}_{ij}$. Specifically, the $(2n-1)$-th and $2n$-th entries of $\mathbf{p}$ are x-axis and y-axis coordinates of Node n, respectively. Without loss of generality, we further assume that

$$A = \begin{bmatrix} I_3 & 0 \\ B & C \end{bmatrix}, \tag{4}$$

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_s \end{bmatrix}, \tag{5}$$

where $\mathbf{p}_a$ denotes the positions of anchor nodes and $\mathbf{p}_s$ denotes that of the remaining nodes. Substituting (4) and (5) into (3) leads to

$$\mathbf{p}_s = (C \otimes I_2)\mathbf{p}_s + (B \otimes I_2)\mathbf{p}_a. \tag{6}$$

Let $M = I - C$, then we have.

$$(M \otimes I_2)\mathbf{p}_s = (B \otimes I_2)\mathbf{p}_a. \tag{7}$$

In the following of this paper, we study the localizability problem by analyzing the existence of solution for $\mathbf{p}_s$ in (7), and propose a necessary and sufficient condition for localizability of a *network*. Moreover, when the network is partially localizable, we show how to use a necessary and sufficient condition for localizability of one *node* to exhaustively detect every localizable node. Regarding how to compute positions in a cooperative way, we design a linear system to recursively compute the $\mathbf{p}_s$ in (7). The convergence of this system is guaranteed by a proper diagonal stabilizer.

IV. A Necessary and Sufficient Condition for the Localizability of a Cascading Quadrilateral Network

Reorganizing (3) leads to

$$(L \otimes I_2)\mathbf{p} = 0, \tag{8}$$

where $L = I - A$. Moreover,

$$L = \begin{bmatrix} 0 & 0 \\ -B & M \end{bmatrix}.$$

We can then state our localizability result.

Theorem 1: A cascading quadrilateral network $\mathcal{N}$ is localizable if and only if rank$(M) = N - 3$, where M is defined in (7).

Proof:

Sufficiency: If M is of full rank, the solution of $\mathbf{p}_s$ is

$$\mathbf{p}_s = (M \otimes I_2)^{-1}(B \otimes I_2)\mathbf{p}_a. \tag{9}$$

Then the network is uniquely localizable as $\mathbf{p}_s$ in (7) has a unique solution,.

Necessity:

Let ξ denote the x-axis value of $\mathcal{N}$, i.e., $\xi = [p_{2n-1}]$ for all $n = 1, \cdots, N$.

According to (1) and (8), we know

$$\begin{cases} L\mathbf{1} = 0, \\ L\xi = 0. \end{cases} \tag{10}$$

Since the positions of any two nodes are not overlapped, then, $\xi_i \neq \xi_j$ for all $i \neq j$. Thus, $\xi \neq \mathbf{1}$.

Next, we complete the proof by contradiction. Suppose by contradiction that det$(M) = 0$. Thus, there exists a nonzero vector $\eta \in \mathbb{R}^{(N-3)}$, such that

$$M\eta = 0.$$

Let $B = [b_1, b_2, \cdots, b_n]$, where $b_i \in \mathbb{R}^{N-3}$, for all $i \in \{1, \cdots, N\}$. Let $\phi \in \mathbb{R}^N$. Denote $\phi = [\phi_1, \phi_2, \phi_3, \phi_r^T]^T$ and $\xi = [\xi_1, \xi_2, \xi_3, \xi_r^T]^T$, where $\phi_r \in \mathbb{R}^{(N-3)}$ and $\xi_r \in \mathbb{R}^{(N-3)}$.

From (10), we know $L(\xi_1\mathbf{1} - \xi) = 0$. Thus, for the following equation,

$$(\xi_2 - \xi_1)b_2 + (\xi_3 - \xi_1)b_3 = M\phi_r, \tag{11}$$

there exists a special solution of ϕ_r, say ϕ_r^*, with a form like

$$\phi_r^* = \xi_1\mathbf{1}_{n-3} - \xi_r.$$

Here, the $\mathbf{1}_{N-3}$ indicates a full-one vector with $(N-3)$ dimension.

Define a set $\mathcal{F}$ as $\mathcal{F} = \{\mathcal{X}|\mathcal{X} = \alpha\mathbf{1} + \beta\xi\}$. Since ξ denotes the true positions of sensor nodes, the set $\mathcal{F}$ denotes all possible positions of sensor nodes up to rotation and translation. Now, we will show that there exists such a vector $\tilde{\xi}$ that $L\tilde{\xi} = 0$ and $\tilde{\xi} \notin \mathcal{F}$.

Since $M\eta = 0$, a general solution of ϕ_r in (11) should be

$$\phi_r = k\eta + \phi_r^*, \tag{12}$$

where $k \in \mathbb{R}$.

Let $\tilde{\xi} = [0, \xi_2 - \xi_1, \xi_3 - \xi_1, \phi_r]^T$. Then, $L\tilde{\xi} = 0$. Now, we show that $\tilde{\xi} \notin \mathcal{F}$. Since $\xi_i \neq \xi_j$, for all $i, j \in \{1, \cdots, N\}$, thus there is no zero entry in ϕ_r^*. Then, there exists a k in (12) such that $\tilde{\xi}_i \neq \tilde{\xi}_j$ for all $i, j \in \{1, \cdots, N\}$.

The first three nodes' positions in $\tilde{\xi}$ are $\{0, \xi_2-\xi_1, \xi_3-\xi_1\}$, which are translated from the $\{\xi_1, \xi_2, \xi_3\}$ with an offset of ξ_1. But, the translation of the rest positions in ϕ_r has a distinct offset with ξ_1. That is, $\tilde{\xi} \neq \xi$ and $\tilde{\xi}$ cannot be obtained from an affine transformation of ξ.

Thus, $\tilde{\xi}$ is not congruent with ξ up to translation and rotation. In other words, $\tilde{\xi}$ becomes a distinct position vector indicating the x-axis coordinate of $\mathcal{N}$. This is contradict with the assumption that $\mathcal{N}$ is uniquely localizable. Now, we can complete the proof. ■

V. Cooperative method for detecting localizable nodes and computing their positions

In this section we propose a method for detecting node localizability in a distributed manner. This method is based on an algorithm for computing node positions, which is proposed in [25]. We summarize this algorithm in Section V-A. Then, in Section V-B, we derive a necessary and sufficient condition for the localizability of each node.

A. A cooperative method for computing node positions

Assume that the measurement noise can be modeled as additive noise in (7), we then have

$$(M \otimes I_2)\mathbf{p}_s = (B \otimes I_2)\mathbf{p}_a + \upsilon, \tag{13}$$

where $\upsilon \sim \mathcal{N}(0, R)$, for some $R > 0$. We can then obtain a recursive algorithm for finding $\mathbf{p}_s$, using the method proposed in [25]. We summarize the idea below.

Let $\Phi = (M \otimes I_2)^T R^{-1}(M \otimes I_2)$ and $\alpha = (M \otimes I_2)^T R^{-1}(B \otimes I_2)\mathbf{p}_a$. We can obtain a weighted least-squares estimation $\hat{\mathbf{p}}_s$ of $\mathbf{p}_s$ as follows.

$$\hat{\mathbf{p}}_s = \Phi^{-1}\alpha. \tag{14}$$

Let $\gamma > 0$ and

$$\tilde{\alpha} = \sqrt{\gamma}\alpha,$$

$$\tilde{\hat{\mathbf{p}}}_s = \frac{z}{\sqrt{\gamma}}.$$

Then, (14) becomes

$$\tilde{\hat{\mathbf{p}}}_s = (\gamma\Phi)^{-1}\tilde{\alpha}. \tag{15}$$

If γ is chosen such that $\|I - \gamma\Phi\| < 1$, we can obtain a recursive algorithm for solving (15) using Richardson's iterations [26]. This yields

$$\tilde{\hat{\mathbf{p}}}_s(t+1) = (I - \gamma\Phi)\tilde{\hat{\mathbf{p}}}_s(t) + \tilde{\alpha}. \tag{16}$$

Then, substituting $\tilde{\alpha}$ and $\tilde{\hat{\mathbf{p}}}_s$ into (16) we obtain

$$\hat{\mathbf{p}}_s(t+1) = (I - \gamma\Phi)\hat{\mathbf{p}}_s(t) + \gamma\alpha. \tag{17}$$

Let $\hat{\mathbf{p}}_s = [\hat{p}_s^{(1)}, \cdots, \hat{p}_s^{(N)}]$, $\alpha = [\alpha^{(1)}, \cdots, \alpha^{(N)}]$ and $\Phi = [\Phi^{(i,j)}]_{i,j=1,\cdots,N}$. Then, at node i we obtain

$$\hat{p}_s^{(i)}(t+1) = \gamma\Phi^{(i)}\hat{p}_s^{(i)}(t) - \gamma\sum_{j\in\mathcal{H}_i}\Phi^{(i,j)}\hat{p}_s^{(j)}(t) + \gamma\alpha^{(i)}, \tag{18}$$

where $\mathcal{H}_i = \{j : \Phi^{(i,j)} \neq 0\}$.

Since $\Phi > 0$, if we choose

$$0 < \gamma < \frac{2}{\|\Phi\|}, \tag{19}$$

then $\|I - \gamma\Phi\| < 1$. Hence, any value of γ satisfying (19) will guarantee that (17) converges. As pointed out in [25], the value leading to the fastest convergence rate of the recursions (17) is

$$\begin{aligned}\gamma &= \frac{2}{\|\Phi\| + \|\Phi^{-1}\|^{-1}} \\ &= \frac{2}{\overline{EIG}(\Phi) + \underline{EIG}(\Phi)}.\end{aligned} \tag{20}$$

The values of $\overline{EIG}(\Phi)$ and $\underline{EIG}(\Phi)$, and thus γ, can be computed in a distributed manner, using the methods [27], [25] based on power iterations. We summarize the idea below.

The largest eigenvalue of Ψ can be estimated in a cooperative way using the power iteration, i.e.,

$$\tilde{b}(k) = \tau_k b(k), \tag{21}$$

$$b(k+1) = \Phi\tilde{b}(k). \tag{22}$$

where $\|b(0)\| = 1$ and τ_k is a re-scaling constant whose role is to avoid that $\|b(k)\|$ either increases or decreases indefinitely. Then, $\overline{EIG}(\Phi)$ can be estimated at node i using

$$\overline{EIG}(\Phi) = \lim_{k\to\infty}\frac{b_i(k+1)}{\tilde{b}_i(k)},$$

where $b_i(k), \tilde{b}_i(k) \in \mathbb{R}^2$ are the components of $b(k)$ and $\tilde{b}(k)$ available at node i.

In order to estimate $\underline{EIG}(\Phi)$, we use the algorithm above to find $\overline{EIG}(\Lambda)$, where $\Lambda = cI - \Phi$, and $c \geq \overline{EIG}(\Phi)$. Then,

$$\underline{EIG}(\Phi) = c - \overline{EIG}(\Lambda).$$

Remark 2: The estimation of eigenvalues can also be addressed by several other techniques, such as the randomized method [28] and spectral analysis method [29], [30].

Specifically, in [29], [30], the eigenvalue estimation problem can be solved by Fast Fourier transform (FFT), but they require the system matrix to be a symmetric Laplacian.

B. Cooperative method for detecting localizable nodes

We have the following immediate corollary of Theorem 1.

Corollary 1: A cascading quadrilateral network is uniquely localizable if and only if $\underline{EIG}(\Phi) > 0$, where $\Phi = M^T M$.

Then, we obtain the following necessary and sufficient condition for node localizability, which we will use as a criterion for detecting un-localizable nodes.

Theorem 2: Let $e_n = [0, \cdots, 0, 1, 1, 0, \cdots, 0] \in \mathbb{R}^{2(N-3)}$ be such that its nonzero entries are in the $(2n-1)$-th and $2n$-th position. Then, Node n is localizable if and only if $\omega^T e_n = 0$, for all $\omega \in \ker(\Phi)$, where $\ker(\Phi)$ denotes the kernel of Φ.

Proof:

Sufficiency: Let $\xi = \mathbf{p}_s$ and $\zeta = (M \otimes I_2)^T (M \otimes I_2)\mathbf{p}_a$. Pre-multiplying by $(M \otimes I_2)^T$ on both sides of (7) leads to

$$\Phi\xi = \zeta,$$

with $\Phi = (M \otimes I_2)^T (M \otimes I_2)$.

Clearly, $\ker(\Phi) = \ker(M)$. Then, the condition of the theorem is equivalent to $\omega^T e_n = 0$, for all $\omega \in \ker(M)$.

Let $\omega \in \ker(\Phi)$. Then,

$$\Phi(\xi + \omega) = \zeta.$$

Let $e_n = [0, \cdots, 0, 1, 1, 0, \cdots, 0] \in \mathbb{R}^{2(N-3)}$ be such that its nonzero entries are in the $(2n-1)$-th and $2n$-th position. Then, Node n is uniquely localizable if

$$e_n^T \omega = 0 \text{ for all } \omega \in \ker(\Phi),$$

and the sufficiency follows as the solution of ξ_n and therefore that of p_n is unique.

Necessity: Notice that, when Node n is localizable, all the solutions $\mathbf{p}_s = [p_1^T, \cdots, p_{n-3}^T]^T$ of (7) have the same components p_n corresponding to Node n. Thus, the proof is complete. ■

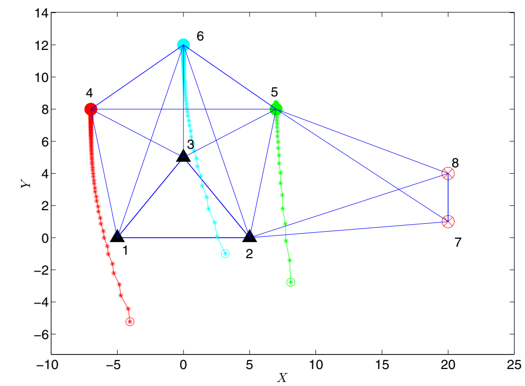
Using Theorem 2, we can devise a method to determine whether an individual node is localizable or not. The idea is to use the algorithm described in Section V-A for estimating $\underline{EIG}(\Phi)$. This algorithm gives as a byproduct, at Node n, a re-scaled version of ω_n, where $\omega = [\omega_1^T, \cdots, \omega_{N-3}^T]^T$ is a vector in the kernel of Φ (notice that Node n only knows ω_n, and not the whole ω). Suppose that the algorithm gives at Node n, that $\underline{EIG}(\Phi) = 0$, i.e., the network is not localizable. Then, if $\omega_n \neq 0$, Theorem 2 asserts that this node is un-localizable.

VI. Simulation and Performance Evaluation

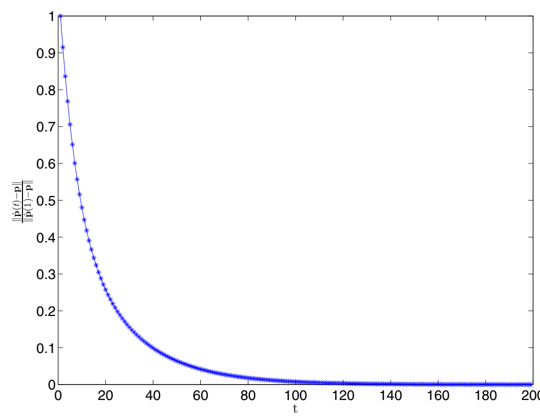
In this section we provide an example to show the application of our method. Fig. 2(a) shows a network with eight nodes, whose positions are given by

$$\mathbf{p} = [-5, 0, 5, 0, 0, 5, -7, 8, 7, 8, 0, 12, 20, 1, 20, 4]^T.$$

Notice that each node position is determined by two contiguous entries from the above vector. The three nodes indicated by solid triangles are anchor nodes, whereas the other ones are normal sensor nodes. The existence of a line segment connecting two nodes indicate that they can measure mutual distances. The estimation is done using the iterations in (18), which are started by a random guess of $\mathbf{p}$.



(a) Trace of the estimated positions.



(b) Convergence process.

Fig. 2. A sensor network of 8 nodes.

Nodes 7 and 8 indicated in Fig. 2(a), are un-localizable nodes. Their coordinates are determined by the last four entries of $\mathbf{p}$. The eigenvalues of the matrix Φ are

$$\text{eig}(\Phi) = \text{diag}\{0, 0.5731, 1.1273, 4.2970, 9.9556\} \otimes \mathbf{1}_2,$$

where $\mathbf{1}_2 = [1\ 1]$. Hence, the kernel of Φ has two dimensions. We use the distributed method summarized in Section V-A, to compute, at Node n, a re-scaled version ω_n of the entries $2n-1$ and $2n$, of a vector $\omega \in \ker(\Phi)$. This gives a vector $\tilde{\omega} \in \mathbb{R}^{2N}$ like

$$\tilde{\omega} = [0\ 0\ 0\ \ -0.7253\delta_1\ \ -0.6884\delta_2]^T \otimes \mathbf{1}_2^T.$$

Here, δ_1 and δ_2 are different re-scaling factors for different ω_n's, which are given by the algorithm at each node. As a result, when we stack all ω_n's together, ω may not be identical with an eigenvector. Then, Theorem 2 asserts that Nodes 4 and 5 are un-localizable, as their corresponding entries in ω are non-zero.

Finally, Fig. 2(b) shows the residual error of the position estimation for localizable nodes.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we provide a novel framework to analyze the localizability of a sensor network and each individual sensor node from the algebraic view. A linear system to cooperatively compute the positions of the localizable sensor nodes is obtained. The required information to achieve these aims are distance measurements and coordinate estimations of neighbors for each node, which belong to local information. Under certain assumptions on the topology of the network, we give a necessary and sufficient condition for the localizability of that network. It is shown that the localizability of the sensor network is strongly related with the algebraic property of its adjacency matrix. We also show how to detect the un-localizable nodes by exploring the null space of the adjacency matrix, which uses local information only.

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