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Three-Dimensional Formation Merging Control of Second-Order Agents under Directed and Switching Topologies

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Abstract— **This paper investigates the formation merging problem for a leader-follower network. The objective is to control a group of agents called followers and modeled by double-integrator dynamics so that they are merged with another group of agents called leaders to form a single globally rigid formation. With the assumption that leaders move in a globally rigid formation with their synchronized velocity known to the followers, we show that a globally rigid formation can be merged. Each follower selects its neighbors and control law according to the target configuration and thus it allows directed and time-varying switching topologies. We provide a necessary and sufficient condition such that a globally rigid formation can be merged asymptotically for the leader-follower network in a setup with directed and time-varying graphs.**

I. Introduction

Formation control of multi-agent systems has received much attention due to its broad applications [1]. In the paper, we consider the formation merging problem in 3D for a leader-follower network. Formation merging means that two sub-formations of agents are merged to become one single globally rigid formation. We assume that a team of agents called leaders already move in a globally rigid formation, and our objective is to control the other team of agents called followers in a distributed way so that they are asymptotically merged into a single globally rigid formation with leaders.

Many strategies exist in the literature for solving the above problem. One way to consider the formation merging problem is to figure out how many distance constraints should be added for the two sub-formations [2], [3] and then a distributed control law guides the agents to asymptotically meet these distance constraints. Considering the inter-agent distance constraints, information flow graph requires globally rigid [4], [5]. The concept of persistence is introduced for directed graph to merge two sub-formations [6]. However, it becomes challenging to analyze the stability of formations in a directed graph setup [7], [8]. Another way to address the formation merging problem is to consider the displacement constraints and utilize relative positions in their own local frames to design a distributed control [9]–[12]. A complex Laplacian based control law is introduced in the 2-dimensional space under a directed and fixed topology [10], [11].

However, when the information flow graph is directed and switches overtime agents may not be able to sense each other mutually and an information flow graph modeling their relative sensing may be varied. Thus, formation merging control has to be done. Few work is reported for such a problem. We initiate this study in our prior work [13], in which a single-integrator model is considered for each follower. The idea is to let each follower select its neighbors to meet a convexity assumption according to the target formation. Then a distributed control law is developed with the control parameters designed also according to the target formation. Moreover, a necessary and sufficient condition is obtained for asymptotically merging the followers with the leaders to form a globally rigid formation. That is, every follower should frequently have a joint path from at least a leader.

The paper generalizes the idea of our prior work [13] for formation merging control with the followers modeled by second-order dynamics, which is more representative in comparison with a single-integrator model that cannot for example model motion in which acceleration is the control input. Nevertheless, stability analysis becomes much more challenging for double-integrator models because the contraction analysis that works for single-integrator models does not work for double-integrator models. Moreover, no common Lyapunov function can be found due to the setup of directed topologies. To overcome this challenge and inspired by the idea of [14] used for consensus analysis, we convert the multi-agent system with double-integrator models to an augmented multi-agent system with single integrator models, for which the transformed velocity component of each agent makes a new virtual agent. We then show that the new graph modeling the augmented multi-agent system preserves certain graphical connectivity properties from the original information flow graph. Thus, the same necessary and sufficient condition is obtained for formation merging control of double-integrator agents.

Notation: $\mathbb{R}$ denotes the set of real numbers. $\mathbf{1}_n$ represents the n-dimensional vector of ones and I_n the identity matrix of order n. The symbol $\otimes$ denotes the Kronecker product.

II. Preliminaries and Problem Formulation

A. Preliminaries

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite set $\mathcal{V}$ of elements called *nodes* and a finite set $\mathcal{E}$ of ordered pairs of nodes called *edges*.

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Let $\mathcal{U} \subset \mathcal{V}$ and we say a node $v \in \mathcal{V} - \mathcal{U}$ is *reachable* from $\mathcal{U}$ if there exists a path from a node in $\mathcal{U}$ to v. And $\mathcal{U}$ is *closed* if any node in $\mathcal{U}$ is not reachable from $\mathcal{V} - \mathcal{U}$.

For a *time-varying graph* denoted by $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, a node v is called *uniformly jointly reachable* from $\mathcal{U} \subset \mathcal{V}$ if there exists $T > 0$ such that for all t, v is reachable from $\mathcal{U}$ in the union graph $\mathcal{G}([t, t+T]) = (\mathcal{V}, \bigcup_{t \in [t,t+T]} \mathcal{E}(t))$.

A *configuration* in $\mathbb{R}^3$ of a set of n nodes is defined by their coordinates in the Euclidean space $\mathbb{R}^3$, denoted as $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal \in \mathbb{R}^{3n}$, where each $p_i \in \mathbb{R}^3$ for $1 \le i \le n$. A *framework* in $\mathbb{R}^3$ is a graph $\mathcal{G}$ equipped with a configuration p, denoted as $\mathcal{F} = (\mathcal{G}, p)$.

We say that two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are *equivalent*, and we write $(\mathcal{G}, p) \sim (\mathcal{G}, q)$, if

$$\|p_i - p_j\| = \|q_i - q_j\|, \ \forall (i,j) \in \mathcal{E}.$$

We say that two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are *congruent*, and we write $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$, if

$$\|p_i - p_j\| = \|q_i - q_j\|, \ \forall i,j \in \mathcal{V}.$$

A framework $(\mathcal{G}, p)$ is called *globally rigid* if

$$(\mathcal{G}, p) \sim (\mathcal{G}, q), \ \forall q \in \mathbb{R}^{3n} \Leftrightarrow (\mathcal{G}, p) \equiv (\mathcal{G}, q).$$

For a directed graph, the Laplacian matrix $L \in \mathbb{R}^{n \times n}$ is defined as follows:

$$L(i,j) = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i \\ \sum\limits_{k \in \mathcal{N}_i} w_{ik} & \text{if } i = j \end{cases}$$

where $w_{ij} > 0$ is called the weight on edge (j,i) and $\mathcal{N}_i$ is the set of neighbors of node i.

A square matrix $E \in \mathbb{R}^{n \times n}$ is *nonnegative*, denoted as $E \ge 0$, if all its entries are nonnegative. And for a matrix $M \in \mathbb{R}^{n \times n}$ we write $E \ge M$ if $E - M \ge 0$. Moveover, E is called *stochastic* if it is nonnegative and every row sum equals 1. The *associated graph* $\mathcal{G}(E)$ consists of n nodes $v_1, \ldots, v_n$ where an edge leads from v_j to v_i if and only if the (i,j)th entry of E is nonzero.

B. Problem Formulation

The paper aims to solve the formation merging problem in a directed and switching topology setting. We assume that the target formation of followers entirely lies in the convex hull spanned by the leaders.

We consider a leader-follower network, with m leaders labeled $1, \ldots, m$ and n followers labeled $m+1, \ldots, m+n$. Let z_i be the 3D position of agent i. Consider a target configuration $p_a = [p_1^\intercal, \ldots, p_m^\intercal]^\intercal \in \mathbb{R}^{3m}$ for the leaders and $p_b = [p_{m+1}^\intercal, \ldots, p_{m+n}^\intercal]^\intercal \in \mathbb{R}^{3n}$ for the followers. Moreover, we assume that agents do not overlap each other in the target configuration.

We say the leaders are in a globally rigid formation p_a if $z_i(t) = A(t)p_i + c(t)$ for $i = 1, \ldots, m$ where $A(t)$ is a unitary matrix representing a rotation and $c(t)$ is a vector corresponding to a translation. Moreover, we say the whole network asymptotically reaches a globally rigid formation $[p_a^\intercal, p_b^\intercal]^\intercal$ if $z_i(t) \to A(t)p_i + c(t)$ for $i = 1, \ldots, m+n$.

We assume $m \ge 4$ and the m leaders move in a globally rigid formation p_a governed by the following dynamics

$$\dot{z}_i(t) = v_r(t), \ i = 1, \ldots, m, \tag{1}$$

where $v_r(t)$ is the synchronized velocity and we assume $v_r(t)$ is known to the followers.

Consider double-integrator dynamics for the followers, i.e.,

$$\begin{cases} \dot{z}_i = v_i \\ \dot{v}_i = u_i, \end{cases} \quad i = m+1, \ldots, m+n, \tag{2}$$

where the position $z_i \in \mathbb{R}^3$ and the velocity $v_i \in \mathbb{R}^3$ represent the states, and the acceleration $u_i \in \mathbb{R}^3$ is the control input. We use a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ to describe the information flow graph, where $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_b$ and an edge $(j,i) \in \mathcal{E}(t)$ means that $z_j - z_i$ is available to agent i at time t.

Remark 2.1: It should be pointed out that if $v_r(t)$ is not known to all followers but partial followers, then $v_r(t)$ can be available to all followers by estimation schemes, such as [15]. And for simplicity in this paper, we just assume the synchronized velocity $v_r(t)$ is known to the followers.

III. Main Results

In this section, we first provide the control law and then show stability analysis for the whole network.

A. Control Design

For each follower i we consider the following control law,

$$u_i = -2\gamma(v_i - v_r(t)) + \dot{v}_r(t) + \sum_{j \in \mathcal{N}_i(t)} k_{ij}(t)(z_j - z_i), \tag{3}$$

where $\mathcal{N}_i(t)$ is the neighbor set of follower i at time t and $k_{ij}(t)$ are control parameters that will be designed, and γ is the damping gain that satisfies $\gamma \ge 1$. To make the problem addressable, we assume that if agent i has neighbors at time t, then its neighbors are selected so that the convex hull spanned by $\{p_j : j \in \mathcal{N}_i(t)\}$ contains p_i in the target configuration $p = [p_a^\intercal, p_b^\intercal]^\intercal$. We call it the *convexity assumption*.

Next we provide a procedure for the design of $k_{ij}(t)$'s. In the following, we omit t for $k_{ij}(t)$'s for simplicity. There are four possible cases for each agent and we provide a procedure for the design of k_{ij}'s for these four cases.

(i) If an agent has no neighbor, then the control law (3) degenerates to

$$u_i = -2\gamma(v_i - v_r(t)) + \dot{v}_r(t).$$

(ii) For the case that the convex hull of an agent's neighbors is a line segment in the target configuration, we first consider that agent i has only two neighbors, say i_1 and i_2. Then we obtain that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2}, \tag{4}$$

where $\alpha_1 = \frac{\|p_{i_2} - p_i\|}{\|p_{i_2} - p_{i_1}\|}$ and $\alpha_2 = \frac{\|p_{i_1} - p_i\|}{\|p_{i_2} - p_{i_1}\|}$. It is clear that $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. Second, if agent i has more than two neighbors, then we can take any two of them containing p_i and obtain the same formula as (4), i.e.,

$$p_i = \alpha_1^l p_{i_1^l} + \alpha_2^l p_{i_2^l},$$

where l enumerates all possible combination of two neighbors containing p_i. Then consider a convex combination of all these representations for p_i, i.e.,

$$p_i = \sum_l \gamma^l(\alpha_1^l p_{i_1^l} + \alpha_2^l p_{i_2^l}) := \sum_{j\in\mathcal{N}_i} \alpha_j p_j,$$

where $\gamma^l \in (0,1)$ and $\sum_l \gamma^l = 1$. It is certain that $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j\in\mathcal{N}_i} \alpha_j = 1$. For this case, we take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

(iii) For the case that the convex hull of an agent's neighbors is a convex polygon, we first consider that agent i has only three neighbors, say $i_1,\ i_2$, and i_3, and they form a triangle in the target configuration. Let the coordinates of $p_{i_1},\ p_{i_2}$, and p_{i_3} be

$$p_{i_1} = (x_{i_1}, y_{i_1}, z_{i_1}),\ p_{i_2} = (x_{i_2}, y_{i_2}, z_{i_2}),\ p_{i_3} = (x_{i_3}, y_{i_3}, z_{i_3}).$$

Denote $x = [x_{i_1}, x_{i_2}, x_{i_3}]^\intercal, y = [y_{i_1}, y_{i_2}, y_{i_3}]^\intercal$ and $z = [z_{i_1}, z_{i_2}, z_{i_3}]^\intercal$. Let $S_{i_1i_2i_3}$ denotes the area of the triangle formed by $p_{i_1},\ p_{i_2}$ and p_{i_3}, then it can be calculated as follows

$$S_{i_1i_2i_3} = \frac{1}{2}\sqrt{S_1^2 + S_2^2 + S_3^2},$$

where $S_1 = \det[x, y, \mathbf{1}_3]^\intercal, S_2 = \det[y, z, \mathbf{1}_3]^\intercal, S_3 = \det[z, x, \mathbf{1}_3]^\intercal$. Then it can be obtained that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2} + \alpha_3 p_{i_3}, \tag{5}$$

where $\alpha_1 = \frac{S_{ii_2i_3}}{S_{i_1i_2i_3}},\ \alpha_2 = \frac{S_{i_1ii_3}}{S_{i_1i_2i_3}}$, and $\alpha_3 = \frac{S_{i_1i_2i}}{S_{i_1i_2i_3}}$. It is known that $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Second, if agent i has more than three neighbors, similar to the procedure of case (ii) we can get the representation for p_i as follows

$$p_i = \sum_{j\in\mathcal{N}_i} \alpha_j p_j,$$

where $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j\in\mathcal{N}_i} \alpha_j = 1$. And for this case we take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

(iv) For the case that the convex hull of an agent's neighbors is a convex polyhedron, we first consider that agent i has only four neighbors, say i_1, i_2, i_3, i_4, and they form a tetrahedron containing p_i in the target configuration. Denote by $V_{i_1i_2i_3i_4}$ the signed volume of the tetrahedron formed by $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$. It can be calculated by the following formula:

$$V_{i_1i_2i_3i_4} = \frac{1}{6}\det([p_{i_2} - p_{i_1}, p_{i_3} - p_{i_1}, p_{i_4} - p_{i_1}]^\intercal).$$

Then it can be obtained that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2} + \alpha_3 p_{i_3} + \alpha_4 p_{i_4}, \tag{6}$$

where

$$\alpha_1 = \frac{V_{ii_2i_3i_4}}{V_{i_1i_2i_3i_4}}, \alpha_2 = \frac{V_{i_1ii_3i_4}}{V_{i_1i_2i_3i_4}}, \alpha_3 = \frac{V_{i_1i_2ii_4}}{V_{i_1i_2i_3i_4}}, \alpha_4 = \frac{V_{i_1i_2i_3i}}{V_{i_1i_2i_3i_4}},$$

where $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. Second, if agent i has more than four neighbors, similar to the procedure of case (ii) we can get the representation for p_i as follows

$$p_i = \sum_{j\in\mathcal{N}_i} \alpha_j p_j,$$

where $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j\in\mathcal{N}_i} \alpha_j = 1$. And for this case we take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

B. Stability Analysis

In this section, we provide stability analysis for the whole network. Let $L(t)$ be the Laplacian matrix for the graph with weights $k_{ij}(t)$'s associated to edges (j, i)'s at time t. Define $z = [z_a^\intercal, z_b^\intercal]^\intercal$ the aggregate state of all z_i's. Throughout the paper we use the subscript a to represent the states for the leaders and the subscript b for the followers. Then the overall system can be described as

$$\begin{bmatrix} \dot z_a \\ \dot z_b \\ \dot v_b \end{bmatrix} = -\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_n \\ L_{lf}(t) & L_{ff}(t) & 2\gamma I_n \end{bmatrix} \otimes I_3\right)\begin{bmatrix} z_a \\ z_b \\ v_b \end{bmatrix} + \begin{bmatrix} \mathbf{1}_m \\ 0 \\ 2\gamma\mathbf{1}_n \end{bmatrix} \otimes v_r(t) + \begin{bmatrix} 0 \\ 0 \\ \mathbf{1}_n \end{bmatrix} \otimes \dot v_r(t), \tag{7}$$

where $L_{lf}(t)$ and $L_{ff}(t)$ are sub-matrices of $L(t)$ which has the following form

$$L(t) = \left[\begin{array}{c|c} 0_{m\times m} & 0_{m\times n} \\ \hline L_{lf}(t) & L_{ff}(t) \end{array}\right] \tag{8}$$

and according to the design procedure given in Subsection III-A, $L(t)$ satisfies

$$(L(t) \otimes I_3)p = 0 \text{ and } L(t)\mathbf{1}_{m+n} = 0. \tag{9}$$

The following result shows that $p = [p_a^\intercal, p_b^\intercal]^\intercal$ is a stable equilibrium formation for the whole network.

Theorem 3.1: Suppose the leaders are in the globally rigid formation p_a. Then

$$\begin{cases} z^*(t) = (I_{m+n} \otimes A)p + \mathbf{1}_{m+n} \otimes (c + \int_0^t v_r(\tau)d\tau) \\ v_b^*(t) = v_r(t) \end{cases}$$

where A is a unitary matrix and c is a constant vector determined by the leaders, is an equilibrium solution of system (7). Moreover, it is stable.

Proof: Let $y_i = z_i - \int_0^t v_r(\tau)d\tau$ and $w_i = v_i - v_r$. Define

$$\begin{aligned}\xi &= [\xi_1^\intercal, \dots, \xi_m^\intercal, \xi_{m+1}^\intercal, \xi_{m+2}^\intercal, \dots, \xi_{m+2n-1}^\intercal, \xi_{m+2n}^\intercal]^\intercal \\ &= [y_1^\intercal, \dots, y_m^\intercal, y_{m+1}^\intercal, x_{m+1}^\intercal, \dots, y_{m+n}^\intercal, x_{m+n}^\intercal]^\intercal,\end{aligned}$$

where $x_i = y_i + \frac{1}{\gamma}w_i,\ i = m+1, \dots, m+n$. Then system (7) can be rewritten in matrix form as

$$\dot\xi = \left[\left(\begin{bmatrix} 0 & 0 \\ 0 & I_n \otimes B \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ L_{lf}(t) \otimes C_1 & L_{ff}(t) \otimes C_2 \end{bmatrix}\right) \otimes I_3\right]\xi, \tag{10}$$

where

$$B = \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix}, C_1 = \begin{bmatrix} 0 \\ \frac{1}{\gamma} \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{\gamma} & 0 \end{bmatrix}.$$

Let

$$L^*(t) = \begin{bmatrix} 0 & 0 \\ L_{lf}(t) \otimes C_1 & L_{ff}(t) \otimes C_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_n \otimes B \end{bmatrix}$$

and it can be checked that $L^*(t)$ can be considered as a Laplacian matrix of a new graph $\mathcal{G}^*(t)$. Then system (10) becomes

$$\dot{\xi} = -(L^*(t) \otimes I_3)\xi. \tag{11}$$

Moreover, it can be verified from (9) that

$$(L^*(t) \otimes I_3)p^* = 0,$$

where $p^* = [p_1^\intercal, \ldots, p_m^\intercal, p_{m+1}^\intercal, p_{m+1}^\intercal, \ldots, p_{m+n}^\intercal, p_{m+n}^\intercal]^\intercal$. To show $[z^*(t)^\intercal, v_b^*(t)^\intercal]^\intercal$ is an equilibrium solution of system (7), it remains to show that

$$\xi^* = (I_{m+2n} \otimes A)p^* + \mathbf{1}_{m+2n} \otimes c$$

is an equilibrium point of system (11). For any t we can get

$$\begin{aligned}&(L^*(t) \otimes I_3)[(I_{m+2n} \otimes A)p^* + \mathbf{1}_{m+2n} \otimes c]\\ &= (L^*(t) \otimes A)p^* = (I_{m+2n} \otimes A)(L^*(t) \otimes I_3)p^* = 0.\end{aligned}$$

Hence, ξ^* is an equilibrium point of system (11).

Next, we show that $[z^*(t)^\intercal, v_b^*(t)^\intercal]^\intercal$ is stable. For any arbitrary $\epsilon > 0$, we choose $\delta = \frac{1}{4\gamma}\epsilon$. Suppose

$$(\forall i)\|z_i^0 - z_i^*\| \le \delta \text{ and } (\forall i)\|v_i^0 - v_i^*\| \le \delta.$$

Then for any i,

$$\|\xi_i^0 - \xi_i^*\| \le \|z_i^0 - z_i^*\| + \frac{1}{\gamma}\|v_i^0 - v_i^*\| \le \delta + \frac{1}{\gamma}\delta \le 2\delta = \frac{\epsilon}{2\gamma}.$$

Consider any $t > 0$, without loss of generality, we say t is in the interval $[t_i, t_{i+1}]$. Then the transition matrix of system (11) can be written as

$$\Phi(t, t_i) = \exp[-(L^*(t_i) \otimes I_3)(t - t_i)] \tag{12}$$

and the solution of system (11) can be described as

$$\xi(t) = \Phi(t, t_i)\Phi(t_i, t_{i-1}) \cdots \Phi(t_1, t_0)\xi^0$$

for an initial state ξ^0. Notice that every transition matrix is stochastic and the product of stochastic matrices is also stochastic ([16], page 51). It then follows that every state $\xi_i(t)$ is a convex combination of $\xi_1^0, \ldots, \xi_{m+2n}^0$, i.e.,

$$\xi_i(t) = \sum_{j=1}^{m+2n} \alpha_j \xi_j^0, \tag{13}$$

where $\alpha_j \ge 0$ and $\sum_{j=1}^{m+2n} \alpha_j = 1$. Since ξ^* is an equilibrium point of system (11), then from (13) it is obtained that $\xi_i^* = \sum_{j=1}^{m+2n} \alpha_j \xi_j^*$. Thus, we have for every i,

$$\|\xi_i(t) - \xi_i^*\| = \|\sum_{j=1}^{m+2n} \alpha_j(\xi_j^0 - \xi_j^*)\| \le \frac{1}{2\gamma}\sum_{j=1}^{m+2n} \alpha_j \epsilon = \frac{\epsilon}{2\gamma}.$$

Then it leads to the fact that

$$(\forall i)\|z_i(t) - z_i^*\| \le \frac{\epsilon}{2\gamma} < \epsilon \text{ and } (\forall i)\|v_i(t) - v_i^*\| \le \epsilon$$

and the conclusion follows. ■

The following result gives a necessary and sufficient graphical condition to ensure that a globally rigid formation $[p_a^\intercal, p_b^\intercal]^\intercal$ can be asymptotically merged.

Theorem 3.2: Suppose the leaders are in the globally rigid formation p_a. A globally rigid formation $[p_a^\intercal, p_b^\intercal]^\intercal$ can be asymptotically merged under the distributed control law (3) if and only if every follower is uniformly jointly reachable from $\mathcal{V}_a$.

To prove Theorem 3.2 we require the following lemmas.

Lemma 3.1: For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $m + n$ nodes, where $\mathcal{V}_a = \{1, \ldots, m\}$ with no incoming edge and $\mathcal{V}_b = \{m + 1, \ldots, m + n\}$, let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a graph with only n nodes and no edges, i.e., $\mathcal{V}' = \{(m + 1)', \ldots, (m + n)'\}$ and $\mathcal{E}' = \varnothing$. Then a new graph $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ is constructed from $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ according to the following rules:

1) $\mathcal{V}^* = \mathcal{V} \cup \mathcal{V}' = \{1, 2, \ldots, m, m+1, (m+1)', \ldots, m+n, (m + n)'\}$, $\mathcal{V}_a^* = \mathcal{V}_a$ and $\mathcal{V}_b^* = \{m + 1, (m + 1)', \ldots, m + n, (m + n)'\}$;
2) There is no edge between node i and node j or node i' and node j' for any $i, j \in \mathcal{V}$ and $i', j' \in \mathcal{V}'$, $i \ne j$, $i' \ne j'$;
3) Edge $(i', i) \in \mathcal{E}^*$ while edge (i, i') may or may not exist in graph $\mathcal{G}^*$ for any $i' \in \mathcal{V}'$, $i = m + 1, m + 2, \ldots, m + n$;
4) Edge $(j, i') \in \mathcal{E}^*$ if and only if edge $(j, i) \in \mathcal{E}$ for any $i, j \in \mathcal{V}$, $i \ne j$.

Then in graph $\mathcal{G}$ every node in $\mathcal{V}_b$ is reachable from $\mathcal{V}_a$ if and only if in graph $\mathcal{G}^*$ every node in $\mathcal{V}_b^*$ is reachable from $\mathcal{V}_a^*$.

Proof: ($\Rightarrow$) Suppose in a graph $\mathcal{G}$ every node in $\mathcal{V}_b$ is reachable from $\mathcal{V}_a$. For any node $i_b \in \mathcal{V}_b$, i_b is reachable from $i_a \in \mathcal{V}_a$. Then there exists a path $(i_a, i_1)(i_1, i_2), \ldots, (i_m, i_b)$ in $\mathcal{G}$. Then for the constructed graph $\mathcal{G}^*$, there exists a path $(i_a, i_1')(i_1', i_1)(i_1, i_2')(i_2', i_2) \cdots (i_m, i_b')(i_b', i_b)$, which means a path $(i_a \to i_b)$ and also a path $(i_a \to i_b')$ exist. Thus every node in $\mathcal{V}_b^*$ is reachable from $\mathcal{V}_a^*$.

($\Leftarrow$) Suppose in graph $\mathcal{G}^*$ every node in $\mathcal{V}_b^*$ is reachable from $\mathcal{V}_a^*$. This means any node in $\mathcal{V}_b^*$, say $i_b \in \mathcal{V}_b$ or $i_b' \in \mathcal{V}_b'$, is reachable from $\mathcal{V}_a^* = \mathcal{V}_a$, say i_a. Note that according to the rules graph $\mathcal{G}$ can be obtained by contracting all the edges (i', i) $(i = m + 1, \ldots, m + n)$ in graph $\mathcal{G}^*$. As a result, a path $(i_a \to i_b)$ exists in graph $\mathcal{G}$. Thus every node in $\mathcal{V}_b$ is reachable from $\mathcal{V}_a$. ■

Lemma 3.2: ([17], page 87) Let E be a nonnegative matrix and denote $e_{ij}^{(k)}$ the (i, j)th entry of E^k. Then $e_{ij}^{(k)} > 0$ if and only if the associated graph $\mathcal{G}(E)$ has a walk from node v_j to node v_i of length k.

Proof of Theorem 3.2: ($\Leftarrow$) Suppose the graph switches at $t_0, t_1, t_2, \ldots$. Recall that the switching signal is regular enough, which means there exists $\tau_D > 0$ such that $t_{i+1} - t_i \ge \tau_D$ for all $i \ge 0$. Moreover, there also exists $\tau_m > \tau_D$ large enough such that $t_{i+1} - t_i \le \tau_m$ for all $i \ge 0$. When for some interval $[t_i, t_{i+1}]$ there is no such a τ_m, we can partition $[t_i, t_{i+1}]$ artificially.

If every follower is uniformly jointly reachable from $\mathcal{V}_a$, by the definition there exists $T > 0$ such that for all t in the union graph $\mathcal{G}([t, t + T])$ every follower is reachable from $\mathcal{V}_a$. Now we generate a subsequence $\{t_{m_k}\}$ of the sequence

$\{t_i\}$ as follows:
(1) Set $m_0 = 0$.
(2) If $t_{m_0} + T \in (t_{i-1}, t_i]$, set $m_1 = i$.
(3) If $t_{m_1} + T \in (t_{i-1}, t_i]$, set $m_2 = i$.
(4) And so on.
Recall the transformed system (11)

$$\dot{\xi} = -(L^*(t) \otimes I_3)\xi.$$

We have at the subsequence of time instants $\{t_{m_k}\}$,

$$\xi(t_{m_{k+1}}) = \Psi(t_{m_k})\xi(t_{m_k}) \tag{14}$$

where $\Psi(t_{m_k}) = \left[\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} L^*(t)dt\right)\right] \otimes I_3$. Denote by Ξ the set of all $\Psi(t_{m_k})$'s derived above. We regard the above evolution as a discrete-time switched system and for simplicity we rewrite (14) as

$$\xi(k+1) = \Psi(k)\xi(k) \text{ with } \Psi(k) \in \Xi. \tag{15}$$

It is known that $L^*(t)$ can be considered as a Laplacian matrix of a new graph $\mathcal{G}^*(t)$. Next we prove that the union graph $\mathcal{G}^*([t, t+T])$ is exactly constructed from $\mathcal{G}([t, t+T])$ according to Lemma 3.1. For any $L^*(t)$, we can decompose it as $-L^*(t) = -D^*(t) + E^*(t)$, where $D^*(t)$ is a diagonal matrix and $E^*(t)$ is a nonnegative matrix with all diagonal entries zero. Similarly we can decompose any $L(t)$ as $-L(t) = -D(t) + E(t)$.

Denote $e^*_{ij}(t)$ the (i,j)th entry of $E^*(t)$ and $e_{ij}(t)$ the (i,j)th entry of $E(t)$. And for simplicity, we omit t. For graph $\mathcal{G}^*(t)$ corresponding to $L^*(t)$, considering any node in $\mathcal{V}^*_b$ we have $e^*_{m+2i-1,m+2i} = \gamma > 0$ $(i = 1, \dots, n)$ and $e^*_{m+2i,m+2i-1} \geq 0$ since $\gamma \geq 1$. Checking the following sub-matrix of $E^*(t)$ we obtain

$$\begin{bmatrix} e^*_{11} & \cdots & e^*_{1m} & e^*_{1,m+1} & \cdots & e^*_{1,m+2n-1} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ e^*_{m1} & \cdots & e^*_{mm} & e^*_{m,m+1} & \cdots & e^*_{m,m+2n-1} \\ e^*_{m+2,1} & \cdots & e^*_{m+2,m} & e^*_{m+2,m+1} & \cdots & e^*_{m+2,m+2n-1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ e^*_{m+2n,1} & \cdots & e^*_{m+2n,m} & e^*_{m+2n,m+1} & \cdots & e^*_{m+2n,m+2n-1} \end{bmatrix}$$
$$\geq \frac{1}{\gamma} E(t).$$

Relabel all the agents in $\mathcal{G}^*(t)$ such that $\mathcal{V}^* = \{1, \dots, m, m+1, (m+1)', \dots, m+n, (m+n)'\}$. It is easy to know that $e_{m+i,m+j} = \gamma e^*_{m+2i,m+2j-1} > 0 (1 \leq i, j \leq n)$ in graph $\mathcal{G}(t)$ if and only if $e^*_{(m+i)',m+j} > 0$ in graph $\mathcal{G}^*(t)$. With the fact that $e^*_{m+i,(m+i)'} > 0$ and $e^*_{(m+i)',m+i} \geq 0$ we know $\mathcal{G}^*(t)$ is exactly constructed from $\mathcal{G}(t)$ according to Lemma 3.1. It uses the similar techniques to show that the union graph $\mathcal{G}^*([t, t+T])$ is also constructed from $\mathcal{G}([t, t+T])$ according to Lemma 3.1. Only to consider the similar sub-matrix of $\tilde{E}^* := \int_t^{t+T} E^*(t)dt$ and $\tilde{E} := \int_t^{t+T} E(t)dt$. So we omit the procedure. Then it follows that in $\mathcal{G}^*([t, t+T])$ every node in $\mathcal{V}^*_b$ is reachable from $\mathcal{V}^*_a$.

Next we show that for all $\Psi(k) \in \Xi$, $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by a constant $\sigma < 1$. We know

$$\Psi(k) = \left[\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} D^*(t)dt\right) \exp\left(\int_{t_{m_k}}^{t_{m_{k+1}}} E^*(t)dt\right)\right] \otimes I_3.$$

We denote $E^* = \int_{t_{m_k}}^{t_{m_{k+1}}} E^*(t)dt$ and it is noted that

$$E^* = E^*(t_{m_k})(t_{m_k+1} - t_{m_k}) + \cdots + E^*(t_{m_{k+1}-1})(t_{m_{k+1}} - t_{m_{k+1}-1}).$$

By the condition that every node in $\mathcal{V}^*_b$ is uniformly jointly reachable from $\mathcal{V}^*_a$, we can then know that every node in $\mathcal{V}^*_b$ is reachable from $\mathcal{V}^*_a$ in the associated graph $\mathcal{G}(E^*)$. Then, considering the following formula:

$$\exp(E^*) = I + E^* + \frac{(E^*)^2}{2!} + \cdots$$

and the fact that $\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} D^*(t)dt\right)$ is a positive diagonal matrix, we can infer by Lemma 3.2 that each row of $\Psi_{lf}(k)$ has a nonzero entry. And with the fact that $\Psi(k)$ is a stochastic matrix, it implies that $\|\Psi_{ff}(k)\|_\infty < 1$. Moreover, $t_{i+1} - t_i$ is uniformly lower-bounded and upper-bounded for any i. And with the fact that $L^*(t_i)$'s are taken in a finite set, there exists a positive constant $\sigma < 1$ such that $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by σ.

Since the m leaders are in a globally rigid formation p_a, from (15) we then have

$$\xi_b(k+1) = \Psi_{ff}(k)\xi_b(k) + \Psi_{lf}(k)\xi^*_a, \tag{16}$$

where $\xi^*_a = (I_m \otimes A)p^*_a + \mathbf{1}_m \otimes c$. By $\|\Psi_{ff}(k)\|_\infty < 1$ we know that $I - \Psi_{ff}(k)$ is invertible. Thus, the system (16) has a unique equilibrium point $\xi^*_b = (I_{2n} \otimes A)p^*_b + \mathbf{1}_{2n} \otimes c$. So by the coordinate transformation $q(k) = \xi_b(k) - \xi^*_b$ we get $q(k+1) = \Psi_{ff}(k)q(k)$. Since $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by $\sigma < 1$, it can be inferred that $\lim_{j\to\infty} \xi_b(t_{m_j}) = \xi^*_b$.

Now we look at the state $\xi_b(t)$ in the interval between any two consecutive switching instants. From the proof of Theorem 3.1, we know that for any $t \in [t_i, t_{i+1})$ and any arbitrary $\epsilon > 0$

$$(\forall i)\|\xi_i(t_i) - \xi^*_i\| \leq \epsilon \Rightarrow (\forall i)\|\xi_i(t) - \xi^*_i\| \leq \epsilon.$$

Therefore, it is known that $\lim_{t\to\infty} \xi_b(t) = \xi^*_b$. Hence, the conclusion follows.

($\Rightarrow$) We prove it in a contrapositive way. Assume that there exists a follower, say i, that is not uniformly jointly reachable from $\mathcal{V}_a$. That is, for any $T > 0$ there exists $t^* \geq 0$ such that in the union graph $\mathcal{G}([t^*, t^* + T])$, i is not reachable from $\mathcal{V}_a$. It follows that in the union graph $\mathcal{G}^*([t^*, t^* + T])$, i is not reachable from $\mathcal{V}^*_a$. Let Θ be the set including all such nodes that are not reachable from $\mathcal{V}^*_a$ in $\mathcal{G}^*([t^*, t^* + T])$. Then it can be known that Θ is a closed set. So the states of these nodes in Θ at $t \in [t^*, t^* + T]$ remain in the convex hull of their states at t^* and ξ will not converge to ξ^*. Hence, a globally rigid formation cannot be merged. ■

IV. Simulations

In this section, we present a simulation to illustrate the correctness of our theoretic results. In the simulation there are 8 leaders moving in a globally rigid formation (a cube in 3D). Then consider 12 followers with any initial states. Denote the set of leaders by $\mathcal{V}_a = \{1, 2, \dots, 8\}$ and denote the set of followers by $\mathcal{V}_b = \{9, 10, \ldots, 20\}$ and the target formation is shown in Fig. 1. For simplicity, we consider a periodic switching graph $\mathcal{G}(t)$, which switches among three different topologies as shown in Fig. 2. And it can be checked that every follower is uniformly jointly reachable from $\mathcal{V}_a$ in $\mathcal{G}(t)$ by taking $T = 3$. A simulation result is shown in Fig. 3 and it is shown that the followers are asymptotically merged with the leaders to reach the target formation.

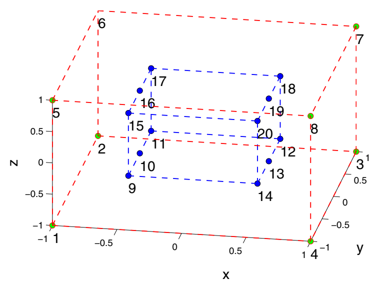

Fig. 1. A target formation for a leader-follower network with 8 leaders and 12 followers.

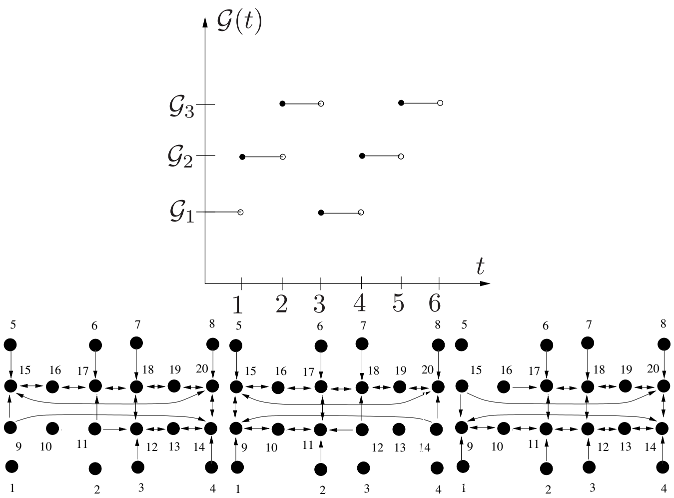

Fig. 2. A periodic switching graph $\mathcal{G}(t)$ that switches among three different topologies $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

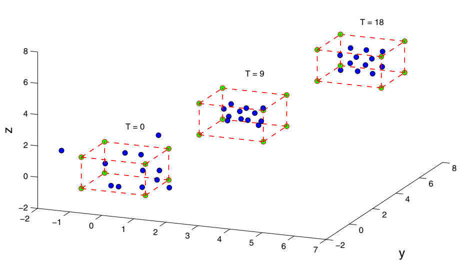

Fig. 3. The followers are asymptotically merged with a formation of leaders to form a larger target formation.

V. Conclusions

This paper investigates the formation merging problem for a leader-follower network. A distributed control law is proposed and the control parameters are designed based on the target configuration. The followers are modeled by double-integrator dynamics and a rule is introduced for followers to select neighbors in order to meet a convexity assumption. Then we present a necessary and sufficient condition so that a group of followers can be asymptotically merged with a group of leaders to form a globally rigid formation. A direction for future research is to design control laws with the assumption that the convexity assumption is relaxed.

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