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Stability of the Kalman Filtering with Two Periodically Switching Sensors over Lossy Networks

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***Abstract*— This paper considers the stability of Kalman filtering of a discrete-time stochastic system using two periodically switching sensors over a network subject to random packet losses, which is modeled by an independent and identically distributed Bernoulli process. It is proved that this problem can be converted into the stability of Kalman filtering using two sensors at each time instant, where the measurements of each sensor are transmitted via an independent lossy channel. Some necessary and sufficient conditions for stability of the estimation error covariance matrices are respectively established, and the effect of the periodic switching on the stability is revealed. Their implications and relationships with related results in the literature are discussed.**

I. Introduction

This work is a contribution to the stability analysis of Kalman filtering of a discrete-time stochastic system using two periodically switching sensors with random packet losses. In contrast to the current literature [1]–[3], the striking feature lies in the use of switching sensors in the networked systems. Sensors of different nature, bandwidth, accuracy and noise levels usually have different performances in specific operating and/or environmental conditions. The use of different sensors may provide richer information to increase the estimation/control performance. In some occasions, a single sensor may not be adequate to obtain sufficient information to observe the state of a dynamical system.

We also study the stability problem of the Kalman filter over a lossy network. See the first networked system in Fig. 1 for an illustration. A motivating example is given by sensor and estimator communicating over a wireless channel for which the quality of the communication channel varies over time because of random fading and congestion. This happens in resource limited wireless sensor networks where communications between devices are power constrained and therefore limited in range and reliability. As in [2], the packet loss process is modeled as an independent and identical distributed (i.i.d.) Bernoulli process. The problem involving the use of switching sensors to transmit data over a lossy network is more complicated than the problem involving only a single sensor. It should be noted that the stability analysis of a switching system is much more involved than that of a single system [4].

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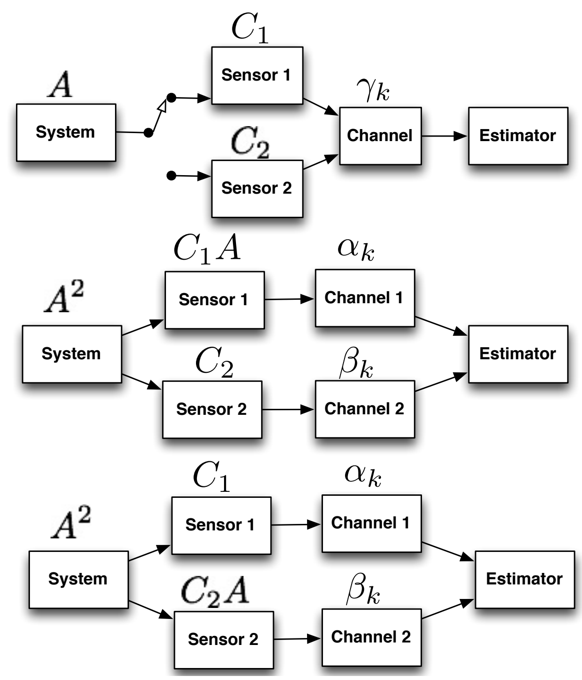

Fig. 1. Networked systems with lossy channels: the open-loop system and the sensor measurement matrices are accordingly denoted above the blocks of systems and sensors. All the lossy channels are subject to the i.i.d. packet loss with the same packet loss rate.

In this paper, it is proved that whether the use of periodically switching sensors requires less stringent network conditions depends on the characteristics of sensors. For the second-order dynamical system, we exactly derive the necessary and sufficient condition for the stability of the estimation error covariance matrices. The interesting finding is that the use of two periodically switching sensors may strictly reduce the network condition to achieve stability of the Kalman filter over a lossy network. Our approach is to convert the original problem into the stability of the Kalman filter of another dynamical system observed by two sensors at each time, and each sensor's measurements are transmitted through an independent lossy channel.

Kalman filtering is of great importance in networked systems due to its various applications ranging from tracking, detection to control. The stability analysis of Kalman filtering with intermittent measurements observed by a single sensor can be tracked back to the influential work [2], which studies the optimal state estimation problem for a discrete-time linear stochastic system under the assumption that the raw measurements are randomly dropped. By modeling the

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packet loss process as an i.i.d. Bernoulli process, they prove the existence of a critical packet loss rate above which the mean state estimation error covariance matrices will diverge. Since their approach lies in the use of two upper and lower bounds for the estimation error covariance matrices, they are unable to exactly quantify the critical loss rate for general systems, and only provide its lower and upper bounds, which are attainable under some special cases, e.g., the lower bound is tight if the observation matrix is invertible. Since then, a large mount of efforts have been made toward finding the critical packet loss rate as to some extent, it characterizes the minimum network condition required for the stability of the Kalman filter.

Recently, a new method has been proposed to evaluate the exact packet loss rate in [1], [5] for a wider class of systems, including second-order systems and the so-called non-degenerate higher-order systems. A remarkable discovery in [5] is that there are examples of second-order systems for which the lower bound given by [2] is not tight. This method requires to exploit the system structure, especially the presence form of the unstable eigenvalue of the open-loop matrix. We extend this approach to the problem involving two periodically switching sensors in this paper, and establish some necessary and sufficient conditions for the stability of the estimation error covariance matrices, respectively.

The rest of the paper is organized as follows. The problem is mathematically formulated in Section II. We derive the stability condition for Kalman filtering using periodically switching sensors without packet losses in Section III. Then, the effect of the lossy network on stability of the Kalman filter using two periodically switching sensors is explored in Section IV. In particular, we are able to exactly characterize the necessary and sufficient condition for the stability of the Kalman filter for the second-order system. Some conclusion remarks are drawn in Section V.

II. Problem formulation

Consider a linear discrete time-invariant stochastic system as follows

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ denotes the system state at time k and w_k is a white Guassian noise with zero mean and positive definite covariance matrix Q. The initial state x_0 is a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix P_0.

There are two switching sensors to cooperatively monitor the system, and at each time, one of them will take a measurement from the system. For instance, the measurement equation is given by

$$y_k = C_{\sigma_k} x_k + v_{\sigma_k}. \tag{2}$$

where $\sigma_k \in \{1, 2\}$ represents the index of which sensor is active to take measurement at time k, and v_{σ_k} is white Gaussian noise with zero mean and positive definite covariance matrix R_{σ_k}. Both C_1 and C_2 are of full rank. The measurement y_k is directly transmitted to a remote estimator via an unreliable communication channel, see the first networked system in Fig. 1. Due to random fading and/or congestion of the communication channel, packets may be lost while in transit inside the network. To examine this phenomena, we use a binary random process γ_k to denote the packet loss process. Precisely, let $\gamma_k = 1$ indicate that the packet containing the information of y_k has been successfully delivered to the estimator while $\gamma_k = 0$ corresponds to the loss of the packet. In this paper, we assume that the communication link is an erasure channel [6], which means that γ_k is an i.i.d. process.

Different from [7], [8], the present work ignores other effects such as quantization, transmission errors and data delays. In comparison with [1], the measurement matrix C_{σ_k} is time-varying. The use of switching sensors is to alleviate the working load of one sensor for the purpose of prolonging the life time of the network or provide richer information for the estimator. As an initial attempt, we consider a periodically switching rule in this work. To be precise,

$$\sigma_k = \begin{cases} 1, & \text{if } k \text{ is odd;} \\ 2, & \text{if } k \text{ is even.} \end{cases} \tag{3}$$

Since the switching rule is deterministic, the estimator is able to know which sensor is in use at each time and whether the packet containing the measurement information is received or not. That is, the information available to the estimator at time k is given as follows

$$\mathcal{F}_k = \{\sigma_i, \gamma_i, y_i\gamma_i, i \leq k\}.$$

Denote the minimum mean square error predictor and estimator by $\hat{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{F}_{k-1}]$ and $\hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{F}_k]$, respectively. Their corresponding estimation error covariance matrices are then given by $P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T|\mathcal{F}_{k-1}]$ and $P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|\mathcal{F}_k]$. In view of [2], the above quantities are able to be computed via the following update recursions.

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - C_{\sigma_k}\hat{x}_{k|k-1}); & (4) \\ P_{k|k} &= P_{k|k-1} - \gamma_k K_k C_{\sigma_k} P_{k|k-1}, & (5) \end{aligned}$$

where the Kalman gain $K_k = P_{k|k-1}C_{\sigma_k}^T(C_{\sigma_k}P_{k|k-1}C_{\sigma_k}^T + R_{\sigma_k})^{-1}$, $\hat{x}_{0|-1} = \bar{x}_0$ and $P_{0|-1} = P_0$. In addition, $P_{k+1|k} = Ap_{k|k}A^T + Q$ and $\hat{x}_{k+1|k} = A\hat{x}_{k|k}$.

Let $P_k := P_{k|k-1}$, it is recursively updated as follows:

$$\begin{aligned} P_{k+1} &= AP_kA^T + Q \qquad (6) \\ &\quad -\gamma_k AP_kC_{\sigma_k}^T(C_{\sigma_k}P_kC_{\sigma_k}^T + R_{\sigma_k})^{-1}C_{\sigma_k}P_kA^T \\ &:= g_k(P_k, R_{\sigma_k}). \end{aligned}$$

The goal of this paper is to derive the necessary and sufficient condition for the mean square stability of the filter, i.e., $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$[1], where the mathematical expectation is taken with respect to the packet loss process $\{\gamma_k\}_{k\in\mathbb{N}}$.

[1]There exists a positive definite matrix $\bar{P}$ such that $P_k \preceq \bar{P}$ for all $k \in \mathbb{N}$. The matrix inequality $A \preceq B$ means that $B - A$ is semi-positive definite. Similar notations will be made for $\prec, \succ$ and $\succeq$ in the sequel.

III. Stability of the Kalman filter without packet losses

In this section, we characterize the stability condition for the Kalman filter using two periodically switching sensors without packet losses. This allows us to focus on the effect of lossy channels on the stability of the Kalman filter in the next section.

For convenience, we first illustrate that the sensor noise levels do not affect the stability analysis of the Kalman filter of the networked system. Denote $R_M = R_1 + R_2$ and $R_m = \min\{\lambda_m(R_1), \lambda_m(R_2)\}I$, where $\lambda_m(R_i)$ is the minimum eigenvalue of R_i. Then, it follows that $R_m \preceq R_{\sigma_k} \preceq R_M, \forall k \in \mathbb{N}$ and

$$g_k(P_k, R_M) \preceq g_k(P_k, R_{\sigma_k}) \preceq g_k(P_k, R_m). \quad (7)$$

This essentially implies that the time-varying R_{σ_k} does not affect the stability analysis for P_k. Thus, there is no loss of generality to assume that $R_1 = R_2 = R$. In comparison with [1], the new challenge solely lies in the time-varying observation matrix C_{σ_k}.

It is known that the stability analysis for a time-varying system is usually much more involved than that of a time-invariant system. Since the focus of this work is on quantifying the effect of the lossy network on the stability of the Kalman filter, we derive the stability condition of the Kalman filter without packet losses, which corresponds to $\gamma_k = 1$ for all $k \in \mathbb{N}$. By virtue of [9], a necessary and sufficient condition for the stability of the Kalman filter without packet losses is that (A, C_{σ_k}) is *uniformly detectable*. This requires the unstable modes of the system *uniformly observable* since all the state variables corresponding to the stable modes of the system will be exponentially stable in the mean square sense. For this purpose, we only need to focus on the unstable modes, and it is sensible to make the following assumption.

Assumption 1: All the eigenvalues of A lie outside or on the unit circle.

Then, (A, C_{σ_k}) is required to be uniformly observable under Assumption 1, i.e., there exist a positive integer h, and $\beta_0 > \alpha_0 > 0$ such that

$$\beta_0 I \succeq \sum_{i=k}^{k+h} (A^{i-k})^T C_{\sigma_i}^T C_{\sigma_i} A^{i-k} \succeq \alpha_0 I \succ 0, \forall k \in \mathbb{N}.$$

Since in this work we only consider a periodically switching rule, the above uniformly observable condition can be further simplified as stated in the following result.

Lemma 1: The system (A, C_{σ_k}) with σ_k given in (3) is *uniformly observable* if and only if both

$$\left(A^2, \begin{bmatrix} C_1 \\ C_2 A \end{bmatrix}\right) \text{ and } \left(A^2, \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix}\right) \quad (8)$$

are observable.

Moreover, if A is nonsingular, the observability property of the following systems

$$\left(A^2, \begin{bmatrix} C_1 \\ C_2 A \end{bmatrix}\right) \text{ and } \left(A^2, \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix}\right) \quad (9)$$

are equivalent.

Proof: The first part directly follows from the definition of observability [10]. We only need to elaborate the second part. Let the observability test matrices $\mathcal{C}_1 = [C_1; C_2 A; \ldots; C_1 A^{2(n-1)}; C_2 A^{2n-1}]$ and $\mathcal{C}_2 = [C_1 A; C_2; \ldots; C_1 A^{2n-1}; C_2 A^{2(n-1)}]$. Consider $\mathcal{C}_1$ and $\mathcal{C}_2 A$, it is clear that the rows of both matrices associated with C_2 are the same. By the Cayley-Hamilton theorem, there exist $a_i \in \mathbb{R}$ such that $A^{2n} = a_0 I + a_1 A^2 + \ldots + a_{n-1} A^{2(n-1)}$. Pre-multiply both sides of the equality by C_1, it follows that the last row of $\mathcal{C}_2 A$ associated with C_1 can be linearly represented by the rows of $\mathcal{C}_1$. Then, it is not difficult to verify that each row of $\mathcal{C}_2 A$ can be represented by the rows of $\mathcal{C}_1$. This implies that $\text{rank}(\mathcal{C}_2 A) \leq \text{rank}(\mathcal{C}_1)$. Similarly, one can argue that $\text{rank}(\mathcal{C}_1 A) \leq \text{rank}(\mathcal{C}_2)$. Since A is nonsingular, it obviously holds that $\text{rank}(\mathcal{C}_1) = \text{rank}(\mathcal{C}_1 A)$ and $\text{rank}(\mathcal{C}_2) = \text{rank}(\mathcal{C}_2 A)$. Combing the above, we obtain that $\text{rank}(\mathcal{C}_1) = \text{rank}(\mathcal{C}_2)$, which completes the proof. ■

Thus, the uniform observability property of the periodically switching system is converted into that of a time-invariant system using two sensors at each time.

In general, the non-singularity condition of A is mild, e.g. it holds for all systems satisfying Assumption 1. To this purpose, we focus on the system satisfying the following observability property in this paper since it is required for the stability of the Kalman filter without packet loss if A satisfies Assumption 1.

Assumption 2: Let $C = [C_1 A; C_2]$, the system (A^2, C) is observable.

Remark 1: By the PHB test [10], the observability of (A^2, C) implies that (A, C) is observable while the observability of (A, C) usually does not imply that (A^2, C) is observable. For instance, $A = \text{diag}(1, -1)$ and $C = [1\ 1]$. This, together with Lemma 1, essentially indicates that using two sensors to observe the system at each time requires a weaker condition for the stability of the Kalman filter than that of periodically using one sensor at each time, which certainly is consistent with our intuition as the former case obtains more information than the later. We also mention that the observability of

$$(A^2, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix})$$

does not imply that of (A^2, C). For example, $A = \text{diag}(1, -1)$, $C_1 = [1\ 1]$ and $C_2 = [1\ \ -1]$. It should be noted that both (A, C_1) and (A, C_2) are observable.

IV. Stability of the Kalman filter with packet losses

In this section, we derive the network condition on the packet loss process γ_k required for the stability of the Kalman filter using two periodically sensors.

Denote the packet receival rate $p = \mathbb{P}\{\gamma_k = 1\}$. In view of Theorem 4 in [1], one can easily establish the following necessary condition.

Theorem 1: Consider the first networked system in Fig. 1, a necessary condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$|\lambda_{\max}|^2(1-p) < 1$, where $\lambda_{\max}$ is the maximum eigenvalue in magnitude of A.

In fact, the above necessary condition has been derived by many authors [1], [2], [11], [12] using a single sensor, and shown to be sufficient under some special cases as well. It is more interesting to investigate whether this condition is sufficient. For a time-invariant observation matrix, i.e., $C_1 = C_2$, it is shown that it is also sufficient if C_1 is invertible on the observable subspace [12] or (A, C_1) is a non-degenerate system [1]. Note that the periodic switching between two stable subsystems may lead to an unstable system due to the destabilizing effect induced by switching. For example, one can verify that the system $x_{k+1} = A_k x_k$ is internally unstable where

$$A_k = \frac{1}{8} \cdot \begin{bmatrix} 0 & 9+7\cdot(-1)^k \\ 9-7\cdot(-1)^k & 0 \end{bmatrix},$$

although A_k has all eigenvalues inside the unit circle for each k.

This intuitively implies that the sufficiency is more involved for the time-varying observation matrices.

In the previous section, the stability condition of the Kalman filter using two periodically switching sensors is lifted into that of a time-invariant system with two measurement sensors if there is no packet loss. This will motivate us to check whether under i.i.d. packet losses, the problem under consideration can be converted into the study of stability condition of the Kalman filter for a time-invariant system using two measurement sensors, each of which is subject to an i.i.d. packet loss process. It turns out to be positive.

Theorem 2: Consider the first networked system in Fig. 1 satisfying Assumption 1, a sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$\mathbb{E}\left(\sum_{i=0}^{\infty} \zeta_i (A^{-2i})^T \begin{bmatrix} A^T C_1^T & C_2^T \end{bmatrix} \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i}\right)^{-1} \prec \infty, \tag{10}$$

where ζ_i is an i.i.d. process with $\mathbb{P}\{\zeta_i = 1\} = p^2$.

To elaborate it, we recall a result in [1].

Lemma 2: [1] Let $\mathcal{O} = \sum_{i=1}^{\infty} \gamma_i (A^{-i})^T C_{\sigma_i}^T C_{\sigma_i} A^{-i}$. Under Assumption 1, there exists two positive numbers α and β such that

$$\beta \mathbb{E}[\mathcal{O}^{-1}] \succeq \sup_{k\in\mathbb{N}} \mathbb{E}[P_{k|k}] \succeq \alpha \mathbb{E}[\mathcal{O}^{-1}]. \tag{11}$$

Proof of Theorem 2:
Note that $P_{k|k-1} \succeq P_{k|k}$ and $P_{k+1|k} = AP_{k|k}A^T + Q$, it is obvious that $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is equivalent to $\sup_{k\in\mathbb{N}} \mathbb{E}[P_{k|k}] \prec \infty$. By Lemma 2, we only need to establish the network condition such that

$$\mathbb{E}[\mathcal{O}^{-1}] \prec \infty. \tag{12}$$

Since γ_k is an i.i.d. process, then $\mathcal{O}$ can also be rewritten by

$$\begin{aligned} \mathcal{O} &= \sum_{i=1}^{\infty} (A^{-2i})^T \begin{bmatrix} \gamma_{2i-1} A^T C_1^T & \gamma_{2i} C_2^T \end{bmatrix} \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i} \\ &\overset{d}{=} \sum_{i=1}^{\infty} (A^{-2i})^T \begin{bmatrix} \alpha_i A^T C_1^T & \beta_i C_2^T \end{bmatrix} \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i}, \end{aligned} \tag{13}$$

where $\overset{d}{=}$ means equal in distribution on both sides, and α_i, β_i are two i.i.d.Bernoulli process with the same statistics with γ_i, i.e., $\mathbb{E}[\alpha_i] = \mathbb{E}[\beta_i] = p$.

This can help us easily establish a sufficient condition for the stability of the Kalman filter with intermittent observations. To be precise, define $\zeta_i = \min\{\alpha_i, \beta_i\}$, which is again an i.i.d. process with $\mathbb{P}\{\zeta_i = 1\} = \mathbb{P}\{\alpha_i = 1\}\mathbb{P}\{\beta_i = 1\} = p^2$.

Combing Lemma 2 and (13), the proof is completed. ■

By Theorem 1 and 2, a simple necessary and sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is obtained.

Theorem 3: Consider the first networked system in Fig. 1 satisfying Assumption 1 and 2. If C is of full rank, a sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$|\lambda_{\max}|^4(1-p^2) < 1. \tag{14}$$

Proof: Since $|\lambda_{\max}|^2(1-p) < 1$, there exists a sufficiently small $\epsilon > 0$ such that $(|\lambda_{\max}| + \epsilon\|A\|)^4(1-p^2) < 1$. Let $\rho = |\lambda_{\max}| + \epsilon\|A\|$, it follows from Lemma 15 [1] that $\|A\|^k \le M\rho^k$ for any $k \in \mathbb{N}$, where $M = \sqrt{n}(1+2/\epsilon)^{n-1}$.

If C is of full rank, it holds that $C^TC \succ \lambda_{\min}(C^TC)I$, where $\lambda_{\min}(C^TC) > 0$ is the minimum eigenvalue of C^TC. This implies that

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{\infty} \zeta_i (A^{-2i})^T C^T C A^{-2i}\right)^{-1} \\ &\prec \frac{1}{\lambda_{\min}(C^TC)} \mathbb{E}\left(\sum_{i=0}^{\infty} \zeta_i (A^{-2i})^T A^{-2i}\right)^{-1} \end{aligned} \tag{15}$$

Note that $\mathbb{P}\{\zeta_1 = 0, \ldots, \zeta_k = 0, \ldots\} = \lim_{k\to\infty}(1-p^2)^k = 0$. Then, the sum $\sum_{i=0}^{\infty} \zeta_i (A^{-2i})^T A^{-2i}$ is positive definite with probability one.

Define a stopping time τ as follows, i.e.,

$$\tau := \inf\{k \in \mathbb{N} | \zeta_k = 1\}, \tag{16}$$

whose probability mass distribution is given by $\mathbb{P}\{\tau = k+1\} = p^2(1-p^2)^k$. Hence,

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{\infty} \zeta_i (A^{-2i})^T A^{-2i}\right)^{-1} \le \mathbb{E}[\zeta_\tau A^{2\tau}(A^{2\tau})^T] \\ &\le (\mathbb{E}[\|A\|^{4\tau}])I \le (M \cdot \mathbb{E}[\rho^{4\tau}])I \\ &= M\rho^4(1-p^2) \sum_{k=0}^{\infty} \rho^{4k}(1-p^2)^k \cdot I, \end{aligned}$$

which is finite since $\rho^4(1-p^2) < 1$. The rest of the proof follows from Theorem 2. ■

Remark 2: The main conservativeness of the sufficient condition lies in the use of Theorem 2. We use a simple example to illustrate the conservativeness, where $A = \mathrm{diag}(\lambda_1, -\lambda_1)$, and $C_1 = C_2 = [1, 1]$. By [1], the necessary and sufficient condition is that $|\lambda_1|^4(1-p) < 1$, which is still weaker than $|\lambda_1|^4(1-p^2) < 1$.

In fact, it follows from Lemma 2 and (13) that $\mathbb{E}[\mathcal{O}^{-1}] \prec \infty$ is equivalent to the stability condition of the Kalman filter of the networked system

$$x_{k+1} = A^2 x_k + w_k, \tag{17}$$

which is observed by two sensors at each time by measurement equations

$$\begin{aligned} y_{k,1} &= C_1 A x_k + v_{k,1}, \\ y_{k,2} &= C_2 x_k + v_{k,2}. \end{aligned} \tag{18}$$

In the above, $v_{k,1}$ and $v_{k,1}$ are two independent white Gaussian noises. The sensor measurements $y_{k,1}$ and $y_{k,2}$ are sent via two independent lossy channels, whose packet loss processes are modeled by two independent process α_k and β_k, respectively. Then, the corresponding Kalman filters to the networked systems in Fig. 1 have the same network condition if A is nonsingular. Thus, it is sufficient to establish the network condition required for the stability of the Kalman filter of the second networked system in Fig. 1.

In general, it is challenging to establish the necessary and sufficient condition for a general vector system. Nonetheless, the following procedures can help to reduce the complexity of the problem. Motivated by [1], we will exploit the system structure under Assumption 2, which is classified as follows.

1. Both $(A^2, C_1 A)$ and (A^2, C_2) are observable.
2. Only one of $(A^2, C_1 A)$ and (A^2, C_2) is observable.
3. Neither $(A^2, C_1 A)$ or (A^2, C_2) is observable.

In fact, it only needs to consider Case 1 since the other two cases can be converted into the combination of Case 1 and that in [2], [5].

For Case 2, it does not lose generality to assume that $(A^2, C_1 A)$ is observable but (A^2, C_2) is not observable. By Kalman canonical decomposition [10], there exists a coordinate transformation such that (A^2, C) is transformed into the following structure

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, C_1 A = [C_{1,1} \ C_{1,2}], C_2 = [0 \ C_{2,2}]. \tag{19}$$

This intuitively means that the state variables corresponding to $A_{1,1}$ is only observed by the sensor with measurement matrix $C_1 A$, and independent of the other sensor measurements[2]. Then, the stability study for the state variables of this part is the same as the case of using only one sensor as in [1], [2], [5]. The new thing is to establish the stability condition for the complement state variables associated with $A_{2,2}$ which are observed by two sensors at each time.

For Case 3, it follows from Proposition III.1 in [13] that there exists a coordinate transformation such that (A^2, C) has the structure either

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, C_1 A = [0 \ C_{1,2}], C_2 = [C_{2,1} \ 0] \tag{20}$$

or

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{bmatrix} \tag{21}$$

$$C_1 A = [0 \ C_{1,2} \ C_{1,3}], C_2 = [C_{2,1} \ 0 \ C_{2,3}]. \tag{22}$$

The first structure indicates that the measurement matrix C_1 can only observe the state subspace corresponding to $A_{2,2}$ and C_2 observes the complement state subspace. While in the second structure, both sensors can observe a common subspace corresponding to $A_{3,3}$. The decomposition in the first structure is very appealing as it helps us to convert the problems under consideration into the case with only an observation matrix, which has been considered in [1], [2], [5]. In the second structure, the common observable subspace associated with $A_{3,3}$ is observed by both sensors, whose stability condition has not be established in the literature.

To sum up, to consider systems satisfying Assumptions 1 and 2, we only need to study the network condition for stability of the Kalman filter over two independent lossy channels for the system satisfying the following assumption.

Assumption 3: Both $(A^2, C_1 A)$ and (A^2, C_2) are observable.

A. Second-order System

Together with [1], we are able to fully characterize the necessary and sufficient condition for the stability of the Kalman filter using two periodically switching sensors over a lossy network for the second-order system, i.e. $A \in \mathbb{R}^{2\times 2}$.

Assumption 4: $A = \text{diag}(\lambda_1, \lambda_2)$ where $\lambda_1 = \lambda_2 \exp(2\pi r I/d)$, $I^2 = -1$ and $d > r > 0$ are irreducible integers.

Theorem 4: Consider the second-order networked system in Fig. 1 satisfying Assumption 3-4. If C or $[C_1; C_2]$ is rank deficient, then a necessary and sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$|\lambda_{\max}|^{2d/(d-1)}(1-p) < 1. \tag{23}$$

Proof: If C is rank deficient, there exist $a \in \mathbb{R}$ such that $C_1 A = a \cdot C_2$. Consider the second networked system in Fig. 1, it is equivalent to the system observed by one sensor but the packet loss process is given by an i.i.d. process $\xi_i = \max\{\alpha_i, \beta_i\}$ with $\mathbb{P}\{\xi_i = 0\} = (1-p)^2$. This is because the measurements from both sensors are the same except for a scaling by a. If $[C_1; C_2]$ is rank deficient, it is equivalent to the case without switching. Then, the rest of the proof follows from [1]. ∎

Note that the study of general vector systems under Assumption 3 is generically challenging and left to our future work. However, if A is with certain form, the necessary and sufficient condition for the stability of the Kalman filter can be easily established.

Assumption 5: $A^{-1} = \text{diag}(J_1, \ldots, J_m)$ and $\text{rank}(C_1) = \text{rank}(C_2) = 1$, where $J_i = \lambda_i^{-1} I_i + N_i \in \mathbb{R}^{n_i \times n_i}$ and $|\lambda_i| > |\lambda_{i+1}|$. I_i is an identity matrix with a compatible dimension and the (j,k)-th element of N_i is 1 if $k = j+1$ and 0, otherwise.

Theorem 5: Consider the first networked system in Fig. 1 satisfying Assumptions 1 - 4. Then, a necessary and sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$|\lambda_{\max}|^2(1-p) < 1. \tag{24}$$

Proof: It can be proved by following a similar line as that of Theorem 13 in [1]. ∎

[2]The rigorous proof shall be included in the full version of the paper.

V. Conclusion

Motivated by the necessity of using switching sensors in the networked system, we have examined the stability of Kalman filtering with i.i.d. packet losses. Some necessary and sufficient conditions have been derived, which are able to characterize the effect of the periodically switching sensors on the stability. It is stressed that the result of this work is very preliminary and the problem requires further investigation.

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