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Kalman Filtering with Intermittent Observations Using Measurements Coding

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Abstract— **This paper studies the state estimation problem of a stochastic discrete-time system over a lossy channel. The packet loss is modeled as an independent and identically distributed (i.i.d.) binary process. To reduce the effect of the random packet losses on the stability of the minimum mean square error estimator, we propose a linear coding method on the measurement of the system. In particular, the linear combination of the current and finite previous measurements is to be transmitted to the estimator over the lossy channel. Some necessary and sufficient conditions for the stability of the estimator are established, and the advantage of the linear coding method is exploited.**

Keywords: Stochastic linear systems; Kalman Filter; Packet loss; Linear coding method; Stability;

I. Introduction

This work proposes a linear coding method to improve the stability condition of the minimum mean square error (MMSE) estimator with random packet losses. The study of the estimation problem of networked systems over a lossy channel has received significant attention in the recent years [1]–[3]. This is motivated by the rapid development of the sensing, communication, and signal processing technologies, which enable the development of the large-scale systems for broad applications such as monitoring, detection, tracking, etc.

Sinopoli et al [1] studies the stability of the Kalman filter under the random packet losses of the raw measurements. By modeling the packet loss process as an independent and identically distributed (i.i.d.) process, they prove the existence of a critical packet loss rate above which the mean state estimation error covariance matrices will diverge. For a general system, they are unable to exactly quantify the packet loss rate and only give an upper and lower bound of it. Motivated but also inspired by the limitation of [1], You, Fu and Xie [4] extend the i.i.d. packet loss model to a Markovian packet loss model, and study the stability of the Kalman filter by exploiting the system structure, from which they derive the necessary and sufficient condition for the stability of Kalman filtering of the second order systems and some special higher-order systems. The establishment of the stability condition for Kalman filtering with lossy raw measurements is usually quite involved. In particular, it still lacks a uniform way to explicitly characterize the necessary and sufficient condition for the stability of Kalman filtering with lossy raw measurements.

Then, Schenate [5] focuses on the stability of the minimum mean square error estimator over the lossy channel by transmitting the output of the Kalman filter in the transmitter side. This leads to a simple necessary and sufficient condition for the stability of the optimal estimator, and implies that the stability of the optimal estimator with packet loss is improved by transmitting the pre-processed measurements. However, the dimension of the estimated state is usually larger than that of raw measurement, which needs more communication resource for the successful transmission, and the estimation process requires more computation capacity. Note that the bandwidth of wireless network is limited and the wireless sensor is unable to be charged.

In this paper, we introduce a simple linear coding method that transmits a linear combination of the current and previous finite measurements to the estimator via the lossy channel. Since the dimension of the linearly coded measurement is the same as raw measurements, it does not increase the communication cost. The required computation power for coding is also less than that of running a Kalman filter. However, the effect of linear coding method on the stability of the MMSE estimator is explored. The good news is that the stability condition transmitting the linearly coded measurements would be weaker than that of the raw measurements, and can even approach the necessary and sufficient condition by transmitting the output of the Kalman filter in the transmitter side.

The rest of the paper is organized as follows. The problem under consideration is formulated in Section 2. In Section 3, the linear coding method is introduced and its improvement of stability is revealed. In Section 4, by transmitting the coded measurements, two main results for stability are given. Base on the results, to a given system, there is a method to design the coding vectors that guarantee the estimation stability. Concluding remarks are drawn in Section 5. To improve the readability of the paper, all the proofs are moved to Appendix.

II. PROBLEM FORMULATION

Consider the following discrete-time stochastic system

$$x_{k+1} = Ax_k + \omega_k; \quad (1)$$
$$y_k = Cx_k + \nu_k, \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$ are system matrices, and A is invertible. $x_k \in \mathbb{R}^n$ denotes the system state at time

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k, and ω_k is a white Guassian noise with zero mean and positive definite covariance matrix Q. The initial state x_0 is a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix P_0. $y_k \in \mathbb{R}^q$ is the measurement vector at time k, and ν_k is a white Guassian noise with zero mean and positive definite covariance matrix R.

We are concerned with a networked system where the measure sensor and the estimator are linked via a lossy channel. Due to the network unreliability, packets from the sensor to the estimator may be lost while in transit. To quantify the effect of packet losses on the estimation performance, we use a binary random process γ_k to denote the packet loss process. Precisely, let $\gamma_k = 1$ indicate that the packet transmitted from the sensor has been successfully delivered to the estimator, while $\gamma_k = 0$ corresponds to the loss of the packet. Here packet loss process is modeled as an i.i.d. process with a packet receival rate $p \in (0, 1)$, e.g. $p = E[\gamma_1]$.

In the scenario of [1], [4], the raw measurement y_k is directly to be transmitted to the estimator at each time because of the very limited bandwidth and computing capability of the sensor. Then, it is clear that the maximum information available to the estimator at time k is given as follows

$$\mathcal{F}_k = \{\gamma_i, y_i\gamma_i, i \leq k\}.$$

Denote the minimum mean square error predictor and estimator by $\hat{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{F}_{k-1}]$ and $\hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{F}_k]$, respectively. Their corresponding estimation error covariance matrices are then given by $P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T|\mathcal{F}_{k-1}]$ and $P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|\mathcal{F}_k]$. In view of [1], the intermittent Kalman Filtering equation is

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + \gamma_k K_k(y_k - C\hat{x}_{k|k-1}); && (3) \\ P_{k|k} &= P_{k|k-1} - \gamma_k K_k C P_{k|k-1}, && (4) \end{aligned}$$

where the Kalman gain $K_k = P_{k|k-1}C^T(CP_{k|k-1}C^T + R)^{-1}$.

Let $\widetilde{P}_k := P_{k|k-1}$, it is recursively updated as follows:

$$\begin{aligned} \widetilde{P}_{k+1} &= A\widetilde{P}_k A^T + Q && (5) \\ &\quad -\gamma_k A\widetilde{P}_k C^T(C\widetilde{P}_k C^T + R)^{-1}C\widetilde{P}_k A^T. \end{aligned}$$

Their goal is to investigate the effect of the packet loss process γ_k on the mean square stability of the filter, i.e., $\sup_{k\in\mathbb{N}} \mathbb{E}[\widetilde{P}_k] < \infty$, where the mathematical expectation is taken with respect to the packet loss process γ_k. Although it has attracted considerable attention from the researchers, e.g. [1], [4], [6], [7], the necessary and sufficient condition for the packet loss process on the stability of the filter is yet to be explicitly characterized, and turns out to closely rely on the system structure. In particular, it depends on the present form of each mode of the open loop system.

To reduce the effect of the packet losses on the stability of the filter, Schenato [5] considers the situation that the sensor has sufficient bandwidth and computing power to estimate the state by using the Kalman filter at the transmitter side, and the state estimator, i.e., $\hat{x}_k^{KF} = \mathbb{E}[x_k|y_1, \ldots, y_k]$ is to be transmitted from the sensor to the estimator over the lossy channel. Then, the optimal estimator in the receiver side is given as follows

$$\hat{x}_k^o = \begin{cases} \hat{x}_k^{KF}, & \text{if } \gamma_k = 1; \\ A\hat{x}_{k-1}^o, & \text{if } \gamma_k = 0. \end{cases} \quad (6)$$

If (A, C) is observable, the necessary and sufficient for the stability of the estimator, i.e., $\sup_{k\in\mathbb{N}} \mathbb{E}[(x_k - \hat{x}_k^o)(x_k - \hat{x}_k^o)^T] < \infty$ is simply given by $|\lambda_{\max}|^2(1 - p) < 1$, where $\lambda_{\max}$ is the maximum eigenvalue of A in magnitude. It should be noted that the transmission of $\hat{x}_k^{KF}$ requires a large communication overhead since the dimension of the state is usually higher than that of the system measurement, and usually results in a larger packet loss rate.

Motivated the above, the goal of the current work is to propose a linear coding algorithm to balance the above scenarioes. In particular, the dimension of the transmitting message is the same as that of raw measurement y_k but the effect of the random packet loss on the stability of the estimator is reduced by comparing with transmitting y_k. A natural idea is to transmit a linear combination of y_k and the previous measurements at time k, which is exactly to be investigated in this paper.

III. LINEAR CODING OF MEASUREMENTS

Suppose that x measurements $y_k, y_{k+i_1}, \ldots, y_{k+i_{x-1}}$ are received respectively at time $k, k+i_1, \ldots, k+i_{x-1}, 1 \leq i_1 < \ldots < i_{x-1}$. Define a matrix by:

$$O(k) = \begin{bmatrix} C \\ CA^{i_1} \\ \ldots \\ CA^{i_{x-1}} \end{bmatrix} \quad (7)$$

Because of possible packet losses, the measurements received by estimator are usually not one-by-one, i.e., $i_{j+1} - i_j \neq 1$. It can be easily proved that if $O(k)$ is full column rank and $i_{x-1} < \infty$, one can obtain an estimator by using the measurements $y_k, y_{k+i_1}, \ldots, y_{k+i_{x-1}}$, and the estimation error covariance is uniformly bounded with respect to k. However, if $O(k)$ is rank deficient, this is impossible. That is, the full rankness of $O(k)$ is essential to the stability of the filter.

We note that whether $O(k)$ is full rank depends on the structure of (A, C) and that of eigenvalues of A, e.g. the existence of a positive integer d such that $\lambda_i^d = \lambda_j^d$, where λ_i and λ_j denote two distinct eigenvalues of A [4]. For example, consider a second-order system given below:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x_k + \omega_k, && (8) \\ y_k &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + \nu_k. && (9) \end{aligned}$$

The eigenvalues of the system are 1 and -1. Then $d = 2$ as $1^2 = (-1)^2$. Suppose that the measurements are received at time $k, k+2, k+4$, the measurement matrix

$$O(k) = \begin{bmatrix} C \\ CA^2 \\ CA^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It still not full column rank after receiving three measurements. To eliminate this issue, the sensor is designed to transmit a linear combination of y_k and the previous measurements.

For the n-dimensional system (1), define a random coding vector $\alpha_k = \begin{bmatrix}\alpha_{k1} & \alpha_{k2} & \dots & \alpha_{kn}\end{bmatrix}$ where each element of α_k is independently generated from a standard Gaussian distribution. The property of coding vector in Lemma 1 is obvious.

Lemma 1: Suppose $\alpha_1, \alpha_2, \dots, \alpha_n$ are all n-dimensional random column vectors and independently generated. Then, the matrix $(\alpha_1^T, \alpha_2^T, \dots, \alpha_n^T)$ is full rank with probability 1. Suppose that α_k is the coding vector at time k, i.e., the coded measurement

$$\tilde{y}_k = \alpha_{k1} y_{k-n+1} + \alpha_{k2} y_{k-n+2} + \dots + \alpha_{kn} y_k \tag{10}$$

is to be transmitted at time k. Compared with transmitting the estimated state, the dimension of the transmitted data is of the same length as that of the raw measurement. It always less than the dimension of x_k if $q < n$ and the coding method consumes less computation power. By transmitting the coded measurements, the condition to guarantee the stability of the filter is weaker than transmitting raw measurements. To elaborate it, we write the observation equation as

$$\tilde{y}_k = \alpha_k \begin{bmatrix} CA^{-n+1} \\ CA^{-n+2} \\ \dots \\ C \end{bmatrix} x_k + n_k, \tag{11}$$

where n_k denotes the measurement noise to the remote estimator at time k. It is a linear combination of $\omega_{k-n+1}, \dots, \omega_k$ and ν_k. Suppose that n coded measurements $\tilde{y}_k, \tilde{y}_{k+i_1}, \dots, \tilde{y}_{k+i_{n-1}}$ are received respectively at time $k, k+i_1, \dots, k+i_{n-1}, 1 \le i_1 < \dots < i_{n-1}$. Similarly, we obtain the following matrix

$$\tilde{O}(k) = \begin{bmatrix} \alpha_k \begin{bmatrix} CA^{-n+1} \\ CA^{-n+2} \\ \dots \\ C \end{bmatrix} \\ \alpha_{k+i_1} \begin{bmatrix} CA^{-n+1+i_1} \\ CA^{-n+2+i_1} \\ \dots \\ CA^{i_1} \end{bmatrix} \\ \dots \\ \alpha_{k+i_{n-1}} \begin{bmatrix} CA^{-n+1+i_{n-1}} \\ CA^{-n+2+i_{n-1}} \\ \dots \\ CA^{i_{n-1}} \end{bmatrix} \end{bmatrix}. \tag{12}$$

If $\tilde{O}(k)$ is full column rank, an estimation algorithm for the state x_k using the coded measurements can be available, and the estimation error covariance matrix is uniformly bounded with respect to k if $i_{n-1} < \infty$. By using the random coding vectors, $\tilde{O}(k)$ will be full column rank with probability one.

Lemma 2: Suppose that the coding vector at each time is independently generated from a standard Gaussian random vector, and (A, C) is observable, then $\tilde{O}(k)$ is full column rank with probability one.

Proof: It is given in Appendix. ■

In fact, x_k can be optimally estimated in the sense of minimizing the mean square estimation error by using the least square method as follows[1]

$$\hat{x}_k = \tilde{O}(k)^+ \begin{bmatrix} \tilde{y}_k \\ \tilde{y}_{k+i_1} \\ \dots \\ \tilde{y}_{k+i_{n-1}} \end{bmatrix}, \tag{13}$$

where the superscript $^+$ denotes the Moore-Penrose pseudo-inverse [8]. Let the corresponding estimation error covariance matrix be

$$P_k = \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | \gamma_1 \tilde{y}_1, \gamma_1, \dots, \gamma_k \tilde{y}_k, \gamma_k].$$

Now, we revisit the second-order system in (8-9). With the un-coded measurements, $O(k)$ is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

if measurements are received at time $k, k+2$ or $k+1, k+3$. Then, it is obvious that $O(k)$ is rank deficient. However, if a set of coding vector

$$\begin{bmatrix} \alpha_k \\ \alpha_{k+1} \\ \alpha_{k+2} \\ \alpha_{k+3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$$

is chosen for the linear coding. $O(k)$ is modified into that

$$\tilde{O}(k) = \begin{bmatrix} 3 & 0 \\ 5 & -1 \\ 5 & 2 \\ 9 & -3 \end{bmatrix},$$

which is obviously full column rank if measurements are received at time $k, k+2$ or $k+1, k+3$. Note that the transmission of the coding measurement has the same dimension as the raw measurement.

IV. Stability Analysis

In this section, we derived the stability condition for the estimator using the coded measurements.

As illustrated in the previous section, the effect of random packet losses on the stability of the estimator can be reduced by using the linear coded measurements. This is formalized in Theorem 1, whose proof is given in Appendix.

Theorem 1: Suppose that the coding vector at each time is independently generated from a standard Gaussian random vector and (A, C) is observable, a necessary and sufficient condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[\|x_k - \hat{x}_k\|^2] < \infty$ is that

$$|\lambda_{\max}|^2 (1-p) < 1. \tag{14}$$

[1]Note that $\hat{x}_k$ can also be written in a recursive form. Due to page limitation, it is not explicitly included here.

The transmission of randomly coded measurement is sufficient to achieve the same stability condition as that of the output of a Kalman filter, which requires the computational capacity of the sensor and increase the communication overhead. However, the assumption that each coding vector is randomly generated may be unreasonable since to estimate from the coded measurements, the estimator must know the sensor's all coding vectors. If the sensor generates coding vectors online, it is difficult for the estimator to learn them without inducing any communication cost. An instant idea is to use the coding vectors generated offline and stored in sensor and estimator in advance. Thus the number of coding vectors should be finite. To this purpose, we use the periodical coding vectors with a period m, i.e., $\alpha_k = \alpha_{k+m}, \forall k \in \mathbb{N}$ and in each period, the coding vectors are different between each other. Then the random assumption in Lemma 1 and Theorem 1 does not hold here.

The new establishment of the stability condition for the estimation with periodic coding vectors is thus required.

Lemma 3: Suppose that $\alpha_1, \dots, \alpha_m$ are independently generated from standard Gaussian random vector and $\alpha_k = \alpha_{k+m}$ for all $k \in \mathbb{N}$(m is the period of coding vectors), and (A, C) is observable. A sufficient condition for $\text{rank}(\tilde{O}(k)) = n$ with probability 1 is that

$$i_{n-1} < m. \tag{15}$$

Proof: Please refer to Appendix. ∎

Base on Lemma 1 and 3, a sufficient condition for the stability of $\hat{x}_k$ is presented in Theorem 2.

Theorem 2: Suppose that $\alpha_1, \dots, \alpha_m$ are independently generated from standard Gaussian random vector and $\alpha_k = \alpha_{k+m}$ for all $k \in \mathbb{N}$(m is the period of coding vectors), and (A, C) is observable. A sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[\|x_k - \hat{x}_k\|^2] < \infty$ is that

$$|\lambda_{\max}|^2 (1-p) \sqrt[m]{P(m)} < 1, \tag{16}$$

where $P(m) = \sum_{i=1}^{n-1} \binom{m}{i} (\frac{p}{1-p})^i, m > n$ and $\binom{m}{i}$ is the number of combinations that select i from m.

Proof: Please refer to Appendix. ∎

By transmitting the output of the Kalman filter, the necessary and sufficient condition for the stability of the estimator in the receiver side is that $|\lambda_{\max}|^2(1-p) < 1$. Since $\lim_{m\to\infty} \sqrt[m]{P(m)} = 1$, then $|\lambda_{\max}|^2(1-p) < 1$ becomes a necessary and sufficient condition as $m \to \infty$. This means that the estimator is bounded for any system satisfying that $|\lambda_{\max}|^2(1-p) < 1$ by using the linear coding on the measurements.

Base on the Theorem 2, the coding period m can be founded as follows. For a system with the property $|\lambda_{\max}|^2(1-p) = \varepsilon < 1$, letting $\sqrt[m]{P(m)} < \frac{1}{\varepsilon}$ leads to the solution of the coding period m. Because $\sqrt[m]{P(m)}$ is a decreasing function in m and $\lim_{m\to\infty} \sqrt[m]{P(m)} = 1$, there always exists a solution m for $\sqrt[m]{P(m)} < \frac{1}{\varepsilon}$.

V. Conclusion

We have introduced a linear coding method to reduce the effect of the packet losses on the stability of the optimal estimator in the receiver side of a lossy channel. The dimension of coded measurement is same as that of the raw measurement. Compared with the transmission of estimated state, the dimension of coded measurement is less and the calculation of coded measurement is simpler. With the random coding vectors, the necessary and sufficient condition for stability is achieved; With periodically random coding vectors, the sufficient condition for stability is achieved. Base on the derived sufficient condition, we are able to design a linear periodic coding method to guarantee the stability of the optimal estimator.

There are two important future research topics. One is to approach the sufficient condition in Theorem 2 to the necessary and sufficient condition. The other is to design deterministic coding vectors that also can improve stability and it won't be periodical.

Appendix

A. Proof of Lemma 2

Proof: Since (A, C) is observable, then

$$\begin{bmatrix} CA^{i-n+1} \\ CA^{i-n+2} \\ \dots \\ CA^{i} \end{bmatrix}$$

is full column rank. Suppose that n coded measurements are received at time $k, k+i_1, \dots, k+i_{n-1}$. Since $\alpha_k, \alpha_{k+i_1}, \dots, \alpha_{k+i_{n-1}}$ are independently generated from standard Gaussian random vector, then

$$\alpha_k \begin{bmatrix} CA^{-n+1} \\ CA^{-n+2} \\ \dots \\ C \end{bmatrix}, \dots, \alpha_{k+i_{n-1}} \begin{bmatrix} CA^{i_{n-1}-n+1} \\ CA^{i_{n-1}-n+2} \\ \dots \\ CA^{i_{n-1}} \end{bmatrix}$$

can be viewed as independently generated.

Base on the Lemma 1, it is not difficult to verify that

$$\begin{bmatrix} \alpha_k \begin{bmatrix} CA^{-n+1} \\ CA^{-n+2} \\ \dots \\ C \end{bmatrix} \\ \alpha_{k+i_1} \begin{bmatrix} CA^{i_1-n+1} \\ CA^{i_1-n+2} \\ \dots \\ CA^{i_1} \end{bmatrix} \\ \dots \\ \alpha_{k+i_{n-1}} \begin{bmatrix} CA^{i_{n-1}-n+1} \\ CA^{i_{n-1}-n+2} \\ \dots \\ CA^{i_{n-1}} \end{bmatrix} \end{bmatrix}$$

is of full column rank with probability one.

B. Proof of Theorem 1

Proof: 1) Sufficiency:

Suppose that n measurements are received at time $k, k+t_1, k+t_2, \dots, k+t_n$. The notion of stability in stopping time is $\sup_{n\in\mathbb{N}} \mathbb{E}[P_{k+t_n}] < \infty$.

By Theorem 6 in [4], the notion of stability in sampling time $\sup_{k\in\mathbb{N}}\mathbb{E}[P_k]<\infty$ and stability in stopping time are equivalent. Denote P_{rn} is the probability of recent n coded measurements make the matrix $\tilde{O}(k)$ full column rank. Then there exists a constant matrix $a>0$ such that

$\mathbb{E}[P_{k+t_{n-1}}]\le a\cdot\mathbb{E}[|\lambda_{\max}|^{2t_{n-1}}]\cdot P_{rn}+\mathbb{E}[|\lambda_{\max}|^{2t_{n-1}}]\cdot\mathbb{E}[P_{k+t_{n-1}}]\cdot(1-P_{rn})$

By Lemma 2, it follows that $P_{rn}=1$. This implies that

$$\mathbb{E}[P_{k+t_{n-1}}]\le a\cdot\mathbb{E}[|\lambda_{\max}|^{2t_{n-1}}]$$

Let $\triangle_i=t_i-t_{i-1}$ be the time interval between the measurement packet $i-1$ and i. We obtain that

$$\mathbb{E}[P_{k+t_{n-1}}]\le a\cdot\mathbb{E}[|\lambda_{\max}|^{2(\triangle_1+\triangle_2+\ldots+\triangle_{n-1})}]$$

Since the packet loss process is i.i.d, the $\triangle_i$ and $\triangle_j$ are independent for $i\neq j$. Then, it follows that $\mathbb{E}[|\lambda_{\max}|^{2\triangle_i}]=\mathbb{E}[|\lambda_{\max}|^{2\triangle_j}]$, and

$$\begin{aligned}\mathbb{E}[P_{k+t_{n-1}}] &\le a\cdot\mathbb{E}[|\lambda_{\max}|^{2\triangle_1}]\cdot\mathbb{E}[|\lambda_{\max}|^{2\triangle_2}]\\ &\cdot\ldots\cdot\mathbb{E}[|\lambda_{\max}|^{2\triangle_{n-1}}]\\ &= a\cdot\mathbb{E}[|\lambda_{\max}|^{2\triangle_1}]^{n-1}\end{aligned}$$

The quantity $\mathbb{E}[|\lambda_{max}|^{2\triangle_1}]$ can also be evaluated by

$$\begin{aligned}\mathbb{E}[|\lambda_{\max}|^{2\triangle_1}] &= p|\lambda_{\max}|^2+p(1-p)|\lambda_{\max}|^4\\ &+ p(1-p)^2|\lambda_{\max}|^6+\ldots\end{aligned}$$

Thus $\mathbb{E}[|\lambda_{\max}|^{2\triangle_1}]<\infty$ if and only if $(1-p)|\lambda_{\max}|^2<1$. Hence, it follows that $(1-p)|\lambda_{\max}|^2<1$ is a sufficient condition for $\sup_{k\in\mathbb{N}}\mathbb{E}[P_k]<\infty$.

2) Necessity. It is straightforward by [5].

C. Proof of Lemma 3

Proof: Because $i_{n-1}<m$, the coding vectors at time $k,k+i_1,k+i_2,\ldots,k+i_{n-1}$ are in the same period. Then, all the n coding vectors are randomly generated. Base on the Lemma 2, the matrix $\tilde{O}(k)$ is of full column rank with probability 1.

D. Proof of Theorem 2

Proof: Denote the time interval making the matrix $\tilde{O}(k)$ full column rank is T. Because the estimation error covariance is only associates with the noise as the recent measurements make the matrix $\tilde{O}(k)$ full column rank. The stability is equivalent to the estimation error covariance is finite between two observable measurement sequences.

$$\sup_{n\in\mathbb{N}}P_{k+t_{n-1}}<\infty\Leftrightarrow\mathbb{E}[|\lambda_{\max}|^{2T}]<\infty$$

We turn to find the sufficient condition for $\mathbb{E}[|\lambda_{\max}|^{2T}]<\infty$:

$$\mathbb{E}[|\lambda_{\max}|^{2T}]=\mathbb{E}[|\lambda_{\max}|^{2T}|0\le T<m]\cdot P_r(0,m)$$
$$+\mathbb{E}[|\lambda_{\max}|^{2T}|m\le T<2m]\cdot P_r(m,2m)+\ldots$$

Where $P_r(im,(i+1)m), i\in\mathbb{N}$ is the probability that $im\le T<(i+1)m$. Then

$$\mathbb{E}[|\lambda_{\max}|^{2T}]=\mathbb{E}[|\lambda_{\max}|^{2T}|0\le T<m]\cdot P_r(0,m)$$
$$+|\lambda_{\max}|^{2m}\cdot\mathbb{E}[|\lambda_{\max}|^{2(T-m)}|m\le T<2m]\cdot P_r(m,2m)+\ldots.$$

As $\mathbb{E}[|\lambda_{\max}|^{2(T-i\times m)}|(i-1)\times m\le T<i\times m]\le|\lambda_{\max}|^{2m},\forall k\in\mathbb{N}$. We could release the condition to a more sufficient one

$$\mathbb{E}[|\lambda_{\max}|^{2T}]<|\lambda_{\max}|^{2m}\cdot P_r(0,m)+|\lambda_{\max}|^{4m}\cdot P_r(m,2m)$$
$$+|\lambda_{\max}|^{6m}\cdot P_r(2m,3m)+\ldots.$$

Denote $\tilde{P}_r(im,(i+1)m)$ is the probability that there are n or more packets received between $im\le T<(i+1)m$. Because the event $im\le T<(i+1)m$ is a sufficient condition for there are no n packets received between $0\le T<m$, $m\le T<2m$, …and $(i-1)m\le T<im$. So

$$P_r(im,(i+1)m)\le\tilde{P}_r(0,m)\tilde{P}_r(m,2m)\cdots\tilde{P}_r((i-1)m,im).$$

The inequality change into

$$\mathbb{E}[|\lambda_{\max}|^{2T}]<|\lambda_{\max}|^{2m}\cdot 1+|\lambda_{\max}|^{4m}\cdot\tilde{P}_r(0,m)$$
$$+|\lambda_{\max}|^{6m}\cdot\tilde{P}_r(0,m)\tilde{P}_r(m,2m)+\ldots.$$

Because the packet loss is a i.i.d. binary process, the probability $\tilde{P}_r((i-1)m,im)=\tilde{P}_r((j-1)m,jm)$ for $i\neq j$. The inequality can be a geometric sequence form as below:

$$\mathbb{E}[|\lambda_{\max}|^{2T}]<|\lambda_{\max}|^{2m}\cdot 1+|\lambda_{\max}|^{4m}\cdot\tilde{P}_r(0,m)$$
$$+|\lambda_{\max}|^{6m}\cdot\tilde{P}_r^2(0,m)+\ldots+|\lambda_{\max}|^{2(i+1)m}\cdot\tilde{P}_r^i(0,m)+\ldots$$

Then research on the equivalent condition

$$\mathbb{E}[|\lambda_{\max}|^{2T}]<\infty\Leftrightarrow|\lambda_{\max}|^{2m}\cdot\tilde{P}_r(0,m)<1.$$

And the probability $\tilde{P}_r(0,m)=\Sigma_{i=1}^{n-1}\binom{m}{i}(p)^i(1-p)^{m-i}=(1-p)^m\Sigma_{i=1}^{n-1}\binom{m}{i}(\frac{p}{1-p})^i$. Where $\binom{m}{i}$ is the number of combinations that select i from m. Denote $P(m):=\Sigma_{i=1}^{n-1}\binom{m}{i}(\frac{p}{1-p})^i$, the sufficient condition for $\sup_{k\in\mathbb{N}}P_k<\infty$ is

$$|\lambda_{\max}|^2\cdot(1-p)\cdot\sqrt[m]{P(m)}<1$$

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