2017 13th IEEE International Conference on Control & Automation (ICCA)
July 3-6, 2017. Ohrid, Macedonia

Stability Analysis for Kalman Filters with Random Measurement Matrices

Damian Marelli[1,4], Tianju Sui[2], Minyue Fu[3,1]

Abstract— **This paper addresses the stability of a Kalman filter when measurements are intermittently available due to an unreliable communication channel between sensors and the estimator. This intermittent behaviour can be modelled as a random and time-varying measurement system. We consider a general discrete-time system with a random measurement matrix and study the stability condition for the associate Kalman filter. By deep dissecting the system structure, the necessary and sufficient stability condition for the Kalman filter is given. We also give methods to checking the stability condition for specific linear systems. Our results generalize previously known stability conditions where the system matrix structure is restricted.**

I. Introduction

Characterizing the behavior of a Kalman filter when measurements are intermittently available has attracted a great interest in the recent years. This is partly due to the development of communications technologies, which today permit distributed control and monitoring in a broad range of applications. When measurements sent through a communication channel are subject to random losses, the estimation accuracy of a Kalman filter deteriorates. In [1], the authors established the mathematical foundations for the estimation stability with measurements loss and pointed out that the covariance of the estimation error may not reach a steady state. Motivated and inspired by this, several authors have studied different aspects of the problem, using different assumptions on network models and protocols.

When a Kalman filter is subject to intermittent observations (KFIO), the error covariance (EC) matrix becomes of random nature. The study of the stochastic properties of the EC is a central issue for performance and stabitity analysis of KFIO. We call the asymptotic expected EC by AEEC and it is important for it provides a hard limit for the design of the components of the system, the communication channel, and the estimator. There exists a rich literature dedicated to finding the stability conditions of the KFIO [1]–[18]. Some authors adopt the stability criterion used in [1], namely, a KFIO is said to be stable if its AEEC is finite [1], and unstable otherwise. Other authors, adopt the concept of peak error covariance, introduced in [3]. More recently, the equivalence between the two notions of stability has been studied in [2], [4].

Despite there are many works focusing on the stability conditions of the KFIO, a necessary and sufficient stability condition for general system is still not available. Most partial answers depend on a particular structure of the system or offer a sufficient not necessary condition. Meanwhile, there are mainly two kinds of packet loss model: The first one considers the dropouts as a sequence of independent and identically distributed (i.i.d.) binary random variables. The second one is known as the Gilbert-Elliott model [19], [20], and models the dropouts using a first order Markov process. To make the packet loss model better describe the packet dropouts, a stationary higher order Markov process (also known as finite state Markov channel (FSMC) [21]) is introduced. In the context of KFIO, this network model has been studied in [22]. Notice that the i.i.d. and the Gilbert-Elliott network models are particular cases of the FSMC model.

A recent result [23] considered the case that the measurements are dropped according to a FSMC model. This model is more general than the i.i.d. and the Gilbert-Elliott models and it permits modeling more complex channels with memory and fading [21]. Focusing on the degenerate systems with its dynamics matrix $\mathbf{A}$ is diagonalizable, they gave out the necessary and sufficient condition for the stability of the KFIO.

In this paper, we tried to get the general necessary and sufficient condition for stability of the associated Kalman filter. We generalizes the result in [23] and the contributions are itemized below:

- The state transition matrix A is general, i.e., not just diagonalizable.
- The measurement equation (i.e., the measurement matrix $\mathbf{C}_t$ and the measurement noise covariance $\mathbf{R}_t$) is totally random, and not just intermittently random.
- We generalized the finite-order Markov assumption on the random model for the measurement equation (formerly the statistical model of the channel). The new condition is (10).
- We generalized the stationary requirement for the statistics of the measurement equation to cyclo-stationary. This requires the introduction of a more suitable notion of stability.

The paper is organized as follows. Section II states the problem formulation. The main result is presented in Section III, as well as the random model of measurement. In Section IV, we explain the generalization of our measurement random model. The computation method of a critical item in stability condition (Theorem 3.1) is given in Section V. We draw our conclusions in Section VII.

II. Problem formulation

Consider the discrete-time linear system

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{w}_t, \tag{1}$$

$$\mathbf{y}_t = \mathbf{C}_t\mathbf{x}_t + \mathbf{v}_t, \tag{2}$$

978-1-5386-2679-5/17/$31.00 ©2017 IEEE

where $\mathbf{x}_t \in \mathbb{C}^n$ is the vector of states, $\mathbf{y}_t \in \mathbb{R}^p$ is the vector of measurements, $\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, with $\mathbf{Q} \geq 0$, is the process noise, $\mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$ with $\mathbf{R}_t \geq 0$, is the measurement noise, $\mathbf{A} \in \mathbb{C}^{n\times n}$ is the state matrix and $\mathbf{C}_t \in \mathbb{C}^{p\times n}$ is the measurement matrix at time t. It is assumed, without loss of generality, that $\mathbf{A}$ is in Jordan normal form. The initial state is $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_0)$, with $\mathbf{P}_0 \geq 0$. Also, $\mathbf{x}_0$, $\mathbf{w}_t$ and $\mathbf{v}_t$ are statistically independent. At time t, the pair $(\mathbf{C}_t, \mathbf{R}_t)$ is randomly drawn from the finite set set $\mathcal{A} = \mathcal{C} \times \mathcal{R}$, where $\mathcal{C} = \left\{\mathbf{C}^{(1)}, \cdots, \mathbf{C}^{(D)}\right\}$ and $\mathcal{R} = \left\{\mathbf{R}^{(1)}, \cdots, \mathbf{R}^{(E)}\right\}$. Let $\Gamma_{t,T} = ((\mathbf{C}_t, \mathbf{R}_t), \cdots, (\mathbf{C}_{t+T-1}, \mathbf{R}_{t+T-1})) \in \mathcal{A}^T$ denote the random sequence of measurement matrices and noise covariances from time T up to time $t+T-1$.

A Kalman filter is used to obtain an estimate $\hat{\mathbf{x}}_{t|t-1}$ of the state $\mathbf{x}_t$ given the knowledge of $\mathbf{y}_0, \cdots, \mathbf{y}_{t-1}$ and Γ_t. The update equation of the expected covariance (EC) $\mathbf{P}_t = \mathbb{E}\left\{\tilde{\mathbf{x}}_t \tilde{\mathbf{x}}_t^*\right\}$, with $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \hat{\mathbf{x}}_{t|t-1}$ and * denoting transpose conjugate, is

$$\mathbf{P}_{t+1} = \psi_{\mathbf{C}_t, \mathbf{R}_t}(\mathbf{P}_t), \tag{3}$$

with

$$\begin{aligned}&\psi_{\mathbf{C}_t,\mathbf{R}_t}(\mathbf{P}_t)\\ =&\mathbf{A}\mathbf{P}_t\mathbf{A}^* + \mathbf{Q} - \mathbf{A}\mathbf{P}_t\mathbf{C}_t^*\left(\mathbf{C}_t\mathbf{P}_t\mathbf{C}_t^* + \mathbf{R}_t\right)^{-1}\mathbf{C}_t\mathbf{P}_t\mathbf{A}^*.\end{aligned}$$

In this work we derive a necessary condition and a sufficient condition, with a trivial gap between them, for the stability of the Kalman filter with a random measurement equation. This is done by studying the asymptotic norm of the expected error covariance (ANEEC). In order to define the ANEEC, we introduce the following notation

$$\mathbf{\Psi}(\mathbf{P}_t, \Gamma_{t,T}) = \psi_{\mathbf{C}_{t+T-1},\mathbf{R}_{t+T-1}} \circ \psi_{\mathbf{C}_{t+1},\mathbf{R}_{t+1}} \circ \psi_{\mathbf{C}_t,\mathbf{R}_t}(\mathbf{P}_t),$$

i.e., $\mathbf{\Psi}(\mathbf{P}_t, \Gamma_{t,T})$ denotes the covariance resulting at time $t+T$, after starting with covariance $\mathbf{P}_t$ at time t, and then applying the sequence of random measurement equations defined by $\Gamma_{t,T}$.

Definition 2.1: The ANEEC is defined as

$$G = \sup_{\substack{t\in\mathbb{Z}\\ \mathbf{P}_t \geq 0}} \limsup_{T\to\infty} \left\|\mathbb{E}\left(\mathbf{\Psi}(\mathbf{P}_t, \Gamma_{t,T})\right)\right\|.$$

III. Main result

Our main result is stated in terms of certain partition of the system (1)-(2) into subsystems which we call finite multiplicative order (FMO) blocks. This partition is introduced next.

Notation 3.1: Consider the following partition of $\mathbf{A}$,

$$\mathbf{A} = \operatorname{diag}(\mathbf{A}_1, \cdots, \mathbf{A}_K), \tag{4}$$

where the sub-matrices $\mathbf{A}_k$ are chosen such that, for any k, the diagonal entries of $\mathbf{A}_k$ have a common FMO up to a constant (i.e., there exists $N_k \in \mathbb{N}$ such that all the entries in the main diagonal of $\mathbf{A}_k^{N_k}$ are equal to $\alpha_k^{N_k}$, with $\alpha_k \in \mathbb{C}$), and for any k and l with $k \neq l$, the diagonal entries of the matrix $\operatorname{diag}(\mathbf{A}_k, \mathbf{A}_l)$ do not have FMO up to any constant.

Let $\bar{J} \in \mathbb{N}$ be the largest size of the Jordan blocks in $\mathbf{A}$, and $\bar{J}_k \in \mathbb{N}$ be the largest size of the Jordan blocks in $\mathbf{A}_k$. For convenience, we assume that the sub-matrices $\mathbf{A}_k$ are ordered such that $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_K|$. Also, when $|\alpha_k| = |\alpha_{k+1}|$, then $\bar{J}_k \geq \bar{J}_{k+1}$.

For each $d = 1, \cdots, D$, consider the partition

$$\mathbf{C}^{(d)} = \left[\mathbf{C}_1^{(d)}, \cdots, \mathbf{C}_K^{(d)}\right],$$

such that, for each k, $\mathbf{C}_k^{(d)}$ have the same number of columns as $\mathbf{A}_k$. Let $\mathcal{C}_k = \left\{\mathbf{C}_k^{(d)} : d = 1, \cdots, D\right\}$.

Definition 3.1 (FMO block): In the above partition, each pair $(\mathbf{A}_k, \mathcal{C}_k)$ is called an FMO block of the system (1)-(2).

Remark 3.1: Notice that if $(\mathbf{A}_k, \mathcal{C}_k)$ is an FMO block, then each sub-matrix $\mathbf{A}_k$ can be written as

$$\mathbf{A}_k = \alpha_k \tilde{\mathbf{A}}_k, \tag{5}$$

$$\tilde{\mathbf{A}}_k = \operatorname{diag}\left\{\exp(i2\pi\theta_{k,1}), \cdots, \exp(i2\pi\theta_{k,K_k})\right\} + \mathbf{U}_k, \tag{6}$$

where $\mathbf{U}_k$ is strictly upper triangular, i.e., its non-zero entries lie above its main diagonal. Also, $\alpha_k \in \mathbb{C}$ and $\theta_{k,j} \in \mathbb{Q}$, for $j = 1, \cdots, K_k$. Notice that for any k and l with $k \neq l$, α_k/α_l is not a root of unity, i.e., $(\alpha_k/\alpha_l)^m \neq 1$ for all $m \in \mathbb{N}$.

The measurements $\mathbf{z}_{t,T}^* = \left[\mathbf{y}_t^*, \cdots, \mathbf{y}_{t+T-1}^*\right]$ available from time t up to $T-1$ can be written as

$$\mathbf{z}_{t,T} = \mathbf{O}(\Gamma_{t,T})\mathbf{x}_t + \mathbf{f}_{t,T}(\Gamma_{t,T}), \tag{7}$$

where the observability matrix $\mathbf{O}(\Gamma_{t,T})$ is given by

$$\mathbf{O}(\Gamma_{t,T}) = \begin{bmatrix}\mathbf{O}_1(\Gamma_{t,T}) & \mathbf{O}_2(\Gamma_{t,T}) & \ldots & \mathbf{O}_K(\Gamma_{t,T})\end{bmatrix}, \tag{8}$$

$$\mathbf{f}_t(\Gamma_{t,T}) = \begin{bmatrix}\mathbf{v}_t\\ \mathbf{C}_{t+1}\mathbf{w}_t + \mathbf{v}_{t+1}\\ \vdots\\ \mathbf{C}_T\sum_{j=t}^{T+t-2}\mathbf{A}^{T+t-2-j}\mathbf{w}_j + \mathbf{v}_{t+T-1}\end{bmatrix},$$

with

$$\mathbf{O}_k(\Gamma_{t,T}) \triangleq \begin{bmatrix}\mathbf{C}_{t,k}\\ \mathbf{C}_{t+1,k}\mathbf{A}_k\\ \vdots\\ \mathbf{C}_{T+t-1,k}\mathbf{A}_k^{T-1}\end{bmatrix}$$

and

$$\mathbf{C}_t = [\mathbf{C}_{t,1}, \cdots, \mathbf{C}_{t,K}],$$

such that, for each k, $\mathbf{C}_{t,k}$ have the same number of columns as $\mathbf{A}_k$.

Our main result is stated in terms of the probability that each matrix $\mathbf{O}_k(\Gamma_{t,T})$ does not have full-column rank (FCR). The following definition identifies the set of sequences $\Gamma_{t,T}$ leading to this property.

Definition 3.2: For $k = 1, \cdots, K$, let

$$\mathcal{N}_k^{t,T} \triangleq \{\Gamma_{t,T} : \mathbf{O}_k(\Gamma_{t,T}) \text{ does not have FCR}\}.$$

We next state our main result. This requires the following assumption.

Definition 3.3: A random process x_t is cyclostationary with period $\tau \in \mathbb{N}$ if

$$\mathbb{P}(X_{t,T}) = \mathbb{P}(X_{t+\tau,T}), \text{ for all } t, T \in \mathbb{N},$$

where $X_{t,T} = \{x_t, \cdots, x_{t+T-1}\}$.

Assumption 3.1: The random process $(\mathbf{C}_t, \mathbf{R}_t)$ is cyclostationary, i.e., there exists $\tau \in \mathbb{N}$, such that

$$\mathbb{P}(\Gamma_{t,T}) = \mathbb{P}(\Gamma_{t+\tau,T}), \text{ for all } t, T \in \mathbb{N}. \tag{9}$$

and [1]

$$\zeta \triangleq \sup_{\substack{T\in\mathbb{N}\\0\le t<\tau}} \sup_{\Gamma_{t,\infty}:\mathbb{P}(\Gamma_{t,T})\neq 0} \frac{\mathbb{P}(\Gamma_{t,T}|\Gamma_{t+T,\infty})}{\mathbb{P}(\Gamma_{t,T})} < \infty. \tag{10}$$

We now state our main result. Its proof is omitted in this conference version.

Theorem 3.1: Consider the system (1)-(2) satisfying Assumption 3.1. For $k \in \{1, \cdots, K\}$, let

$$\Phi_k = \max_{0\le t<\tau} \limsup_{T\to\infty} \mathbb{P}\left(\mathcal{N}_k^{t,T}\right)^{1/T}.$$

If

$$|\alpha_k|^2 \Phi_k < 1, \text{ for all } k \in \{1, \cdots, K\}, \tag{11}$$

then $G < \infty$, and if

$$|\alpha_k|^2 \Phi_k > 1, \text{ for some } k \in \{1, \cdots, K\}, \tag{12}$$

then $G = \infty$.

Remark 3.2: Notice that Theorem 3.1 is inconclusive in the case when $|\alpha_k|^2 \Phi_k = 1$. Trivial gaps of this kind are common in the literature [1], [6].

IV. About Assumption 3.1

In this section we give an interpretation of the technical condition stated in Assumption 3.1.

We start by giving the following definition.

Definition 4.1: A random process x_t is Markov of order $\nu \in \mathbb{N}$ if, for all $\mu \geq 1$,

$$\mathbb{P}(x_t|x_{t-\nu-\mu}, \cdots, x_{t-1}) = \mathbb{P}(x_t|x_{t-\nu}, \cdots, x_{t-1}).$$

Furthermore, it is independent if $\nu = 0$.

Notice that, if $(\mathbf{C}_t, \mathbf{R}_t)$ is independent, then (10) holds trivially. Hence, (10) can be interpreted as a relaxation of the notion of independence. Also, if the random process $(\mathbf{C}_t, \mathbf{R}_t)$ is stationary, condition (9) holds with $\tau = 1$. Hence, (9) can be interpreted as relaxations of the notion of stationarity. Furthermore, the following lemma states that, whenever (9) holds, (10) is guaranteed if $(\mathbf{C}_t, \mathbf{R}_t)$ is finite-order Markov. Thus, (10) can be interpreted as a generalization of the finite-order Markov property (under the condition that (9) holds).

Lemma 4.1: If $(\mathbf{C}_t, \mathbf{R}_t)$ satisfies (9) and is finite-order Markov, then it also satisfies (10).

Proof: Without loss of generality, let $T > \nu$, where ν is the Markov order of $(\mathbf{C}_t, \mathbf{R}_t)$. We have

$$\begin{aligned}
&\frac{\mathbb{P}(\Gamma_{t,T}|\Gamma_{t+T,\infty})}{\mathbb{P}(\Gamma_{t,T})} \\
&= \frac{\mathbb{P}(\Gamma_{t,T}|\Gamma_{t+T,\nu})}{\mathbb{P}(\Gamma_{t,T})} \\
&= \frac{\mathbb{P}(\Gamma_{t,T-\nu}|\Gamma_{t+T-\nu,\nu})\,\mathbb{P}(\Gamma_{t+T-\nu,\nu}|\Gamma_{t+T,\nu})}{\mathbb{P}(\Gamma_{t,T-\nu}|\Gamma_{t+T-\nu,\nu})\,\mathbb{P}(\Gamma_{t+T-\nu,\nu})} \\
&= \frac{\mathbb{P}(\Gamma_{t+T-\nu,\nu}|\Gamma_{t+T,\nu})}{\mathbb{P}(\Gamma_{t+T-\nu,\nu})}.
\end{aligned}$$

Hence

$$\begin{aligned}
\zeta &= \sup_{t,T} \sup_{\Gamma_{t,\infty}:\mathbb{P}(\Gamma_{t,T})\neq 0} \frac{\mathbb{P}(\Gamma_{t,T}|\Gamma_{t+T,\infty})}{\mathbb{P}(\Gamma_{t,T})} \\
&= \sup_{0\le s<\tau} \sup_{\Gamma_{s,2\nu}:\mathbb{P}(\Gamma_{s,\nu})\neq 0} \frac{\mathbb{P}(\Gamma_{s,\nu}|\Gamma_{s+\nu,\nu})}{\mathbb{P}(\Gamma_{s,\nu})} \\
&< \infty,
\end{aligned}$$

where the inequality follows since the supremum operations are taken over finite sets. ∎

V. Computing Φ_k

Our result in Theorem 3.1 is stated in terms of the quantity Φ_k. We study below how to compute this quantity. Since in our study the choice of $k \in \{1, \cdots, K\}$ is fixed, to remove k from the notation, we consider a generic FMO block $(\mathbf{A}, \mathcal{C})$. We classify FMO blocks in two categories, namely, degenerate and non-degenerate. This definition is given below.

Definition 5.1: Let $\mathcal{F} = \left\{\mathbf{C}^{(d)} \in \mathcal{C}, \mathbf{C}^{(d)} \text{ has FCR}\right\}$ and $\overline{\mathcal{F}} = \mathcal{C} \setminus \mathcal{F}$. Let $\mathbf{D} = \operatorname{stack}\left\{\mathbf{C}^{(d)} \in \overline{\mathcal{F}}\right\}$ be the matrix obtained after stacking together all matrices $\mathbf{C}^{(d)}$ in $\overline{\mathcal{F}}$. The FMO block $(\mathbf{A}, \mathcal{C})$ is said to be non-degenerate if $\mathcal{F} \neq \emptyset$ and if the pair $(\mathbf{A}, \mathbf{D})$ is non-observable. Otherwise, the block $(\mathbf{A}, \mathcal{C})$ is said to be degenerate.

Below we state methods to compute Φ for both, degenerate and non-degenerate blocks.

A. Non-degenerate FMO blocks

The following lemma states a closed form expression of Φ for non-degenerate FMO blocks. Its proof is deferred to Appendix.

Lemma 5.1: Let $(\mathbf{A}, \mathcal{C})$ be a non-degenerate FMO block. If the random process $\mathbf{C}_t$ is Markov of order ν, then,

$$\Phi = \limsup_{T\to\infty} \left(\prod_{\tau=1}^{T} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}}|\mathbf{C}_{\tau-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{\tau-1} \in \overline{\mathcal{F}}\right)\right)^{1/T}. \tag{13}$$

B. Degenerate FMO blocks

A closed-form expression of Φ for general degenerate systems in terms of the network parameters is difficult to derive. Nevertheless, Lemma 5.2 below states a numerical method to evaluate this quantity.

[1] See Section IV for an interpretation of the following condition.

We start by introducing some necessary notation.

Notation 5.1: Let $N \in \mathbb{N}$ be the smallest positive integer such that $\mathbf{A}^N = \alpha^N \mathbf{I}$. Let $\mathbb{K} = \left\{\ker\left(\mathbf{O}\left(\Gamma\right)\right) : \Gamma \in \mathcal{N}^{0,N}\right\}$ be the set of all possible kernels of $\mathbf{O}\left(\Gamma\right)$, for sequences Γ of length N, for which $\mathbf{O}\left(\Gamma\right)$ has a non-trivial kernel. Notice that $\mathbb{K}$ includes all possible non-trivial kernels of $\mathbf{O}\left(\Gamma\right)$ for sequences Γ of any positive length. For any $T \in \mathbb{N}$, define the map $\psi : \mathcal{N}^{0,T} \to \mathbb{K}$ by

$$\psi\left(\Gamma\right) = \ker\left(\mathbf{O}(\Gamma)\right).$$

Let $\mathcal{K}_i$, $i = 1, \cdots, I$ denote all the elements in $\mathbb{K}$. The elements $\mathcal{K}_i$ are numerated such that, if $\mathcal{K}_i \subset \mathcal{K}_j$, then $i > j$. Let $\tilde{N} \in \mathbb{N}$ be the smallest positive integer, greater than or equal to ν, which is multiple of both, N and τ. For each $i = 1, \cdots, I$, let

$$\mathcal{L}_i^{t,n} = \left\{\Gamma \in \mathcal{N}^{t,n\tilde{N}} : \psi\left(\Gamma\right) = \mathcal{K}_i\right\}. \tag{14}$$

The next lemma provides a method for the numerical evaluation of Φ. The proof is omitted in this conference version.

Lemma 5.2: Let $(\mathbf{A}, \mathcal{C})$ be an FMO block. If the random process $\mathbf{C}_t$ is Markov of order ν and cyclostationary with period τ, then,

$$\Phi = \max_{0 \leq t < \tau} \rho_t^{1/\tilde{N}}, \tag{15}$$

where

$$\rho_t = \max_{\substack{1 \leq i < I: \\ \mathbb{P}\left(\mathcal{L}_i^{t,1}\right) \neq 0}} \mathbb{P}\left(\mathcal{L}_i^{t,2} \middle| \mathcal{L}_i^{t,1}\right).$$

VI. Example

Consider a system whose dynamics is given by (1), with

$$\mathbf{A} = \operatorname{diag}\left\{\alpha_1 \alpha_2, \alpha_3\right\},$$

for some $\alpha_1 > \alpha_2 > \alpha_3 > 0$. There are two sensors. For each $s = 1, 2$, sensor s measures

$$\mathbf{u}_t^{(s)} = \mathbf{H}^{(s)} \mathbf{x}_t + \mathbf{e}_t^{(s)}, \tag{16}$$

with $e_t^{(s)} \sim \mathcal{N}\left(0, \mathbf{E}^{(s)}\right)$, and $\mathbf{H}^{(1)} = [2, 0, 1]$ and $\mathbf{H}^{(2)} = [0, -1, 0]$. Due to communication constraints, the measurements from both sensors are alternatively transmitted., i.e.,

$$\mathbf{M}_t = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix} & t \text{ even}, \\ \begin{bmatrix} 0 & 1 \end{bmatrix} & t \text{ odd}. \end{cases}$$

We assume that the communication channel has a i.i.d. packet loss model, i.e., $\mathbf{L}_t = l_t$, with $\mathbb{P}\left(l_t = 0\right) = \lambda$ and $\mathbb{P}\left(l_t | l_{t-1}\right) = \mathbb{P}\left(l_t\right)$. We can use the result of Theorem 3.1 to determine the stability of the Kalman filter.

We have

$$\mathbf{B}_t = \begin{cases} \begin{bmatrix} l_t & 0 \end{bmatrix} & t \text{ even}, \\ \begin{bmatrix} 0 & l_t \end{bmatrix} & t \text{ odd}. \end{cases}$$

Therefore

$$\mathbf{C}_t = \begin{cases} \begin{bmatrix} 2l_t & 0 & l_t \end{bmatrix} & t \text{ even}, \\ \begin{bmatrix} 0 & -l_t & 0 \end{bmatrix} & t \text{ odd}. \end{cases} \tag{17}$$

Hence, the measurement equation of the aggregated system is given by (2), with $\mathbf{C}_t$ given by (17).

From Definition 3.1 the FMO blocks of the above system are $([\alpha_1], \mathcal{C}_1)$, $([\alpha_2], \mathcal{C}_2)$ and $([\alpha_2], \mathcal{C}_3)$ with

$$\begin{aligned} \mathcal{C}_1 &= \{[0], [2]\}, \\ \mathcal{C}_2 &= \{[0], [-1]\}, \\ \mathcal{C}_1 &= \{[0], [1]\}. \end{aligned}$$

Our next step is then to decide whether each FMO block is degenerate or not.

Consider the block $([\alpha_1], \mathcal{C}_1)$. Following Definition 5.1, we have

$$\mathcal{F}_1 = \{[2]\}, \text{ and } \overline{\mathcal{F}}_1 = \{[0]\}.$$

Hence

$$\begin{aligned} \mathcal{F}_1 &\neq \emptyset, \\ \mathbf{D} &= [0], \end{aligned}$$

and $(\mathbf{A}, \mathbf{D})$ is non-observable. Thus, $([\alpha_1], \mathcal{C}_1)$ is non-degenerate. It then follows that we can compute Φ_1 using Lemma 5.1. Since $\nu = 0$, we have

$$\begin{aligned} &\mathbb{P}\left(\mathbf{C}_{t,1} \in \overline{\mathcal{F}}_1 | \mathbf{C}_{t-\nu,1} \in \overline{\mathcal{F}}_1, \cdots, \mathbf{C}_{t-1,1} \in \overline{\mathcal{F}}_1\right) \\ &\qquad = \mathbb{P}\left(\mathbf{C}_{t,1} \in \overline{\mathcal{F}}_1\right) = \begin{cases} \lambda & t \text{ even}, \\ 1 & t \text{ odd}. \end{cases} \end{aligned}$$

Hence, from (13),

$$\Phi_1 = \lambda^{1/2}.$$

Following a similar argument, we can easily show that

$$\Phi_2 = \Phi_3 = \lambda^{1/2}.$$

It then follows from Theorem 3.1 that

$$\begin{aligned} \alpha_1^4 \lambda < 1 &\Rightarrow G < \infty, \\ \alpha_1^4 \lambda > 1 &\Rightarrow G = \infty. \end{aligned}$$

VII. Conclusion

We studied the necessary and sufficient stability condition for the Kalman filter associated with a general linear system with random measurement matrix. The condition is expressed in terms of the quantities Φ_k. A computation method to ealuate them for a particular system was also introduced.

Appendix

Notation 1.1: Let L be the number of Jordan blocks of $\mathbf{A}$. Consider the following partition of $\mathbf{A}$

$$\mathbf{A} = \operatorname{diag}\left(\mathbf{A}_1, \cdots, \mathbf{A}_L\right), \tag{18}$$

where $\mathbf{A}_l$ is a Jordan block with size J_l and eigenvalue α_l.

Proof: [Proof of Lemma 5.1] Let $t \leq s < t+T$ be such that $\mathbf{C}_s \in \mathcal{F}$. For each $l = 1, \cdots, L$, we have

$$\mathbf{A}_l^s = a_l^s \begin{bmatrix} 1 & p^{(1)}(s) & \cdots & p^{(J_l-1)}(s) \\ 0 & 1 & \ddots & p^{(J_l-2)}(s) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $p^{(j)}(s)$ is a polynomial in s of degree j, given by $p^{(j)}(s) = a_l^{-j}\binom{s}{j}$. Hence, $\mathbf{A}_l^s$ is non-singular, and therefore, so is $\mathbf{A}^s$. This means that $\mathbf{C}_s\mathbf{A}^s$ has FCR, and therefore so does $\mathbf{O}(\Gamma_{t,T})$. This means that $\mathbf{O}(\Gamma_{t,T})$ has FCR if and only if $\#(\Gamma_{t,T}) \neq 0$. Hence

$$\mathbb{P}\left(\mathcal{N}^{t,T}\right) \Leftrightarrow \mathbb{P}\left(\#(\Gamma_{t,T}) = 0\right), \quad (19)$$

where

$$\#(\Gamma_{t,T}) = |\{t \leq s < t+T : \mathbf{C}_s \in \mathcal{F}\}|,$$

denotes the number of times $t \leq s < t+T$ in which $\mathbf{C}_t$ has FCR.

The remainder of the proof is based on the argument of [23, Lemma 11]. Without loss of generality, let $T \geq \nu$. We then have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{N}^{t,T}\right) \\
=&\mathbb{P}\left(\#(\Gamma_{t,T}) = 0\right) \\
=&\mathbb{P}\left(\#(\Gamma_{t,\nu}) = 0\right)\mathbb{P}\left(\#(\Gamma_{t+\nu,T-\nu}) = 0 | \#(\Gamma_{t,\nu}) = 0\right) \\
=&\mathbb{P}\left(\#(\Gamma_{t,\nu}) = 0\right) \\
&\cdot \prod_{\tau=t+\nu}^{t+T-1} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}} | \mathbf{C}_{t-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{t-1} \in \overline{\mathcal{F}}\right) \\
=&\frac{\mathbb{P}\left(\#(\Gamma_{t,\nu}) = 0\right)}{\prod_{\tau=t}^{t+\nu-1} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}} | \mathbf{C}_{t-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{t-1} \in \overline{\mathcal{F}}\right)} \\
&\cdot \prod_{\tau=t}^{t+T-1} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}} | \mathbf{C}_{t-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{t-1} \in \overline{\mathcal{F}}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\limsup_{T\to\infty} \\
&\left(\frac{\mathbb{P}\left(\#(\Gamma_{t,\nu}) = 0\right)}{\prod_{\tau=t}^{t+\nu-1} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}} | \mathbf{C}_{t-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{t-1} \in \overline{\mathcal{F}}\right)}\right)^{1/T} \\
&= 1,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\left(\mathcal{N}^{t,T}\right)^{1/T} \\
=&\limsup_{T\to\infty} \\
&\left(\prod_{\tau=t}^{t+T-1} \mathbb{P}\left(\mathbf{C}_\tau \in \overline{\mathcal{F}} | \mathbf{C}_{t-\nu} \in \overline{\mathcal{F}}, \cdots, \mathbf{C}_{t-1} \in \overline{\mathcal{F}}\right)\right)^{1/T}.
\end{aligned}$$

And the result follows after noticing from the above that $\limsup_{T\to\infty} \mathbb{P}\left(\mathcal{N}^{t,T}\right)^{1/T}$ is independent of the starting time t. ■

References

[1] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry. Kalman filtering with intermittent observations. *IEEE Trans. Autom. Control*, 49(9):1453–1464, Sep. 2004.

[2] L. Xie. Stochastic comparison, boundedness, weak convergence, and ergodicity of a random riccati equation with markovian binary switching. *SIAM J. on Control and Optimization*, 50(1):532–558, 2012.

[3] M. Huang and S. Dey. Stability of kalman filtering with markovian packet losses. *Automatica*, 43(4):598–607, 2007.

[4] K. You, M. Fu, and L. Xie. Mean square stability for kalman filtering with markovian packet losses. *Automatica*, 47(12):2647–2657, 2011.

[5] L. Xie. Stability of a random riccati equation with markovian binary switching. *IEEE Trans. Autom. Control*, 53(7):1759–1764, Aug. 2008.

[6] Y. Mo and B. Sinopoli. Kalman filtering with intermittent observations: Tail distribution and critical value. *IEEE Trans. Autom. Control*, 57(3):677–689, Mar. 2012.

[7] K. Plarre and F. Bullo. On kalman filtering for detectable systems with intermittent observations. *IEEE Trans. Autom. Control*, 54(2):386–390, Feb. 2009.

[8] Y. Mo and B. Sinopoli. A characterization of the critical value for kalman filtering with intermittent observations. In *IEEE Conf. Decision Control*, pages 2692–2697, Cancun, Mexico, Dec. 2008.

[9] Li Xie and Lihua Xie. Peak covariance stability of a random riccati equation arising from kalman filtering with observation losses. *J. of Systems Science and Complexity*, 20(2):262–272, 2007.

[10] E. Rohr, D. Marelli, and M. Fu. Statistical properties of the error covariance in a kalman filter with random measurement losses. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 5881–5886, Atlanta, USA, 15-17 2010.

[11] E. Rohr, D. Marelli, and M. Fu. On the error covariance distribution for kalman filters with packet dropouts. In Mario Alberto Jordán, editor, *Discrete Time Systems*, pages 71–92. InTech, 2010.

[12] X. Liu and A. Goldsmith. Kalman filtering with partial observation losses. In *proc. IEEE Conf. Decision Control*, pages 4180–4186, Paradise, The Bahamas, Dec. 2004.

[13] A. F. Dana, V. Gupta, J. P. Hespanha, B. Hassibi, and R. M. Murray. Estimation over communication networks: Performance bounds and achievability results. In *Proc. Amer. Control Conf.*, pages 3450–3455, New York, NY, 9-13 2007.

[14] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry. Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1):163–187, 2007.

[15] L. Schenato. Optimal estimation in networked control systems subject to random delay and packet drop. *IEEE Trans. Autom. Control*, 53(5):1311–1317, Jun. 2008.

[16] E. Rohr, D. Marelli, and M. Fu. Kalman filtering for a class of degenerate systems with intermittent observations. In *Proc. IEEE Conf. Decision Control and European Control Conf.*, pages 2422–2427, Orlando, FL, 2011.

[17] S. Y. Park and A. Sahai. Intermittent kalman filtering: Eigenvalue cycles and nonuniform sampling. In *Proc. Amer. Control Conf.*, pages 3692–3697, San Francisco, CA, 06 2011.

[18] D. Quevedo, A. Ahlen, and K. Johansson. State estimation over sensor networks with correlated wireless fading channels. *IEEE Trans. Autom. Control*, to be published.

[19] E. N. Gilbert. Capacity of a burst-noise channel. *Bell Syst. Tech. J*, 39(9):1253–1265, 1960.

[20] E. O. Elliott. Estimates of error rates for codes on burst-noise channels. *Bell Syst. Tech. J*, 42(9):1977–1997, 1963.

[21] P. Sadeghi, R. Kennedy, P. Rapajic, and R. Shams. Finite-state markov modeling of fading channels: a survey of principles and applications. *IEEE Signal Processing Mag.*, 25(5):57–80, 2008.

[22] A. Censi. Kalman filtering with intermittent observations: Convergence for semi-markov chains and an intrinsic performance measure. *IEEE Trans. Autom. Control*, 56(2):376–381, Feb. 2011.

[23] Eduardo Rath Rohr, Damin Marelli, and Minyue Fu. Kalman filtering with intermittent observations: On the boundedness of the expected error covariance. *Automatic Control, IEEE Transactions on*, 59(10):2724–2738, 2014.