Adaptive Consensus of Nonlinearly Parameterized Multi-Agent Systems

Imil Hamda Imran , Zhiyong Chen , Yamin Yan , and Minyue Fu

Abstract—**This letter studies the distributed adaptive consensus problem for multi-agent systems (MASs) subject to nonlinearly parameterized uncertainties. In the existing research, an adaptive consensus scheme was developed for the same scenario but under the assumption that the uncertainties must be linearly parameterized in their unknown parameters. In this letter, the assumption is removed by using a novel distributed adaptive update law, which makes the scheme applicable for more general nonlinear MASs. This letter contains a general design paradigm using the certainty equivalence principle and specific implementation of it on nonlinear second-order MASs. The results are supported by rigorous mathematical analysis and numerical simulation.**

Index Terms—**Networked control systems, adaptive control, distributed control.**

I. INTRODUCTION

OVER the past decades, control of MASs has attracted increasing research; see, e.g., the survey paper [1]. The research on consensus of linear MASs is relatively mature with tremendous results in the existing literature, for example, [2]–[6]. In practice, agent dynamics are usually subject to nonlinear uncertainties. MASs with uncertain nonlinearities are mainly studied by the internal model approach [7]–[9]. Adaptive control has also been proved to be a useful tool for handling systems uncertainties, in particular, unknown parameters. The basic idea is to estimate the unknown parameters online and hence achieve desired control performance using the certainty equivalence principle. Technically, it aims to propose an adaptive controller in a superposition form on top of an existing controller such that the behavior of the system is still satisfactory when system uncertainties appear. In the traditional adaptive control scheme, a parameter update law along the gradient of a proper Lyapunov function of the nominal system is proved to be effective.

Recently, the adaptive control technique has been investigated for the consensus problem of MASs with uncertainties.

For instance, a first-order MAS was first studied in [10] under an undirected communication graph and a more general framework for a group of continuous-time systems was considered in [11]. A similar adaptive technique was used in [12] for both first-order and second-order MASs with a Nussbaum gain added to deal with unknown control direction. The undirected fixed topology has been extended to undirected jointly connected switching topology in [13] for leader-following case and in [14] for leaderless case. For a network of directed topology, [15] considered the leader-following case for higher-order MASs and achieved consensus with a residual error.

In the above schemes, although the adaptive law along the gradient of a Lyapunov function is effective, it has the limitation that the technique cannot be generalized to handle second-order MASs with a directed graph. It essentially requires all the agent states to construct a global Lyapunov function. This intrinsic methodology limitation makes a general distributed implementation fashion difficult for tackling the consensus problems for complicated MASs. In practice, developing an effective distributed adaptive control protocol that is able to maintain the system's collective behavior is necessary, because communication constraints among agents are inevitable. This challenge has been overcome in the previous work [16] for the special case with an assumption that the uncertain parameters are linearly parameterized. The key idea was to introduce a compensation for the adaptive law such that the steady state of the estimation error is in a manifold of the state space of agent states and estimated parameters. This method does not rely on the gradient of a Lyapunov function and thus provides advantages in distributed implementation.

Most of the existing results in adaptive control are under an essential assumption that the uncertainties are linearly parameterized. However, nonlinearly parameterized systems are commonly encountered in practical systems [17], [18]. Handling nonlinearly parameterized uncertainties is always a difficult issue in adaptive control even for a single system. There exist some results in literature. For example, the research based on convex/concave nonlinear functions is one of the major research lines for adaptive control of nonlinearly parameterized models; see, e.g., [19]–[21]. In [22], the so-called immersion and invariance adaptive control was proposed by construction of a monotone mapping. An adaptive control method for the class of non-affine nonlinearly parameterized systems was studied in [23] by introducing a biasing vector function into parameter estimation. Another novel adaptive

control approach based on forward/backward adaptation law was established to achieve system stability and parameter convergence in [24] which does not rely on the expression of system nonlinearities.

However, none of these results have been successfully applied in a networked setting. It is for the first time to pursue a distributed adaptive consensus controller for nonlinearly parameterized systems. More specifically, we will study the same distributed adaptive consensus problem formulated in [16], but for the MASs subject to nonlinearly parameterized uncertainties. The linear parameterization assumption will be removed by a novel distributed adaptive update law, which makes the scheme applicable for more general nonlinear MASs. In this scheme, the adaptive estimation error is driven to a deliberately designed manifold in the space of agent states and estimated parameters. The new adaptive scheme will be designed for more general nonlinearly parameterized uncertainties. With this new scheme, we will be able to solve an open consensus problem for a leaderless second-order MAS under a directed network.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider an MAS of n agents under properly designed controllers represented by

$$\dot{x}_i = f_i(x), \ i = 1, \ldots, n \tag{1}$$

where $x_i \in \mathbb{R}^l$ is the state of the i-th agent and $x = [x_1^\mathsf{T}, x_2^\mathsf{T}, \ldots, x_n^\mathsf{T}]^\mathsf{T}$. The dependence of the function f_i on x (not only x_i) indicates the interconnection among agents. Let $f(x) = [f_1^\mathsf{T}(x), f_2^\mathsf{T}(x), \ldots, f_n^\mathsf{T}(x)]^\mathsf{T}$. Then, the nominal uncertainty-free MAS (1) can be put in a compact form as $\dot{x} = f(x)$. Suppose the MAS has achieved a certain collective behavior, specifically, with a property in terms of a Lyapunov-like function. Throughout this letter, the notation $\|\cdot\|$ means the Euclidean norm and $\|x\|_R = \|Rx\|$ for a real matrix R.

Assumption 1: For the system (1), there exists a Lyapunov-like function $V(x)$ satisfying $\underline{\alpha}(\|x\|_R) \leq V(x) \leq \bar{\alpha}(\|x\|_R)$ for a matrix $R \in \mathbb{R}^{\bar{n}l \times nl}$ with $\bar{n} \leq n$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$, such that,

$$\frac{\partial V(x)}{\partial x} f(x) \leq -\alpha(\|x\|_R) \tag{2}$$

for a class $\mathcal{K}_\infty$ function α. Furthermore,

$$\left\| \frac{\partial V(x)}{\partial x} \right\|^2 \leq \sigma \alpha(\|x\|_R) \tag{3}$$

for some constant $\sigma > 0$.

Remark 1: Two typical selections of R in Assumption 1 are explained as follows. (i) If R is a nonsingular matrix, i.e., $\bar{n} = n$, then $\|x\|_R = 0$ implies $\|x\| = 0$. So, the function $V(x)$ is a Lyapunov function and Assumption 1 implies $\lim_{t \to \infty} \|x(t)\| = 0$, i.e., asymptotic stability about the equilibrium at the origin. (ii) If $R \in \mathbb{R}^{(n-1)l \times nl}$, i.e., $\bar{n} = n - 1$, is a full row rank matrix and the rows are perpendicular to span$\{\mathbf{1} \otimes I_l\}$ where $I_l \in \mathbb{R}^{l \times l}$ is an identity matrix and $\mathbf{1} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$, then $\|x\|_R = 0$ implies $x = \mathbf{1} \otimes x_o$ for some $x_o \in \mathbb{R}^l$. So, the function $V(x)$ is a Lyapunov function for the Rx-subsystem and Assumption 1 implies

$\lim_{t \to \infty} \|x(t)\|_R = 0$, i.e., $\lim_{t \to \infty} [x(t) - \mathbf{1} \otimes x_o(t)] = 0$, which is a typical consensus phenomenon.

The research target is to propose an adaptive law in a superposition form on top of the existing controller, such that the behavior of the nominal system is still maintained when the system is subject to uncertainties. Such a superposition design is called the certainty equivalence principle. Specifically, the uncertainties and the additional adaptive control are taken into the system (1) in the manner described by

$$\dot{x}_i = f_i(x) + f_i^\delta(x_i, w_i, \mu_i), \ i = 1, \ldots, n \tag{4}$$

where $w_i \in \mathbb{R}^{s_i}$ denotes the unknown constant parameters and $\mu_i \in \mathbb{R}^{s_i}$ the additional adaptive control input to handle these uncertainties. It is assumed that $\mu_i = w_i$ could cancel the uncertainties if the parameter w_i were known, that is,

$$f_i^\delta(x_i, w_i, \mu_i) := g_i(x_i, w_i) - g_i(x_i, \mu_i)$$

for some function g_i, throughout this letter. In the practical scenario that w_i is unknown, an adaptive law is required for μ_i to dynamically cancel the uncertainties associated with the parameter w_i.

There are two major challenges in designing a distributed adaptive law in the present networked setting. First, an adaptive law usually depends on the gradient of the established Lyapunov function for the nominal system, which is $V(x)$ in Assumption 1 for the present case. More specifically, the adaptive law depends on $\partial V(x)/\partial x_i$ for each agent i, which hence depends on not only the local state of agent i, but also the full network state x. This is an obstacle for a distributed adaptive law. This challenge has been overcome in the previous research [16] for the special case under the linearly parameterized constraint, i.e.,

$$g_i(x_i, w_i) = h_i(x_i)w_i, \ g_i(x_i, \mu_i) = h_i(x_i)\mu_i \tag{5}$$

for some function $h_i(x_i)$. The main result is summarized as follows.

Theorem 1 [16]: Consider the system (4) with (5) under Assumption 1. Let the distributed adaptive controller be

$$\mu_i = \hat{w}_i - \beta_i(x_i)$$
$$\dot{\hat{w}}_i = -\lambda_i h_i^\mathsf{T}(x_i) f_i(x) \tag{6}$$

where $\beta_i(x_i)$ is any continuously differentiable function satisfying

$$\frac{\partial \beta_i(x_i)}{\partial x_i} = -\lambda_i h_i^\mathsf{T}(x_i) \tag{7}$$

for some $\lambda_i > 0$. Then, the derivative of

$$U(x, z) = V(x) + \frac{\sigma}{4(1-k)} \sum_{i=1}^{n} z_i^\mathsf{T} z_i / (2\lambda_i) \tag{8}$$

with

$$z_i = \beta_i(x_i) - \tilde{w}_i, \ \tilde{w}_i = \hat{w}_i - w_i, \ z = [z_1^\mathsf{T}, z_2^\mathsf{T}, \ldots, z_n^\mathsf{T}]^\mathsf{T} \tag{9}$$

satisfies

$$\dot{U}(x, z) \leq -k\alpha(\|x\|_R), \tag{10}$$

for any $0 < k < 1$, along the trajectory of the closed-loop system (4)+(5)+(6).

The second challenge is to handle more general nonlinearly parameterized uncertainties. Together with the first challenge, the question is how to remove the linearly parameterized constraint (5) in Theorem 1. This is not a trivial extension. Without the constraint (5), the function $h_i(x_i)$ required for the adaptive controller (6) does not exist. In the next section, we will find a strategy to construct a suitable $h_i(x_i)$ that plays the same role, but is capable of tackling nonlinearly parameterized uncertainties.

III. A NEW DISTRIBUTED ADAPTIVE SCHEME

The main result of this section is to give the explicit condition for $h_i(x_i)$ and hence (6) and (7) for nonlinearly parameterized systems such that Theorem 1 still holds without the constraint (5). It is noted that for a linearly parameterized $g_i(x_i, w_i)$, the following property holds, for any vector z_i of the same dimension of w_i,

$$g_i(x_i, w_i) - g_i(x_i, w_i - z_i) = g_i(x_i, z_i) = h_i(x_i)z_i.$$

When $g_i(x_i, w_i)$ is nonlinearly parameterized, we will construct two functions $h_i(x_i)$ and $\tau_i(z_i)$ satisfying the following condition

$$\begin{aligned} &\left[h_i(x_i)\tau_i(z_i) - g_i(x_i, w_i) + g_i(x_i, w_i - z_i)\right]^{\mathsf{T}} \\ &\quad \times \left[g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\right] \geq 0. \end{aligned} \quad (11)$$

Remark 2: The condition (11) has two-fold meanings. On one hand, $g_i(x_i, w_i) - g_i(x_i, w_i - z_i)$ represents the change direction of the function $g_i(x_i, w_i)$ along the parameter w_i. The selection of $h_i(x_i)\tau_i(z_i)$ is along the change direction in the sense of

$$[h_i(x_i)\tau_i(z_i)]^{\mathsf{T}} \cdot \left[g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\right] \geq 0. \quad (12)$$

On the other hand, $h_i(x_i)\tau_i(z_i)$ determines the boundary of $g_i(x_i, w_i) - g_i(x_i, w_i - z_i)$ in the sense of

$$\begin{aligned} &\|[h_i(x_i)\tau_i(z_i)]^{\mathsf{T}}[g_i(x_i, w_i) - g_i(x_i, w_i - z_i)]\| \\ &\quad \geq \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|^2, \end{aligned} \quad (13)$$

that requires

$$\|h_i(x_i)\tau_i(z_i)\| \geq \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|. \quad (14)$$

It is easy to verify that (11) is equivalent to (12) +(13). For many classes of functions g_i, the two functions $h_i(x_i)$ and $\tau_i(z_i)$ can be constructed to satisfy (12) and (13), without knowing the exact value of w_i. An example is given in Section V.

The role of the condition (11) will be seen in the main result summarized in the following theorem.

Theorem 2: Consider the system (4) under Assumption 1. Suppose there exist two functions $h_i(x_i)$ and $\varrho_i(\cdot) > 0$ and a continuously differentiable positive definite function $W_i(z_i)$, such that (11) holds for $\tau_i(z_i) = \varrho_i(W_i(z_i))(\partial W_i(z_i)/\partial z_i)^{\mathsf{T}}$. Let the distributed adaptive controller be (6) where $\beta_i(x_i)$ is any continuously differentiable function satisfying (7) for some $\lambda_i > 0$. Then, the derivative of

$$U(x, z) = V(x) + \frac{\sigma}{4(1-k)} \sum_{i=1}^{n} \int_0^{W_i(z_i)} \frac{\varrho_i(s)}{\lambda_i} ds \quad (15)$$

with (9) satisfies (10) for any $0 < k < 1$, along the trajectory of the closed-loop system (4)+(6).

Proof: We can rewrite the closed-loop system composed of (4) and (6) as follows, noting $\mu_i = w_i - z_i$,

$$\dot{x}_i = f_i(x) + g_i(x_i, w_i) - g_i(x_i, w_i - z_i). \quad (16)$$

By direct calculation, we have the time derivative of the function $V(x)$, along the trajectory of (16), as follows

$$\begin{aligned} \dot{V}(x) \leq {}&-\alpha(\|x\|_R) \\ &+ \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i}(g_i(x_i, w_i) - g_i(x_i, w_i - z_i)). \end{aligned}$$

For any $0 < k < 1$, pick $a = (1-k)/\sigma$. Then,

$$a\left\|\frac{\partial V(x)}{\partial x}\right\|^2 \leq (1-k)\alpha(\|x\|_R). \quad (17)$$

Furthermore,

$$\begin{aligned} \dot{V}(x) \leq {}&-\alpha(\|x\|_R) + \sum_{i=1}^{n} a\left\|\frac{\partial V(x)}{\partial x_i}\right\|^2 \\ &+ \sum_{i=1}^{n} \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|^2/4a \\ \leq {}&-k\alpha(\|x\|_R) \\ &+ \frac{1}{4a} \sum_{i=1}^{n} \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|^2. \end{aligned} \quad (18)$$

Next, noting that

$$\dot{\hat{w}}_i = \frac{\partial\beta_i(x_i)}{\partial x_i}f_i(x), \quad (19)$$

the dynamics of z_i can be written as

$$\begin{aligned} \dot{z}_i &= \frac{\partial\beta_i(x_i)}{\partial x_i}\dot{x}_i - \dot{\hat{w}}_i \\ &= \frac{\partial\beta_i(x_i)}{\partial x_i}(f_i(x) + g_i(x_i, w_i) - g_i(x_i, w_i - z_i)) \\ &\quad - \frac{\partial\beta_i(x_i)}{\partial x_i}f_i(x) \\ &= \frac{\partial\beta_i(x_i)}{\partial x_i}(g_i(x_i, w_i) - g_i(x_i, w_i - z_i)). \end{aligned} \quad (20)$$

Therefore, we can verify that the time derivative of $W_i(z_i)$, along the trajectory of the aforementioned z_i-dynamics, satisfies

$$\dot{W}_i(z_i) = \frac{\partial W_i(z_i)}{\partial z_i}\frac{\partial\beta_i(x_i)}{\partial x_i}(g_i(x_i, w_i) - g_i(x_i, w_i - z_i)). \quad (21)$$

From above, the time derivative of $U(x, z)$, along the trajectory of the closed-loop system, is

$$\begin{aligned} \dot{U}(x, z) \leq {}&-k\alpha(\|x\|_R) \\ &+ \frac{1}{4a} \sum_{i=1}^{n} \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|^2 \\ &+ \frac{1}{4a} \sum_{i=1}^{n} \varrho_i(W_i(z_i))\frac{\partial W_i(z_i)}{\partial z_i}\frac{\partial\beta_i(x_i)}{\partial x_i} \\ &\times (g_i(x_i, w_i) - g_i(x_i, w_i - z_i))/\lambda_i \end{aligned} \quad (22)$$

and hence

$$\dot{U}(x, z) \leq -k\alpha(\|x\|_R)$$
$$+ \frac{1}{4a} \sum_{i=1}^{n} \|g_i(x_i, w_i) - g_i(x_i, w_i - z_i)\|^2$$
$$- \frac{1}{4a} \sum_{i=1}^{n} \tau_i^\mathsf{T}(z_i) h_i^\mathsf{T}(x_i)(g_i(x_i, w_i) - g_i(x_i, w_i - z_i))$$
$$\leq -k\alpha(\|x\|_R) \tag{23}$$

due to (11). The proof is thus completed. ∎

Remark 3: With the constraint (5), the condition (11) automatically holds for $\varrho_i(\cdot) = 1$, $W_i(z_i) = z_i^\mathsf{T} z_i/2$, and $\tau_i(z_i) = z_i$. Then, the function $U(x, z)$ in (15) reduces to that in (8), and hence Theorem 2 to Theorem 1. In other words, there is no additional conservativeness applied on Theorem 2, compared with its special version Theorem 1.

Remark 4: When the function $g_i(x_i, w_i)$ is a general non-linear function, it is not difficult to find two functions $h_i(x_i)$ and $\tau_i(z_i)$ to satisfy (11). However, it should be noted that the existence of $\beta_i(x_i)$ satisfying (7) is not always guaranteed for every $h_i(x_i)$. In other words, it is challenging to find a suitable $h_i(x_i)$ for both (7) and (11) simultaneously. The solution will be found on a case-by-case basis.

Remark 5: The adaptive controller (6) in Theorem 1 or Theorem 2 is applied to every agent i. This distributed control protocol only relies on the local agent state x_i and its nominal dynamics $f_i(x)$. The MAS with the nominal dynamics $f_i(x)$ is implemented in distributed fashion before-hand when the nonlinear function $g_i(x_i, w_i)$ has not been taken into consideration.

IV. ADAPTIVE CONSENSUS OF A SECOND-ORDER MAS

In this section, we study adaptive consensus for a second-order MAS using the scheme in Theorem 2. Consider $n \geq 2$ autonomous agents governed by the set of equations

$$\dot{p}_i = v_i$$
$$\dot{v}_i = \alpha_1 p_i + \alpha_2 v_i + \xi_i(v_i, w_i) + u_i, \quad i = 1, \dots, n, \tag{24}$$

where $p_i, v_i \in \mathbb{R}$ are the states, the constants $\alpha_1, \alpha_2 \in \mathbb{R}$ are the known parameters, $u_i \in \mathbb{R}$ is the control input of agent i, and $\xi_i(v_i, w_i)$ is a bounded nonlinear function with unknown constant parameter w_i. For convenience of presentation, we define $A = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}$, $x_i = \begin{bmatrix} p_i \\ v_i \end{bmatrix}$ and

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}. \tag{25}$$

In this section, the network topology is represented by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. A finite non-empty set of nodes is denoted by $\mathcal{V} = \{1, 2, \dots, n\}$ and the set of directed edges is presented by $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Let $\mathcal{A} = [a_{ij}]$ denote the adjacency matrix where $a_{ij} > 0$ if the edge $(j, i) \in \mathcal{E}$, $i \neq j$, $a_{ij} = 0$ if $i = j$. So, there exists no self-loop. Define the Laplacian matrix as $L = [L_{ij}]$ that has elements of $L_{ij} = -a_{ij}$, $j \neq i$ and $L_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij}$. Let L_i be the i-th row of L, then the

distributed information from the network to the agent i can be written as

$$L_i p = -\sum_{j=1}^{n} a_{ij}(p_j - p_i), \quad L_i v = -\sum_{j=1}^{n} a_{ij}(v_j - v_i).$$

In this section, we investigate a general directed leaderless MAS with the following assumption.

Assumption 2: The network topology contains a directed spanning tree.

Under Assumption 2, the Laplacian matrix L has one zero eigenvalue and all the remaining eigenvalues are with positive real parts. Let $r \in \mathbb{R}^n$ and $\mathbf{1}$ be the left and right eigenvectors corresponding to the eigenvalue 0. We have $r^\mathsf{T} L = 0$, $L\mathbf{1} = 0$, and $r^\mathsf{T} \mathbf{1} = 1$. There exist matrices $M \in \mathbb{R}^{(n-1) \times n}$, $N \in \mathbb{R}^{n \times (n-1)}$ such that

$$T = \begin{bmatrix} r^\mathsf{T} \\ M \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} \mathbf{1} & N \end{bmatrix}. \tag{26}$$

Then, the Laplacian matrix L can be transformed into

$$TLT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \tag{27}$$

where $J = MLN \in \mathbb{R}^{(n-1) \times (n-1)}$ is the matrix with positive eigenvalues of L on the diagonal. Let us define the matrix R as follows

$$\begin{bmatrix} Mp \\ Mv \end{bmatrix} = Rx \tag{28}$$

where R has a full row rank and the rows of R are perpendicular to span $\{\mathbf{1} \otimes I_2\}$. There are two technical lemmas regarding the property of the nominal system (24) with $\xi_i(v_i, w_i) = 0$.

Lemma 1 [16]: Under Assumption 2, by selecting sufficiently large γ_1 and $\gamma_2 = c\gamma_1 > 1$ with $c > 0$, then the matrix $\bar{A} = \begin{bmatrix} 0 & I \\ \alpha_1 I - \gamma_1 J & \alpha_2 I - \gamma_2 J \end{bmatrix}$ is Hurwitz.

Lemma 2 [16]: Consider the MAS (24) with $\xi_i(v_i, w_i) = 0$ under Assumption 2. Let the control protocol be

$$u_i = -\gamma_1 L_i p - \gamma_2 L_i v, \tag{29}$$

where γ_1 and γ_2 are properly selected such that the matrix $\bar{A}$ is Hurwitz. Let $P = P^\mathsf{T} > 0$ be a unique solution to the Lyapunov equation such that $P\bar{A} + \bar{A}^\mathsf{T} P = -I$. The Lyapunov function $V(x) = x^\mathsf{T} R^\mathsf{T} P R x$ satisfies

$$\lambda_{\min}(P)\|x\|_R^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|_R^2. \tag{30}$$

The derivative of $V(x)$ along the closed-loop system satisfies $\dot{V}(x) = -\|x\|_R^2$.

Now, the main result on a distributed adaptive controller for the MAS (24) is stated in the following theorem.

Theorem 3: Consider the MAS (24) under Assumption 2. Suppose there exist two functions $\hbar_i(v_i)$ and $\varrho_i(\cdot) > 0$ such that, with $\tau_i(\varsigma_i) = \varrho_i(\varsigma_i^2/2)\varsigma_i$,

$$[\hbar_i(v_i)\tau_i(\varsigma_i) - \xi_i(v_i, w_i) + \xi_i(v_i, w_i - \varsigma_i)]^\mathsf{T}$$
$$\times [\xi_i(v_i, w_i) - \xi_i(v_i, w_i - \varsigma_i)] \geq 0. \tag{31}$$

Let the controller be

$$u_i = -\gamma_1 L_i p - \gamma_2 L_i v - \xi_i(v_i, \mu_i) \tag{32}$$

and

$$\mu_i = \hat{w}_i - \rho_i(v_i)$$
$$\dot{\hat{w}}_i = -\lambda_i \hbar_i^\mathsf{T}(v_i)[\alpha_1 p_i + \alpha_2 v_i - \gamma_1 L_i p - \gamma_2 L_i v], \quad (33)$$

where γ_1 and γ_2 are selected according to Lemma 2 and $\rho_i(v_i)$ is a continuously differentiable function satisfying

$$\frac{\partial \rho_i(v_i)}{\partial v_i} = -\lambda_i \hbar_i^\mathsf{T}(v_i) \quad (34)$$

for some $\lambda_i > 0$. Then, the closed-loop system (24)+(32)+(33) achieves consensus in the sense of

$$\lim_{t\to\infty} p_i(t) - p_o(t) = 0, \quad \lim_{t\to\infty} v_i(t) - v_o(t) = 0 \quad (35)$$

for some functions $p_o(t), v_o(t):[0,\infty) \mapsto \mathbb{R}$.

Proof: First of all, the system composed of (24) and (32) can be written as follows,

$$\dot{p}_i = v_i$$
$$\dot{v}_i = \alpha_1 p_i + \alpha_2 v_i - \gamma_1 L_i p - \gamma_2 L_i v + \xi_i(v_i, w_i)$$
$$\quad - \xi_i(v_i, \mu_i), \quad i = 1, \dots, n.$$

It can also be put in the following compact form,

$$\dot{x}_i = f_i(x) + g_i(x_i, w_i) - g_i(x_i, \mu_i) \quad (36)$$

where the nominal dynamics $\dot{x}_i = f_i(x)$ is given by

$$\dot{p}_i = v_i$$
$$\dot{v}_i = \alpha_1 p_i + \alpha_2 v_i - \gamma_1 L_i p - \gamma_2 L_i v, \quad i = 1, \dots, n \quad (37)$$

and

$$g_i(x_i, w_i) = \begin{bmatrix} 0 \\ \xi_i(v_i, w_i) \end{bmatrix}, \quad g_i(x_i, \mu_i) = \begin{bmatrix} 0 \\ \xi_i(v_i, \mu_i) \end{bmatrix}. \quad (38)$$

The system (36) takes the form (4). Applying Lemma 2 verifies Assumption 1 for $\dot{x}_i = f_i(x)$. Furthermore, we can verify that

$$\frac{\left\| \frac{\partial V(x)}{\partial x} \right\|^2}{\|x\|_R^2} = \frac{\|2x^\mathsf{T} R^\mathsf{T} PR\|^2}{\|x\|_R^2} \le 4\|PR\|^2 < \infty. \quad (39)$$

By Theorem 2 with $\varsigma_i = \rho_i(v_i) - \hat{w}_i + w_i$, $h(x_i) = \hbar_i(v_i)$, $\tau_i(z_i) = \tau_i(\varsigma_i)$, $\beta_i(x_i) = \rho_i(v_i)$, and

$$U(x, \varsigma) = V(x) + \frac{\sigma}{4(1-k)} \sum_{i=1}^{n} \int_0^{\varsigma_i^2/2} \frac{\varrho_i(s)}{\lambda_i} ds, \quad (40)$$

one has $\dot{U}(x, \varsigma) \le -k\|x\|_R^2$.

It is noted that $U(x(t), \varsigma(t))$ and hence $\|x(t)\|_R$ are bounded. Because of

$$R\dot{x} = \bar{A}Rx + R(g(x, w) - g(x, w - \varsigma)),$$

with $w = [w_1^\mathsf{T}, \dots, w_n^\mathsf{T}]^\mathsf{T}$, $\varsigma = [\varsigma_1^\mathsf{T}, \dots, \varsigma_n^\mathsf{T}]^\mathsf{T}$, and $g(x, w) = [g_1^\mathsf{T}(x_1, w_1), \dots, g_n^\mathsf{T}(x_n, w_n)]^\mathsf{T}$, $\|\dot{x}(t)\|_R$ is bounded and hence $-k\|x(t)\|_R^2$ uniformly continuous in t. Also, a finite limit

$$\lim_{t\to\infty} \int_0^t -k\|x(t)\|_R^2 \ge \lim_{t\to\infty} \int_0^t \dot{U}(x(t), \varsigma(t)) \ge -U(x(0), \varsigma(0))$$

exists. By Barbalat's Lemma, one has $\lim_{t\to\infty} \|x(t)\|_R = 0$, that is,

$$\lim_{t\to\infty} \begin{bmatrix} Mp(t) \\ Mv(t) \end{bmatrix} = 0. \quad (41)$$

Let $p_o(t) = r^\mathsf{T} p(t)$ and $v_o(t) = r^\mathsf{T} v(t)$. One has

$$\lim_{t\to\infty} p(t) - p_o(t)\mathbf{1} = N \lim_{t\to\infty} Mp(t) = 0$$
$$\lim_{t\to\infty} v(t) - v_o(t)\mathbf{1} = N \lim_{t\to\infty} Mv(t) = 0.$$

This completes the proof. ∎

Remark 6: If the parameter w_i in (24) were known, the following controller

$$u_i = -\gamma_1 L_i p - \gamma_2 L_i v - \xi_i(v_i, w_i) \quad (42)$$

could be designed to achieve consensus by directly canceling $\xi_i(v_i, w_i)$ in (24). In the practical case with w_i unknown, the real controller takes the form (32), which is equivalent to (42) with w_i replaced by its estimation μ_i. Also, the estimation is determined by the adaptive law (33). The design approach in Theorem 3 constitutes the certainty equivalence principle.

V. A NUMERICAL EXAMPLE

We consider the following six-agent systems

$$\dot{p}_i = v_i$$
$$\dot{v}_i = -p_i + \xi_i(v_i, w_i) + u_i, \quad i = 1, \dots, 6. \quad (43)$$

The nonlinear functions $\xi_i(v_i, w_i)$ are given as follows

$$\xi_i(v_i, w_i) = \begin{cases} \tanh(w_i v_i^2), & i = 1, 2, 3, 4 \\ v_i \tanh(w_i v_i^2), & i = 5, 6 \end{cases}, \quad (44)$$

and the unknown constant parameters w_i are arbitrarily selected. The communication network for the MAS (43) has a fixed topology with its Laplacian represented by

$$L = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 0 \\ -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -4 & 9 & -5 \\ -1 & 0 & -2 & 0 & 0 & 3 \end{bmatrix}. \quad (45)$$

By Lemma 1, we select $\gamma_1 = 10$ and $\gamma_2 = 15$ such that $\bar{A}$ is Hurwitz. Pick the function $\hbar_i(v_i)$ as follows,

$$\hbar_i(v_i) = \begin{cases} v_i^2 + 1, & i = 1, 2, 3, 4 \\ v_i^3 + v_i, & i = 5, 6. \end{cases} \quad (46)$$

First, we consider the agents $i = 1, 2, 3, 4$. For $\varsigma_i \ge 0$, one has

$$\hbar_i(v_i)\varsigma_i \ge \varsigma_i v_i^2 \ge \tanh(w_i v_i^2) - \tanh((w_i - \varsigma_i)v_i^2) \ge 0$$

and hence

$$\left[\hbar_i(v_i)\varsigma_i - \tanh(w_i v_i^2) + \tanh((w_i - \varsigma_i)v_i^2) \right]$$
$$\times \left[\tanh(w_i v_i^2) - \tanh((w_i - \varsigma_i)v_i^2) \right] \ge 0, \quad (47)$$

which verifies (31) with $\varrho_i(\cdot) = 1$. For $\varsigma_i \le 0$, a similar argument follows. Also, we can verify (31) with $\varrho_i(\cdot) = 1$. for the agents $i = 5, 6$.
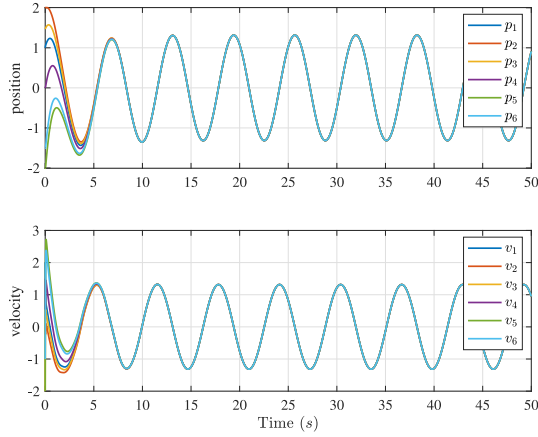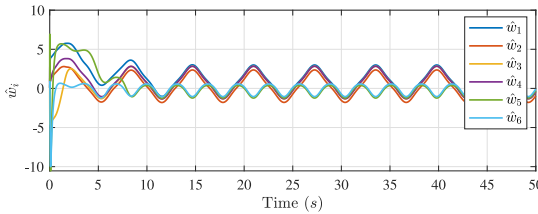
Fig. 1.   Profile of consensus of the MAS.

Fig. 2.   Profile of the state $\hat{w}_i$ of the update law.

Now, we pick the function $\rho_i(v_i)$, satisfying (34) with $\lambda_i = 1$, as follows

$$\rho_i(v_i) = \begin{cases} -\frac{1}{3}v_i^3 - v_i, & i = 1, 2, 3, 4 \\ -\frac{1}{4}v_i^4 - \frac{1}{2}v_i^2, & i = 5, 6. \end{cases} \quad (48)$$

As a result, the controller (32)+(33) can be explicitly constructed and Theorem 3 guarantees the achievement of consensus.

It is observed in Fig. 1 that the states of the six agents achieve consensus on a sinusoid of frequency 1 rad/s determined by the nominal dynamics $\dot{p}_i = v_i$, $\dot{v}_i = -p_i$. The profile of the state $\hat{w}_i$ is plotted in Fig. 2, which converges, not to the real value of w_i as in traditional adaptive control, but to $w_i + \rho_i(v_i)$ for a deliberately designed $\rho_i(v_i)$. For $i = 1, 2, 3, 4$, $\rho_i(v_i)$ contains v_i and v_i^3, so $\hat{w}_i$ demonstrates the fundamental frequency 1 rad/s of v_i at the steady state. For $i = 5, 6$, $\rho_i(v_i)$ contains v_i^2 and v_i^4, so $\hat{w}_i$ demonstrates the fundamental frequency 2 rad/s of v_i^2.

VI. CONCLUSION

In this letter, we have presented a distributed adaptive consensus protocol for an MAS with uncertain nonlinearities to maintain system's nominal collective behavior. The new adaptive scheme is effective for general nonlinearly parameterized systems. To demonstrate its effectiveness, we have solved a consensus problem for a leaderless second-order MAS with a directed network. It will be interesting to apply the proposed adaptive scheme for more collective control scenarios in future research.

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