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Comparison of Periodic and Event-Based Sampling for Linear State Estimation ⋆

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Abstract: In this paper, the state estimation problem for continuous-time linear systems with two types of sampling is considered. First, the optimal state estimator under periodic sampling is presented. Then the state estimator with event-based updates is designed, i.e., when an event occurs the estimator is updated linearly by using the measurement of output, while between the consecutive event times the estimator is updated by minimum mean-squared error criteria. The average estimation errors under both sampling schemes are compared quantitatively for first and second order systems, respectively. A numerical example is given to compare the effectiveness of two state estimators.

Keywords: Event-based estimation, Kalman filter, periodic sampling, event-based sampling, average estimation errors

1. INTRODUCTION

Recently, event-based control and estimation have received a lot of attention in the system and control community (Lemmon, 2010; Heemels et al., 2012). Event-based mechanism occurs naturally in many situations, such as relay systems (Dodds, 1981), pulse modulation (Holmes and Lipo, 2003), and biological systems (Keener and Sneyd, 2008); particularly it has been boosted in wireless sensor networks where the communication resource is limited and even scare (Xu and Hespanha, 2004; Imer and Basar, 2005). Åström and Bernhardsson (2002) highlighted some advantages of event-based sampling over periodic sampling and motivated the development of systematic design and analysis of event-based controllers and estimators (Tabuada, 2007; Henningsson et al., 2008; Rabi et al., 2008; Li et al., 2010; Wang and Lemmon, 2011; Meng and Chen, 2012; Meng et al., 2013; Wang et al., 2014). In this paper, we focus on the event-based estimation problem.

To reduce data transmission rate in sensor networks, some estimation schemes and algorithms on event-based mechanism such as “Send on Delta” were presented and considered (Miskowicz, 2006; Suh et al., 2007). Åström (2007) provided some comments on Kalman filter under event-based sampling. Rabi (2006) and Rabi et al. (2012) investigated the state estimation problem for first order linear systems with a constraint on the sampling rate, and compared the estimation errors under event-based and periodic sampling. However, to our knowledge, there has been no work carrying out the comparison of periodic and event-based sampling for state estimation of higher order systems, which is definitely worth studying.

The previous work (Sijs and Lazar, 2011; Wu et al., 2013; You and Xie, 2013) gave some approximating algorithms for event-based estimator under the assumption that the predicted estimate or the innovation is Gaussian. Sijs and Lazar (2011) considered the state estimation under general event based sampling, and provided a stable approximation algorithm by using Gaussian sums. Wu et al. (2013), and You and Xie (2013) devised scheduling schemes that the output is communicated based on measurement innovation, and provided recursive algorithms for the discrete-time state estimation with the assumption of Gaussian prediction. However, the above Gaussian assumption is not guaranteed to be true. Without the assumption, Shi et al. (2013) considered the discrete-time event-triggered state estimation problem in the framework of maximum likelihood estimation. In this paper, some results for continuous-time linear state estimation are provided without the Gaussion assumption.

The objective of the paper is to compare periodic and event-based sampling for the state estimation of continuous-time linear stochastic systems. First, we present the optimal state estimator under periodic sampling. Then we design the state estimator with event-based updates. Specifically, when an event occurs the estimator is updated linearly by using the upcoming measurement, while between the consecutive event times the estimator is updated by MMSE (minimum mean-squared error) criteria. The main idea of the event-based sampling is that the output measurement is transmitted to the estimator only if the measurement innovation is significant enough. Later, we compare the mean-square estimate errors of two state

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estimators under the same average sampling rates for first order systems and symmetric second order systems, respectively. Finally, we give a numerical example to compare the estimation errors of two state estimators.

The following notations will be used in the paper. For a sequence of numbers $a_j, j = 1, \ldots, m$, $\text{diag}\{a_1, \ldots, a_m\}$ denotes the diagonal matrix with a_j in the diagonal and zero elsewhere. For a stochastic process $x(t), t \geq 0$, $\sigma(x(s), s \leq t)$ denotes the σ-algebra generated by $x(s), s \leq t$. $\mathbb{E}[\cdot]$ and $\mathbb{E}[\cdot|\cdot]$ denote the mathematical expectation and the conditional expectation, respectively.

2. PROBLEM FORMULATION

Consider the following linear stochastic system:

$$dx(t) = Ax(t) + DdW(t), \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^m$ are the state and the output, respectively. $E[x(0)] = \mu$, $E[(x(0) - \mu)(x(0) - \mu)^T] = \Sigma > 0$. The stochastic disturbance $W(t) \in \mathbb{R}^d$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume $C \in \mathbb{R}^{m\times n}$ has full row rank. (A, D) is controllable. The output $y(t)$ is transmitted to the estimator by a sensor with limited computational capability at sampling times $\tau_0, \ldots, \tau_N, \ldots$.

In this paper, we will design and compare two types of estimators for the state $x(t)$ based on the measurements $y(\tau_0), \ldots, y(\tau_k)$, and sampling times $\tau_0, \ldots, \tau_k$, where $\tau_k \leq t < \tau_{k+1}, k = 0, 1, \ldots$. To be specific, for the first estimator the sampling times are deterministic and periodic, while for the second estimator the sampling times are a sequence of random times based on the event occurrence.

3. STATE ESTIMATORS

From (1), we have for any $k = 0, 1, \ldots$,

$$x(t) = e^{A(t-\tau_k)}x(\tau_k) + \int_{\tau_k}^{t} e^{A(t-s)}DdW(s), \quad \tau_k \leq t. \tag{3}$$

Let $\mathcal{F}_t = \sigma\{x(s), s \leq t\}$, and $\mathcal{Y}_t = \sigma\{y(s), s \leq t\}$. Let $\mathcal{H}_t = \sigma\{y(\tau_0), \ldots, y(\tau_k), \tau_0, \ldots, \tau_k\}$ for $\tau_k \leq t < \tau_{k+1}$. Noting $\mathcal{H}_t \subset \mathcal{Y}_t \subset \mathcal{F}_t$ and $\mathcal{H}_t = \mathcal{H}_{\tau_k}$, $\tau_k \leq t < \tau_{k+1}$, from (3) we obtain that

$$\begin{aligned}\mathbb{E}[x(t)|\mathcal{H}_t] &= \mathbb{E}[e^{A(t-\tau_k)}x(\tau_k)|\mathcal{H}_{\tau_k}] \\ &\quad + \mathbb{E}\big[\mathbb{E}[\int_{\tau_k}^{t} e^{A(t-s)}DdW(s)|\mathcal{F}_{\tau_k}]\big|\mathcal{H}_{\tau_k}\big] \\ &= e^{A(t-\tau_k)}\mathbb{E}[x(\tau_k)|\mathcal{H}_{\tau_k}], \ \tau_k \leq t < \tau_{k+1}. \end{aligned} \tag{4}$$

From the fundamental result of state estimation (Åström, 2006; Oksendal, 2003), we have the MMSE (minimum mean-squared error) update between two consecutive sampling times:

$$\hat{x}(t) = e^{A(t-\tau_k)}\hat{x}(\tau_k), \quad \tau_k \leq t < \tau_{k+1},$$

where $\hat{x}(t)$ is the estimator for $x(t)$.

In what follows, we develop the recursive algorithms for $\hat{x}(\tau_k), k = 0, 1, \ldots$. By (2) and (3), it follows that

$$\begin{aligned} x(\tau_{k+1}) &= e^{A(\tau_{k+1}-\tau_k)}x(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} e^{A(\tau_{k+1}-s)}DdW(s), \\ y(\tau_k) &= Cx(\tau_k), \ k = 0, 1, \ldots. \end{aligned} \tag{5}$$

First, consider the case of periodic sampling, i.e., $\tau_k = kh$. Following the argument to the standard Kalman filter derivation (Anderson and Moore, 1979; Åström, 2006), we get the algorithm for the periodic estimator:

Algorithm I. Periodic estimator:

$$\begin{aligned} \hat{x}[(k+1)h] &= \hat{x}'[(k+1)h] \\ &\quad + K_{k+1}\big\{y[(k+1)h] - C\hat{x}'[(k+1)h]\big\}, \\ \hat{x}'[(k+1)h] &= e^{Ah}\hat{x}(kh), \\ K_{k+1} &= P'_{k+1}C^T(CP'_{k+1}C^T)^{-1}, \\ P_{k+1} &= P'_{k+1} - K_{k+1}CP'_{k+1}; \\ P'_{k+1} &= e^{Ah}P_k e^{A^T h} \\ &\quad + \int_{kh}^{(k+1)h} e^{A[(k+1)h-s]}DD^T e^{A^T[(k+1)h-s]}ds, \\ \hat{x}'(0) &= \mu, \quad P'_0 = \Sigma. \end{aligned} \tag{6}$$

We now consider the case that the sampling times are the following event-based times:

$$\begin{aligned} &\tau_0 = 0, \\ &\tau_{k+1} = \inf\{t > \tau_k; \|z(t-)\| \geq \delta\}, \quad k = 0, 1, \ldots, \end{aligned} \tag{7}$$

where $\delta > 0$ is a tuning parameter, $z(t) = y(t) - C\hat{x}(t)$ is the so-called innovation process, and $z(t-) = \lim_{s\uparrow t} z(s)$ is the left limit of $z(t)$.

Remark 3.1. The main idea of the specification for the event-based times $\tau_k, k = 0, 1, \ldots$ is that the output is transmitted to the estimator only when the measurement innovation (the error between the measurement and the predicted value of output) is large enough; otherwise it is not regarded "significant enough" and hence not be sent to the estimator. It is still not clear whether the innovation process $z(t)$ is approximately Gaussian or not. However, $\|z(\tau_k-)\| = \delta$, and $z(\tau_k-)$ is non-Gaussian. Actually, for the one-dimensional case, $z(\tau_k-)$ is binomial, and for the two-dimensional case, $z(\tau_k-)$ is supported on a circle.

From (4), we have

$$\hat{x}(\tau_{k+1}-) \triangleq \lim_{t\uparrow\tau_{k+1}} \hat{x}(t) = e^{A(\tau_{k+1}-\tau_k)}\hat{x}(\tau_k). \tag{8}$$

At the time τ_{k+1}, an event occurs and a new measurement is received by the estimator. Then by Theorem 3.2.1 in Anderson and Moore (1979), we get the linear MMSE update as follows:

$$\hat{x}(\tau_{k+1}) = \hat{x}(\tau_{k+1}-) + K_{k+1}z(\tau_{k+1}-), \ k = 0, 1, \ldots \tag{9}$$

where

$$K_{k+1} = P^-_{k+1}C^T(CP^-_{k+1}C^T)^{-1}, \tag{10}$$

$$\begin{aligned} P^-_{k+1} &= e^{A(\tau_{k+1}-\tau_k)}P_k e^{A^T(\tau_{k+1}-\tau_k)} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} e^{A(\tau_{k+1}-s)}DD^T e^{A^T(\tau_{k+1}-s)}ds, \end{aligned} \tag{11}$$

$$P_k = P^-_k - K_k CP^-_k. \tag{12}$$

Here, (11) is obtained from

$$\tilde{x}(\tau_{k+1}-) = e^{A(\tau_{k+1}-\tau_k)}\tilde{x}(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} e^{A(\tau_{k+1}-s)}DdW(s),$$

which follows by (5) and (8).

By the above result we get the following algorithm for the event-based estimator.

Algorithm II. Event-based estimator:

$$\begin{aligned}\hat{x}(\tau_k) &= \hat{x}(\tau_k-) + K_k\big[y(\tau_k) - C\hat{x}(\tau_k-)\big],\\ \hat{x}(\tau_{k+1}-) &= e^{A(\tau_{k+1}-\tau_k)}\hat{x}(\tau_k),\\ K_k &= P_k^- C^T(CP_k^-C^T)^{-1},\\ P_k &= P_k^- - K_kCP_k^-,\\ P_{k+1}^- &= e^{A(\tau_{k+1}-\tau_k)}P_k^- e^{A^T(\tau_{k+1}-\tau_k)}\\ &\quad+\int_{\tau_k}^{\tau_{k+1}} e^{A[\tau_{k+1}-s]}DD^Te^{A^T[\tau_{k+1}-s]}ds,\\ \hat{x}(0-) &= \mu,\quad P_0^- = \Sigma. \end{aligned} \tag{13}$$

Different from Algorithm I, the updating times $\tau_k, k = 1, 2, \dots$ in Algorithm II are random. From (7), they depend on the output $y(t)$, which implies that the gains $K_k, k = 1, 2, \dots$ are also random and dependent on $y(t)$.

Remark 3.2. Noting $\tau_0 = 0, \Sigma > 0$, it follows that $z(\tau_0) = y(\tau_0) - C\hat{x}(\tau_0) = 0$, which leads to $\tau_1 > 0$. Since (A, D) is controllable, it follows that for any $t > 0$, $\int_0^t e^{As}DD^Te^{A^Ts}ds$ is positive definite, which gives that $\int_{\tau_0}^{\tau_1} e^{A(\tau_1-s)}DD^Te^{A^T(\tau_1-s)}ds$ is positive definite. This together with the fact that C has full row rank implies that $CP_1^-C^T$ is invertible and positive definite. From (13), it follows that $z(\tau_1) = y(\tau_1) - C\hat{x}(\tau_1) = 0$, which together with (7) leads to $\tau_2 > \tau_1$. Thus, by the induction we obtain that τ_{k+1} is strictly greater than τ_k, which guarantees the sequence of sampling times $\tau_k, k = 0, 1, \dots$ are well defined.

4. THE CASE OF FIRST ORDER SYSTEMS

In this section, consider the first order system described by

$$dx(t) = ax(t)dt + \sigma dw(t), \tag{14}$$
$$y(t) = cx(t), \tag{15}$$

where $x(t), y(t) \in \mathbb{R}$, $c \neq 0$, and $\sigma \neq 0$. $w(t)$ is a one-dimensional Wiener process. We will compare the average estimation errors for both sampling schemes.

For the system above, by Algorithm II we obtain that

$$\hat{x}(\tau_k) = \frac{1}{c}y(\tau_k),\quad K_k = \frac{1}{c},\quad P_k = 0.$$

From (4), it follows that

$$\hat{x}(t) = \frac{1}{c}e^{a(t-\tau_k)}y(\tau_k),\ \tau_k \le t < \tau_{k+1}.$$

We now check the average estimation errors

$$J = \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T |x(t) - \hat{x}(t)|^2dt\right], \tag{16}$$

for periodic and event-based sampling, respectively. First, under the periodic sampling scheme (i.e., $\tau_k = kh$), the average estimation error is

$$J_P = \limsup_{n\to\infty}\frac{1}{nh}\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\mathbb{E}\left[|x(t) - \hat{x}(t)|^2\right]dt$$
$$= \limsup_{n\to\infty}\frac{1}{nh}\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\mathbb{E}\left[\Big|\int_{kh}^t \sigma e^{a(t-s)}dw(s)\Big|^2\right]dt.$$

Thus, we have for the case $a = 0$,

$$J_P = \limsup_{n\to\infty}\frac{1}{nh}\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\sigma^2(t-kh)dt = \frac{\sigma^2h}{2},$$

and for the case $a \neq 0$,

$$\begin{aligned}J_P &= \limsup_{n\to\infty}\frac{1}{nh}\sum_{k=0}^{n-1}\sigma^2\int_{kh}^{(k+1)h}\frac{e^{2a(t-kh)}-1}{2a}dt\\ &= \frac{\sigma^2(e^{2ah}-2ah-1)}{4a^2h}.\end{aligned}$$

In what follows, we calculate the average estimation error under the event-based sampling. In this case, the sampling times are $\tau_0 = 0$, and

$$\tau_{k+1} = \inf\{t > \tau_k; |z(t-)| \ge \delta\},\quad k = 0, 1, \dots,$$

where $z(t) = y(t) - c\hat{x}(t) = \sigma\int_{\tau_k}^t e^{a(t-s)}dw(s), \tau_k \le t < \tau_{k+1}$. It can be verified that $z(\tau_k) = 0$, and $z(t)$ satisfies

$$dz(t) = az(t) + c\sigma dw(t),\quad \tau_k \le t < \tau_{k+1}. \tag{17}$$

Let $\mathcal{Z}_t = \sigma(z(s), s \le t)$. Noting $z(t)$ is a time-homogeneous strong Markovian process (Oksendal, 2003) and $z(\tau_k) = 0$, $k = 0, 1, 2, \dots$, we obtain that for any Borel-measurable function f,

$$\begin{aligned}\mathbb{E}\Big[\int_{\tau_k}^{\tau_{k+1}} f(z(t))dt\Big|\mathcal{Z}_{\tau_k}\Big] &= \mathbb{E}\Big[\int_{\tau_k}^{\tau_{k+1}} f(z(t))dt\Big|z(\tau_k)\Big]\\ &= \mathbb{E}\Big[\int_{\tau_0}^{\tau_1} f(z(t))dt\Big|z(\tau_0)\Big]\\ &= \mathbb{E}\Big[\int_0^{\tau_1} f(z(t))dt\Big].\end{aligned}$$

Thus, $\int_{\tau_k}^{\tau_{k+1}} f(z(t))dt, k = 0, 1, \dots$ are independent and identically distributed. From this together with (16), the average estimation error is

$$\begin{aligned}J_E &= \limsup_{n\to\infty}\frac{1}{c^2\tau_n}\sum_{k=0}^{n-1}\mathbb{E}\Big[\int_{\tau_k}^{\tau_{k+1}}|z(t)|^2dt\Big]\\ &= \limsup_{n\to\infty}\frac{n}{c^2\tau_n}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}\Big[\int_{\tau_k}^{\tau_{k+1}}|z(t)|^2dt\Big]\\ &= \frac{\mathbb{E}\big[\int_0^{\tau_1}|z(t)|^2dt\big]}{c^2\mathbb{E}[\tau_1]}.\end{aligned} \tag{18}$$

From the probabilistic representation of solutions to differential equations (See Lemma 3.1 in Wang et al. (2014) or Oksendal (2003)), it follows that $\mathbb{E}[\tau_1|z(0) = x]$ is the solution of

$$\frac{c^2\sigma^2}{2}h''(x) + axh'(x) = -1,$$

with boundary conditions $h(\delta) = h(-\delta) = 0$, which gives

$$\begin{aligned}\mathbb{E}[\tau_1] =& 2\int_0^\delta\frac{1}{c^2\sigma^2}\int_0^y\exp\big[-\frac{a(y^2-z^2)}{c^2\sigma^2}\big]dzdy\\ =& \sum_{k=1}^\infty\frac{2^{2k-1}(-a)^{k-1}(k-1)!\delta^{2k}}{(2k)!}.\end{aligned}$$

Similarly, we have

$$\mathbb{E}\Big[\int_0^{\tau_1}|z(t)|^2dt\Big] = 2\int_0^\delta\int_0^y\frac{z^2}{c^2\sigma^2}\exp\big[-\frac{a(y^2-z^2)}{c^2\sigma^2}\big]dzdy.$$

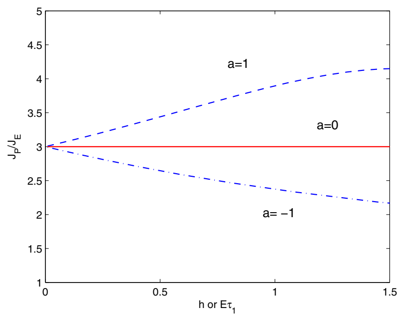


Fig. 1. J_P/J_E for $a = -1, 0, 1$ with the same average sampling period

From this together (18), we get

$$J_E = \frac{\int_0^\delta \int_0^y z^2 \exp\big[-\frac{a(y^2-z^2)}{c^2\sigma^2}\big]dzdy}{c^2\int_0^\delta \int_0^y \exp\big[-\frac{a(y^2-z^2)}{c^2\sigma^2}\big]dzdy},$$

which coincides with the result in (Rabi, 2006, Chapter 3). Particularly, for the case $a = 0$,

$$\mathbb{E}[\tau_1] = \frac{\delta^2}{c^2\sigma^2}, \qquad J_E = \frac{\delta^2}{6c^2}.$$

We now compare the estimation effectiveness under both sampling schemes. Specifically, the estimation error with periodic sampling is compared to one with event-based sampling under the assumption that average sampling rates are the same, i.e. $h = h_E$. For the case $a = 0$, we have $h = \mathbb{E}[\tau_1] = \delta^2/(c^2\sigma^2)$ and

$$\frac{J_P}{J_E} = \frac{\delta^2/2c^2}{\delta^2/6c^2} = 3.$$

The ratio J_P/J_E as a function of h or $\mathbb{E}[\tau_1]$ is plotted in Figure 1 for $a = -1, 0, 1$. The figure shows that event-based sampling gives substantially smaller estimation errors under the same sampling rates. It can be seen that the ratio for unstable systems is larger than one for stable systems.

5. THE CASE OF SYMMETRIC SECOND ORDER SYSTEMS

In this section, consider the following second order system described by

$$\begin{aligned} dx(t) &= Ax(t)dt + Ddw(t),\\ y(t) &= Cx(t), \end{aligned} \tag{19}$$

where $x(t), y(t) \in \mathbb{R}^2$, $A = \text{diag}\{a, a\}$, $C = \text{diag}\{c, c\}$ and $D = \text{diag}\{\sigma, \sigma\}$. $w(t) = (w_1(t), w_2(t))^T$ is a two-dimensional Wiener process. Without loss of generality, assume $c = \sigma = 1$. In this case, by Algorithm II and (4), it follows that

$$K_k = I, \quad P_k = 0, \quad \hat{x}(\tau_k) = x(\tau_k),$$
$$\hat{x}(t) = e^{A(t-\tau_k)}x(\tau_k), \ \tau_k \le t < \tau_{k+1}.$$

First, for the case of the periodic sampling the average estimation error is

$$\begin{aligned} J_P &= \limsup_{n\to\infty} \frac{1}{nh}\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h} \mathbb{E}\big[\|x(t)-\hat{x}(t)\|^2\big]dt\\ &= \limsup_{n\to\infty} \frac{1}{nh}\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h} 2\mathbb{E}\Big[\Big|\int_{kh}^{t} e^{a(t-s)}dw(s)\Big|^2\Big]dt\\ &= \begin{cases} h & a = 0\\ \dfrac{e^{2ah}-2ah-1}{2a^2h} & a \ne 0. \end{cases} \end{aligned}$$

In what follows, we calculate the average estimation error under the event-based sampling. In this case, the sampling times are $\tau_{k+1} = \inf\{t > \tau_k; \|z(t-)\| \ge \delta\}$, where $z(t) = x(t) - \hat{x}(t)$. Then

$$z(t) = \Big(\int_{\tau_k}^{t} e^{a(t-s)}dw_1(s), \int_{\tau_k}^{t} e^{a(t-s)}dw_2(s)\Big)^T$$

satisfies

$$dz(t) = Az(t) + dw(t), \quad z(\tau_k) = (0,0)^T, \quad \tau_k \le t < \tau_{k+1}.$$

As in Gardiner (2004) and Meng and Chen (2012), set

$$z_1(t) = r(t)cos[\phi(t)], \quad z_2(t) = r(t)sin[\phi(t)],$$

with

$$r(t) = \sqrt{z_1^2(t) + z_2^2(t)}.$$

Then by using Ito's formula, we have that

$$dr(t) = [ar(t) + \frac{1}{2r(t)}]dt + dv(t), \ r(\tau_k) = 0, \ \tau_k \le t < \tau_{k+1},$$

where

$$v(t) = w_1(t)cos[\phi(t)] + w_2(t)sin[\phi(t)].$$

It can be verified that $v(t)$ is a standard Wiener process, since it is an orthogonal transformation of $w(t)$.

From Lemma 3.1 in Wang et al. (2014) or Oksendal (2003), we obtain that $\mathbb{E}[\tau_1|r(0) = x]$ satisfies

$$\frac{1}{2}g''(x) + (ax + \frac{1}{2x})g'(x) = -1,$$

with boundary conditions $g(\delta) = g(-\delta) = 0$, which gives

$$\begin{aligned} \mathbb{E}[\tau_1] &= 2\int_0^\delta \int_0^y \frac{z}{y}\exp\big[-a(y^2-z^2)\big]dzdy\\ &= \sum_{k=1}^{\infty} \frac{(-a)^{k-1}\delta^{2k}}{2kk!}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathbb{E}\Big[\int_0^{\tau_1} \|z(t)\|^2 dt\Big] &= 2\int_0^\delta \int_0^y \frac{z^3}{y}\exp\big[-a(y^2-z^2)\big]dzdy\\ &= \sum_{k=1}^{\infty} \frac{(-a)^{k-1}\delta^{(2k+2)}}{(2k+2)(k+1)!}. \end{aligned}$$

From this together (18), we get

$$J_E = \frac{\int_0^\delta \int_0^y \frac{z^3}{y}\exp\big[-a(y^2-z^2)\big]dzdy}{\int_0^\delta \int_0^y \frac{z}{y}\exp\big[-a(y^2-z^2)\big]dzdy}.$$

Particularly, for the case $a = 0$,

$$\mathbb{E}[\tau_1] = \frac{\delta^2}{2}, \qquad J_E = \frac{\delta^2}{4}.$$

We now compare the estimation errors for both sampling under the assumption that average sampling rates are

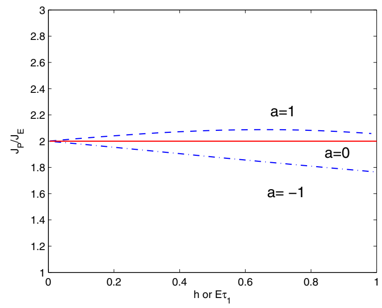


Fig. 2. J_P/J_E for $a = -1, 0, 1$ with the same average sampling period

the same, i.e. $h = \mathbb{E}[\tau_1]$. For the case $a = 0$, we have $h = \mathbb{E}[\tau_1] = \delta^2/2$, and

$$\frac{J_P}{J_E} = \frac{\delta^2/2}{\delta^2/4} = 2.$$

The ratio J_P/J_E as a function of h or $\mathbb{E}[\tau_1]$ is plotted in Figure 2 for $a = -1, 0, 1$. The figure shows that event-based sampling gives smaller estimation errors and the ratio for unstable systems is larger than one for stable systems. Different from the case of first order systems, the improvement of event-based sampling over periodic sampling declines for unstable systems when the average sampling period exceeds around 0.7.

6. AN ILLUSTRATIVE EXAMPLE

In this section, an example of the double integrator is provided to demonstrate and compare the effectiveness of two state estimators.

For the system in (1)-(2), set

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad C = (1 \;\; 0).$$

Then it follows that

$$dx(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t)dt + \begin{pmatrix} 0 \\ dw_2(t) \end{pmatrix}$$
$$y(t) = (1 \;\; 0)\; x(t). \tag{20}$$

Here $x(t) = (x_1(t), x_2(t))^T$ is characterized as the position and speed of an object, while only the position can be measured. Noting

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad C(x(\tau_k) - \hat{x}(\tau_k)) = 0,$$

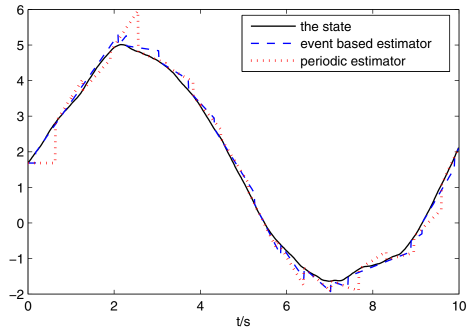
we get that for $\tau_k \le t < \tau_{k+1}$, $k = 0, 1, \ldots,$

$$\begin{aligned} z(t) &= Ce^{A(t-\tau_k)}(x(\tau_k) - \hat{x}(\tau_k)) + C\int_{\tau_k}^t e^{A(t-s)}Ddw(s) \\ &= (1 \;\; 0)\begin{pmatrix} 1 & t-\tau_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1(\tau_k) \\ \tilde{x}_2(\tau_k) \end{pmatrix} \\ &\quad + (1 \;\; 0)\int_{\tau_k}^t \begin{pmatrix} 1 & t-s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ dw_2(s) \end{pmatrix} \\ &= (t-\tau_k)\tilde{x}_2(\tau_k) + \int_{\tau_k}^t (t-s)dw_2(s) \\ &= \int_{\tau_k}^t [\tilde{x}_2(\tau_k) + w_2(s)]ds, \end{aligned} \tag{21}$$

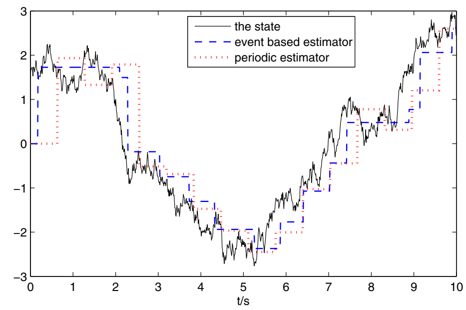
where $\tilde{x}(\tau_k) = x(\tau_k) - \hat{x}(\tau_k)$. From this together with (7),

$$\tau_{k+1} = \inf\Big\{t > \tau_k; \Big|\int_{\tau_k}^t [\tilde{x}_2(\tau_k) + w_2(s)]ds\Big| = \delta\Big\}.$$

Take $\delta = 0.3$. By implementing Algorithm II, we get the trajectory of the event-based estimator, which together with the state $x(t)$ is shown in Figure 3. Meanwhile, we have $\lim_{k\to\infty} \tau_k/k = 0.64$. Let $h = 0.64$. From Algorithm I, we get the trajectory of the the periodic estimator, which is also plotted in Figure 3. It can be seen that the event-based estimator is more consistent with the state trajectory than the periodic estimator, irrespective of the first state or the second state.



(a) The first state



(b) The second state

Fig. 3. Trajectories of the state, event-based estimator and periodic estimator

The squared estimation errors $\|\tilde{x}(t)\|^2$ under event-based and periodic sampling are shown in Figure 4. It can be seen that the average estimation errors under event-based sampling are smaller compared with the periodic sampling. Indeed, the time-average value of squared estimation errors under event-based and periodic samplings are 0.4107 and 0.5636, respectively.

7. CONCLUDING REMARKS

This paper considered the state estimation problem of continuous-time linear systems with two types of sampling. We first presented the optimal state estimator under periodic sampling, and designed the state estimator with

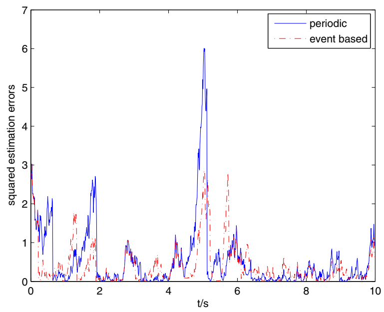


Fig. 4. Squared estimation errors under event-based and periodic sampling

event-based updates. Then, we compared the average estimation errors for both sampling schemes in first and second order systems, respectively. A numerical example was provided to compare the effectiveness of two state estimators. It can be shown that in these cases the event-based sampling does outperform periodic sampling for state estimation. Specifically, for critically stable first order systems, the ratio of the average estimation error with periodic sampling to one with event-based sampling is 3; for critically stable symmetric second order systems, the ratio is 2. However, due to severe mathematical difficulties, the quantitative comparison of average estimation errors for the general case under both sampling schemes was not provided in this paper, which needs to be investigated further.

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