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Formation Merging Control in 3D under Directed and Switching Topologies

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Abstract: The paper studies the formation merging problem for a leader-follower network. That is, how to control a team of agents called followers so that they are merged with a team of agents called leaders to form a larger globally rigid formation. Under the premise that a group of leaders move in a globally rigid formation with their synchronized velocity known to the followers, we show that the followers can asymptotically merge themselves to the formation for arbitrarily initial configurations. Each follower selects its neighbors and also its control law according to the target formation they aim to achieve and thus it allows directed and time-varying switching topologies. It is shown that a globally rigid formation can be merged asymptotically for the leader-follower network in a setup with directed and time-varying graphs if and only if every follower frequently has a joint path from at least a leader.

Keywords: Multi-agent systems, formation control, leader-follower networks

1. INTRODUCTION

Multi-agent systems represent a group of autonomous agents communicating locally with each other and cooperating to execute a task. Formation control is a significant research problem of multi-agent systems. In recent years, it has received much attention due to its broad civil (Leonard et al. (2007)) and military applications (Murray (2007)).

In the paper, we consider a leader-follower network and the formation merging problem in 3D. By formation merging we mean two sub-formations of agents are merged to form one single globally rigid formation. In the paper, we assume that a group of agents called leaders moves as a whole in a globally rigid formation while the other group of agents called followers are initially in an arbitrary configuration. The objective is to control the followers in a distributed way so that they asymptotically merge into a single globally rigid formation with the sub-formation of leaders. The work is mainly motivated by the necessity of performing some basic operations such as rejoint/split maneuvers of distributed formations in a distributed manner.

One way to address the formation merging problem is to figure out how many new distance constraints should be imposed for agent pairs in the two groups in order to form a single globally rigid formation and then work out a distributed control law for the agents to meet these new distance constraints asymptotically. Considering the inter-agent distance constraints, graph rigidity is a basic requirement because if the sensing graph is not globally rigid, then there are non-congruent formations whose inter-agent distances satisfy the specified values (Anderson and Yu (2011); Cao et al. (2011)). From this perspective, Eren et al. (2004) consider merging two globally rigid formations to get a single globally rigid one in both $\mathbb{R}^2$ and $\mathbb{R}^3$; Yu et al. (2006a) aim to control the merging efficiently and optimally in the sense of minimizing the number of added distance constraints; and Yu et al. (2006b) extend the idea to merge more than two (minimally) rigid formations to obtain a single (minimally) rigid formation. For directed graphs, the concept of persistence is introduced for merging two sub-formations (Hendrickx et al. (2008)). However, it becomes challenging in analyzing the stability of formations in a directed graph setting (Cao et al. (2008); Guo et al. (2010)).

Another way to address the formation merging problem is to consider the displacement constraints between agent pairs and use relative positions measured in their own local frames to design a linear control law for the purpose as for formation control by linear coordination laws (Lin et al. (2004, 2013b); Han et al. (2013)). In Lin et al. (2013b) and Han et al. (2013), a complex Laplacian based control law is introduced to address formation control problems in the plane under a directed and fixed topology, for which

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no global knowledge of reference frame is required. Lin et al. (2013a) extend this idea to solve formation control problems in d-dimensional space.

Borrowing these ideas, we aim to solve the formation merging problem by exploring procedures for control law design as well as necessary and sufficient conditions for asymptotic formation merging. In practical applications, switching of an information flow graph may be induced by some unpredictable changes in the system. So we consider the scenario of directed and switching topologies in the paper. To our best knowledge, there is rare work addressing formation control problems in a switching topology setting. In this paper, to make the problem solvable, the neighbors of each agent are selected to meet a certain convexity assumption according to the target formation they aim to achieve. Then a distributed control law is proposed for formation merging with the control parameters designed based on the specific target formation. A necessary and sufficient condition for solving the problem of asymptotically merging a group of followers with a group of leaders to form a globally rigid formation is obtained. That is, every follower should frequently have a joint path from at least a leader. The analysis of the asymptotic merging behavior in a switching topology setting mainly relies on graph Laplacian and an idea similar to the one based on joint spectral radius for switched systems.

Notation: $\mathbb{R}$ denotes the set of real numbers. $\mathbf{1}_n$ represents the n-dimensional vector of ones and I_n represents the identity matrix of order n. The symbol $\otimes$ denotes the Kronecker product.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Preliminaries

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite set $\mathcal{V}$ of elements called *nodes* and a finite set $\mathcal{E}$ of ordered pairs of nodes called *edges*. A *walk* in a graph $\mathcal{G}$ is an alternating sequence

$$\mathcal{W}: \ v_1 e_1 v_2 e_2 \cdots v_{k-1} e_{k-1} v_k$$

of nodes v_i and edges e_i such that $e_i = (v_i, v_{i+1})$ for every $i = 1, 2, \dots k-1$. We say that $\mathcal{W}$ is a walk from v_1 to v_k. The *length* of a walk is the number of the edges in the walk.

Let $\mathcal{R} \subset \mathcal{V}$ be a subset of nodes in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A node $v \in \mathcal{V} - \mathcal{R}$ is said to be *reachable* from $\mathcal{R}$ if there exists a walk from a node in $\mathcal{R}$ to v. Moreover, $\mathcal{R}$ is said to be *closed* in $\mathcal{G}$ if any node in $\mathcal{R}$ is not reachable from $\mathcal{V} - \mathcal{R}$.

When the edge set in a directed graph changes over time, we call it a *time-varying graph*, denoted as $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$. For a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, a node v is said to be *uniformly jointly reachable* from $\mathcal{R} \subset \mathcal{V}$ if there exists $T > 0$ such that for all t, v is reachable from $\mathcal{R}$ in the union graph $\mathcal{G}([t, t+T])$, whose edge set is the union of the edge set of $\mathcal{G}(t)$ over the time interval $[t, t+T]$.

A *configuration in* $\mathbb{R}^3$ (or simply called a configuration in this paper) of a set of n nodes is defined by their coordinates in the Euclidean space $\mathbb{R}^3$, denoted as $p = [p_1^\top, \dots, p_n^\top]^\top$, where each $p_i \in \mathbb{R}^3$ for $1 \le i \le n$. A *framework in* $\mathbb{R}^3$ (or simply called a framework in this paper) is a graph $\mathcal{G}$ equipped with a configuration p in $\mathbb{R}^3$, denoted as $\mathcal{F} = (\mathcal{G}, p)$.

We say that two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are *equivalent*, and we write $(\mathcal{G}, p) \sim (\mathcal{G}, q)$, if $\|p_i - p_j\| = \|q_i - q_j\|, \forall (i,j) \in \mathcal{E}$. We say that two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are *congruent*, and we write $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$ (or simply p and q are congruent, $p \equiv q$), if $\|p_i - p_j\| = \|q_i - q_j\|, \forall i, j \in \mathcal{V}$. A framework $(\mathcal{G}, p)$ is called *globally rigid* if

$$(\mathcal{G}, p) \sim (\mathcal{G}, q), \ \forall q \in \mathbb{R}^{3n} \Leftrightarrow (\mathcal{G}, p) \equiv (\mathcal{G}, q).$$

Denote by $\mathcal{N}_i$ the set of neighbors of node i. For a directed graph, the Laplacian matrix $L \in \mathbb{R}^{n \times n}$ is defined as follows:

$$L(i,j) = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i \\ \sum_{k \in \mathcal{N}_i} w_{ik} & \text{if } i = j. \end{cases}$$

where $w_{ij} > 0$ is called the weight on edge (j, i). According to the definition, L satisfies $L\mathbf{1}_n = 0$.

A square matrix $E \in \mathbb{R}^{n \times n}$ is called *stochastic* if it is nonnegative and every row sum equals 1, i.e., $E\mathbf{1}_n = \mathbf{1}_n$. And the product of stochastic matrices is also stochastic (Lin (2008), page 34). For an $n \times n$ nonnegative matrix E, the *associated graph* $\mathcal{G}(E)$ consists of n nodes $v_1, \dots, v_n$ where an edge leads from v_j to v_i if and only if the (i,j)-th entry of E is not zero.

2.2 Problem statement

In the paper, we study the control problem of formation merging. That is, under the premise that the leaders are already in a globally rigid formation, how do we control the followers such that they are merged with the leaders to form a globally rigid formation? The paper aims to solve the formation merging problem in a directed and switching topology setting. That is, the information flow graph for the followers is directed and switches over time. As a first step towards the general formation merging control problem, we assume in the paper that the target formation of followers entirely lies in the three-dimensional convex hull spanned by the leaders.

We consider a leader-follower network, with m leaders labeled $a_1, \dots, a_m$ and n followers labeled $b_1, \dots, b_n$ in three dimension. Let z_i denote the 3D position of agent i. We consider a target configuration $p_a = [p_{a_1}^\top, \dots, p_{a_m}^\top]^\top$ in $\mathbb{R}^{3m}$ for the leaders and a target configuration $p_b = [p_{b_1}^\top, \dots, p_{b_n}^\top]^\top$ in $\mathbb{R}^{3n}$ for the followers, where each $p_i \in \mathbb{R}^3$ for $i = a_1, \dots, a_m, b_1, \dots, b_n$. We assume that agents do not overlap each other in the target configuration.

We say the leaders are in a globally rigid formation p_a if it holds for all t that $z_i(t) = A(t)p_i + c(t)$ for $i = a_1, \dots, a_m$ where $A(t)$ at any time t is a unitary matrix corresponding to a rotation and $c(t)$ is a vector in $\mathbb{R}^3$ representing a translation. Moreover, we say the whole network asymptotically reaches a globally rigid formation $[p_a^\top, p_b^\top]^\top$ if it holds that $z_i(t) \to A(t)p_i + c(t)$ for $i = a_1, \dots, a_m, b_1, \dots, b_n$.

Since at least four agents are required to form a formation in 3D, we assume $m \ge 4$. Suppose the m leaders are in a globally rigid formation p_a and is governed by the following dynamics

$$\dot{z}_i(t) = v_r(t),\ i = a_1, \ldots, a_m \tag{1}$$

where $v_r(t)$ is a common reference velocity, which is also known to the followers.

Remark 2.1. If $v_r(t)$ is not known to all followers but partial followers, then $v_r(t)$ can be available to all followers by some consensus schemes, such as (Wieland and Allgower (2009)). We do not focus on the velocity consensus problem in this paper, so we assume that the synchronized velocity $v_r(t)$ is known to the followers.

Consider a single integrator model for the followers, i.e.,

$$\dot{z}_i = u_i,\ i = b_1, \ldots, b_n \tag{2}$$

where $u_i \in \mathbb{R}^3$ represents the velocity control input of follower i. Define the aggregate state $z_b = \left[z_{b_1}^\top, \cdots, z_{b_n}^\top\right]^\top$, as a column vector in $\mathbb{R}^{3n}$. Suppose each follower i has an onboard sensor allowing it to measure the relative positions of its neighbors. We use a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ to describe the sensing topology, where $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_b$ with $\mathcal{V}_a = \{a_1, \ldots, a_m\}$ and $\mathcal{V}_b = \{b_1, \ldots, b_n\}$, and an edge $(j, i) \in \mathcal{E}(t)$ means that $z_j - z_i$ is available to agent i at time t. Without loss of generality, we assume that every leader does not have an incoming edge from any others. Moreover, it is assumed that the system is under a dwell-time constraint, i.e., the interval between any two switching instants is greater than a constant.

3. MAIN RESULTS

In this section, we first provide a procedure for the design of formation control law and next present stability analysis for the formation merging control problem.

3.1 Control design

We consider the following control law for each follower i,

$$u_i = v_r(t) + \sum_{j \in \mathcal{N}_i(t)} k_{ij}(t)(z_j - z_i), \tag{3}$$

where $\mathcal{N}_i(t)$ is the neighbor set of follower i at time t and $k_{ij}(t)$ are control parameters that will be designed in the following. To make the problem addressable, we assume that if agent i has neighbors at time t, then its neighbors are selected so that the convex hull spanned by $\{p_j : j \in \mathcal{N}_i(t)\}$ contains p_i in the target configuration $p = [p_a^\top, p_b^\top]^\top$. We call it the *convexity assumption.*

Remark 3.1. To meet convexity assumption, each follower i disregards all the neighbors if their convex hull does not contain i in the target configuration. Technically, the communication range can be increased so that the followers can find neighbors to meet this assumption.

Next we present a procedure for the design of $k_{ij}(t)$'s and for simplicity, we omit t in all the statements unless it is necessary. As it is assumed that agents do not overlap in the target configuration, then to meet the convexity assumption an agent cannot have only one neighbor. So there four possible cases and we provide a procedure for the design of k_{ij}'s for these four cases.

(i) If an agent has no neighbor, then the control law (3) degenerates to

$$u_i = v_r(t).$$

(ii) For the case that an agent's neighbors form a one-dimensional convex hull in the target configuration, we first consider that agent i has only two neighbors, say i_1 and i_2. Then it can be obtained that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2}, \tag{4}$$

where $\alpha_1 = \frac{\|p_{i_2} - p_i\|}{\|p_{i_2} - p_{i_1}\|}$ and $\alpha_2 = \frac{\|p_{i_1} - p_i\|}{\|p_{i_2} - p_{i_1}\|}$. It is clear that $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. Second, if agent i has more than two neighbors, then we can take any two of them containing p_i and obtain the same formula as (4), i.e.,

$$p_i = \alpha_1^l p_{i_1^l} + \alpha_2^l p_{i_2^l}$$

where l enumerates all possible combination of two neighbors containing p_i. Then, consider a convex combination of these representations for p_i. That is, using γ^l's that satisfy $\gamma^l \in (0, 1)$ and $\sum_l \gamma^l = 1$, we can have

$$p_i = \sum_l \gamma^l (\alpha_1^l p_{i_1^l} + \alpha_2^l p_{i_2^l}) := \sum_{j \in \mathcal{N}_i} \alpha_j p_j.$$

It is certain that $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j \in \mathcal{N}_i} \alpha_j = 1$. For this case, we take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

(iii) For the case that an agent's neighbors form a two-dimensional convex hull, we first consider that agent i has only three neighbors, say i_1, i_2, and i_3, and they form a triangle in the target configuration. Suppose the coordinates of p_{i_1}, p_{i_2}, and p_{i_3} are as follows: $p_{i_1} = (x_{i_1}, y_{i_1}, z_{i_1})$, $p_{i_2} = (x_{i_2}, y_{i_2}, z_{i_2})$, $p_{i_3} = (x_{i_3}, y_{i_3}, z_{i_3})$. Denote $x = [x_{i_1}, x_{i_2}, x_{i_3}]^\top, y = [y_{i_1}, y_{i_2}, y_{i_3}]^\top$ and $z = [z_{i_1}, z_{i_2}, z_{i_3}]^\top$. We let $S_{i_1 i_2 i_3}$ denote the area of the triangle formed by p_{i_1}, p_{i_2} and p_{i_3} and it can be calculated by the following formula:

$$S_{i_1 i_2 i_3} = \frac{1}{2}\sqrt{S_1^2 + S_2^2 + S_3^2}$$

where $S_1 = \det[x, y, \mathbf{1}_3]^\top, S_2 = \det[y, z, \mathbf{1}_3]^\top, S_3 = \det[z, x, \mathbf{1}_3]^\top$. Then it can be obtained that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2} + \alpha_3 p_{i_3} \tag{5}$$

where $\alpha_1 = \frac{S_{i i_2 i_3}}{S_{i_1 i_2 i_3}}$, $\alpha_2 = \frac{S_{i_1 i i_3}}{S_{i_1 i_2 i_3}}$, and $\alpha_3 = \frac{S_{i_1 i_2 i}}{S_{i_1 i_2 i_3}}$. It is true that $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Second, if agent i has more than three neighbors, we can do the similar procedure as for case (ii) to get the representation for p_i in terms of all its neighbor's coordinates, i.e.,

$$p_i = \sum_{j \in \mathcal{N}_i} \alpha_j p_j$$

where $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j \in \mathcal{N}_i} \alpha_j = 1$. We then take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

(iv) Similarly, for the case that an agent's neighbors form a three-dimensional convex hull, we first consider that agent i has only four neighbors and these four neighbors, say i_1, i_2, i_3, i_4, form a tetrahedron containing p_i inside in the target configuration. Denote by $V_{i_1 i_2 i_3 i_4}$ the signed volume of the tetrahedron formed by $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$. It can be calculated by the following formula:

$$V_{i_1 i_2 i_3 i_4} = \frac{1}{6}\det([p_{i_2} - p_{i_1}, p_{i_3} - p_{i_1}, p_{i_4} - p_{i_1}]^\top).$$

Then it can be obtained that

$$p_i = \alpha_1 p_{i_1} + \alpha_2 p_{i_2} + \alpha_3 p_{i_3} + \alpha_4 p_{i_4}, \tag{6}$$

where

$$\alpha_1 = \frac{V_{i i_2 i_3 i_4}}{V_{i_1 i_2 i_3 i_4}}, \alpha_2 = \frac{V_{i_1 i i_3 i_4}}{V_{i_1 i_2 i_3 i_4}}, \alpha_3 = \frac{V_{i_1 i_2 i i_4}}{V_{i_1 i_2 i_3 i_4}}, \alpha_4 = \frac{V_{i_1 i_2 i_3 i}}{V_{i_1 i_2 i_3 i_4}},$$

for which $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. Second, if agent i has more than four neighbors, we can do the similar procedure as for case (ii) to get the representation for p_i in terms of all its neighbor's coordinates, i.e.,

$$p_i = \sum_{j \in \mathcal{N}_i} \alpha_j p_j$$

where $\alpha_j > 0$ for all $j \in \mathcal{N}_i$ and $\sum_{j \in \mathcal{N}_i} \alpha_j = 1$. We then take $k_{ij} = \alpha_j$ for $j \in \mathcal{N}_i$.

Remark 3.2. With the convexity assumption, $\alpha_j > 0$ and $\sum_{j \in \mathcal{N}_i} \alpha_j = 1$. This is important for the stability analysis as otherwise it becomes very challenging due to possibly negative weights in the interaction control law.

3.2 Stability analysis

In this section, we present stability analysis for the system under the proposed control law (3).

Let $L(t)$ be the Laplacian matrix for the graph with weights $k_{ij}(t)$'s associated to edges (j, i)'s at time t. Under the control laws (1) and (3), the overall system can be described as

$$\dot{z} = -(L(t) \otimes I_3)z + \mathbf{1}_{m+n} \otimes v_r(t), \tag{7}$$

where z is the aggregated state of all z_i's. By our assumption on the information flow graph $\mathcal{G}(t)$, we know that $L(t)$ has the following form

$$L(t) = \left[\begin{array}{c|c} 0_{m \times m} & 0_{m \times n} \\ \hline L_{lf}(t) & L_{ff}(t) \end{array} \right]. \tag{8}$$

Also, by the design procedure given in Subsection 3.1, we know that $L(t)$ satisfies

$$(L(t) \otimes I_3)p = 0 \text{ and } L(t)\mathbf{1}_{m+n} = 0. \tag{9}$$

The following result shows that $p = [p_a^\mathsf{T}, p_b^\mathsf{T}]^\mathsf{T}$ is a stable equilibrium formation for the leader-follower system under the proposed control law.

Theorem 1. Suppose the leaders are in the globally rigid formation p_a. Then

$$z^*(t) = (I_{m+n} \otimes A)p + \mathbf{1}_{m+n} \otimes (c + \int_0^t v_r(\tau)d\tau),$$

where A is a unitary matrix and c is a constant vector determined by the leaders, is an equilibrium solution of system (7). Moreover, it is stable.

Proof: Let $y = z - \mathbf{1}_{m+n} \otimes \int_0^t v_r(\tau)d\tau$. Then system (7) can be transformed to

$$\dot{y} = -(L(t) \otimes I_3)y. \tag{10}$$

To show $z^*(t)$ is an equilibrium solution of system (7), it remains to show $y^* = (I_{m+n} \otimes A)p + \mathbf{1}_{m+n} \otimes c$ is an equilibrium point of system (10). From (9) we can get

$$\begin{aligned} &(L(t) \otimes I_3)[(I_{m+n} \otimes A)p + \mathbf{1}_{m+n} \otimes c] \\ &= (L(t) \otimes A)p = (I_{m+n} \otimes A)(L(t) \otimes I_3)p = 0. \end{aligned}$$

Hence, y^* is an equilibrium point of system (10).

Next, we show that $z^*(t)$ is stable, which is equivalent to show y^* is a stable equilibrium point of system (10). Suppose the switching time is $t_0, t_1, t_2, \ldots$. Consider any $t > 0$. Without loss of generality, say t is in the interval $[t_i, t_{i+1}]$. Thus, the transition matrix can be written as

$$\Phi(t, t_i) = \exp(-(L(t_i) \otimes I_3)(t - t_i)) \tag{11}$$

and the solution of system (10) can be described by

$$y(t) = \Phi(t, t_i)\Phi(t_i, t_{i-1}) \cdots \Phi(t_1, t_0)y^0$$

for an initial state y^0. Note that every transition matrix in the above formula is stochastic (Lin (2008), page 51) and recall that the product of stochastic matrices is also stochastic. It follows that every state $y_i(t)$ is a convex combination of $y^0_{a_1}, \ldots, y^0_{a_m}, y^0_{b_1}, \ldots, y^0_{b_n}$. That is,

$$y_i(t) = \sum_{j=1}^m \alpha_{a_j} y^0_{a_j} + \sum_{k=1}^n \alpha_{b_k} y^0_{b_k}, \tag{12}$$

where $\alpha_{a_j} \geq 0$ $(j = 1, \ldots, m)$, $\alpha_{b_k} \geq 0$ $(k = 1, \ldots, n)$ and $\sum_{j=1}^m \alpha_{a_j} + \sum_{k=1}^n \alpha_{b_k} = 1$. For any arbitrary $\epsilon > 0$, we choose $\delta = \epsilon$. Suppose

$$(\forall i) \parallel y^0_i - y^*_i \parallel \leq \delta.$$

Since y^* is an equilibrium point, then from (12) it follows that

$$y^*_i = \sum_{j=1}^m \alpha_{a_j} y^*_{a_j} + \sum_{k=1}^n \alpha_{b_k} y^*_{b_k}.$$

Thus, we have for every i,

$$\begin{aligned} \parallel y_i(t) - y^*_i \parallel &= \parallel \sum_{j=1}^m \alpha_{a_j}(y^0_{a_j} - y^*_{a_j}) + \sum_{k=1}^n \alpha_{b_k}(y^0_{b_k} - y^*_{b_k}) \parallel \\ &\leq \sum_{j=1}^m \alpha_{a_j}\delta + \sum_{k=1}^n \alpha_{b_k}\delta = \delta = \epsilon \end{aligned}$$

and the conclusion follows. ■

The next result presents a necessary and sufficient graphical condition to ensure that a globally rigid formation of $[p_a^\mathsf{T}, p_b^\mathsf{T}]^\mathsf{T}$ can be asymptotically merged in the leader-follower network.

Theorem 2. Suppose the leaders are in the globally rigid formation p_a. A globally rigid formation of $[p_a^\mathsf{T}, p_b^\mathsf{T}]^\mathsf{T}$ can be asymptotically merged under the distributed control law (3) if and only if every follower is uniformly jointly reachable from $\mathcal{V}_a$.

The proof requires a lemma from graph theory.

Lemma 3. (Beineke and Wilson (1997), page 87) Let E be a nonnegative matrix and denote $e^{(k)}_{ij}$ the (i, j)th entry of E^k. Then $e^{(k)}_{ij} > 0$ if and only if the associated graph $\mathcal{G}(E)$ has a walk from node v_j to node v_i of length k.

Proof of Theorem 2: ($\Leftarrow$) Suppose the graph switches at $t_0, t_1, t_2, \ldots$. Recall the dwell-time constraint, which means that there exists a $\tau_D > 0$ such that $t_{i+1} - t_i \geq \tau_D$ for all $i \geq 0$. Moreover, we can always find a $\tau_m > \tau_D$ large enough such that $t_{i+1} - t_i \leq \tau_m$ for all $i \geq 0$. When there are no switching in $[t_i, \infty]$ for some i, we can choose any $\tau_m > \tau_D$ to construct virtual switching instants.

If every follower is uniformly jointly reachable from $\mathcal{V}_a$, by the definition there exits a $T > 0$ such that for all t in the union graph $\mathcal{G}([t, t + T])$ every follower is reachable from $\mathcal{V}_a$. Now we generate a subsequence $\{t_{m_k}\}$ of the sequence $\{t_i\}$ as follows:

(1) Set $m_0 = 0$.
(2) If $t_{m_0} + T \in (t_{i-1}, t_i]$, set $m_1 = i$.
(3) If $t_{m_1} + T \in (t_{i-1}, t_i]$, set $m_2 = i$.
(4) And so on.

Thus, for the transformed system

$$\dot{y}(t) = -(L(t) \otimes I_3) y(t),$$

we have at the subsequence of time instants t_{m_k},

$$y(t_{m_{k+1}}) = \Psi(t_{m_k}) y(t_{m_k}) \tag{13}$$

where $\Psi(t_{m_k}) = \left[\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} L(t)dt\right)\right] \otimes I_3$. Denote by Ξ the set of all $\Psi(t_{m_k})$'s derived above. We regard the above evolution as a discrete-time switched system and for simplicity we rewrite (13) as

$$y(k+1) = \Psi(k) y(k) \text{ with } \Psi(k) \in \Xi. \tag{14}$$

Note that, due to the special structure of $L(t)$ described in (8), $\Psi(k)$ has the following form

$$\Psi(k) = \left[\begin{array}{c|c} I_{3m\times 3m} & 0_{3m\times 3n} \\ \hline \Psi_{lf}(k) & \Psi_{ff}(k) \end{array}\right].$$

Next we show that for all $\Psi(k) \in \Xi$, $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by a constant $\sigma < 1$. For any $L(t)$, we can decompose it as $-L(t) = -D(t) + E(t)$ where $D(t)$ is a diagonal matrix and $E(t)$ is a nonnegative matrix with all diagonal entries zero. Thus,

$$\Psi(k) = \left[\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} D(t)dt\right) \exp\left(\int_{t_{m_k}}^{t_{m_{k+1}}} E(t)dt\right)\right] \otimes I_3.$$

We denote $E = \int_{t_{m_k}}^{t_{m_{k+1}}} E(t)dt$ and it is noted that

$$E = E(t_{m_k})(t_{m_k+1} - t_{m_k}) + \cdots + E(t_{m_{k+1}-1})(t_{m_{k+1}} - t_{m_{k+1}-1}).$$

By the condition that every follower is uniformly jointly reachable from $\mathcal{V}_a$, we can then know that every follower is reachable from $\mathcal{V}_a$ in the associated graph $\mathcal{G}(E)$. Then, considering the equality

$$\exp(E) = I + E + \frac{E^2}{2!} + \cdots$$

and the fact that $\exp\left(-\int_{t_{m_k}}^{t_{m_{k+1}}} D(t)dt\right)$ is a positive diagonal matrix, we can infer by Lemma 3 that each row of $\Psi_{lf}(k)$ has a nonzero entry because each row in the corresponding block of $\exp(E)$ has a nonzero entry. On the other hand, as shown in Theorem 1, we know that $\Psi(k)$ is a stochastic matrix. The above two conclusions together imply that $\|\Psi_{ff}(k)\|_\infty < 1$. Moreover, recall that $\tau_D \le t_{i+1} - t_i \le \tau_m$. And with the fact that $L(t_i)$'s are taken in a finite set as there are only a finite number of graphs with different connectivity, it follows that there is a positive constant $\sigma < 1$ such that $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by σ.

Since by assumption that the m leaders are kept in a globally rigid formation p_a, from (14) we then have

$$y_b(k+1) = \Psi_{ff}(k) y_b(k) + \Psi_{lf}(k) y_a^*, \tag{15}$$

where y_b is the aggregated state of y_i's associated to the followers and $y_a^* = (I_m \otimes A)p_a + \mathbf{1}_m \otimes c$ for a constant unitary matrix A and a constant vector c, representing the fixed globally rigid formation of the leaders. Due to the fact that $\|\Psi_{ff}(k)\|_\infty < 1$ we can obtain that $I - \Psi_{ff}(k)$ is invertible. Then system (15) has a unique equilibrium point $y_b^* = (I_n \otimes A)p_b + \mathbf{1}_n \otimes c$. So by the coordinate transformation $q(k) = y_b(k) - y_b^*$ we get

$$q(k+1) = \Psi_{ff}(k) q(k). \tag{16}$$

Thus, to show whether a globally rigid formation $[p_a^\top, p_b^\top]^\top$ can be asymptotically merged, it is necessary to show that $q(k)$ asymptotically converges to 0. Since we just showed that $\|\Psi_{ff}(k)\|_\infty$ is uniformly upper-bounded by $\sigma < 1$, it follows straightforward that $q(k)$ asymptotically converges to 0. So we can reach the conclusion that

$$\lim_{j\to\infty} y_b(t_{m_j}) = y_b^*.$$

Now let us look at the evolution of the continuous state $y_b(t)$ in the interval between any two consecutive switching instants. From the proof of Theorem 1, we know that for any $t \in [t_i, t_{i+1})$ and any arbitrary $\epsilon > 0$

$$\|y_i(t_i) - y_i^*\| \le \epsilon \Rightarrow \|y_i(t) - y_i^*\| \le \epsilon.$$

Therefore, it is known that $\lim_{t\to\infty} y_b(t) = y_b^*$. And the conclusion follows.

($\Rightarrow$) We prove it in a contrapositive way. Assume that there exists a follower, say b_i, that is not uniformly jointly reachable from $\mathcal{V}_a$. That is, for any $T > 0$ there exists $t^* \ge 0$ such that in the union graph $\mathcal{G}([t^*, t^* + T])$, b_i is not reachable from $\mathcal{V}_a$. Let Θ be the set including all such followers that are not reachable from $\mathcal{V}_a$ in $\mathcal{G}([t^*, t^* + T])$. Then it can be known that Θ is a closed set in $\mathcal{G}([t^*, t^* + T])$. So the states of these followers at $t \in [t^*, t^* + T]$ remain in the convex hull of their states at t^* and do not converge to form a globally rigid formation with other agents. ■

4. SIMULATION

In this section, we present a simulation to validate our theoretic results. We suppose there are 8 leaders who are moving in a globally rigid formation (a cube in 3D) as shown in Fig. 1. Consider 12 followers and we expect the leader-follower network achieves the target formation as a whole (Fig. 1).

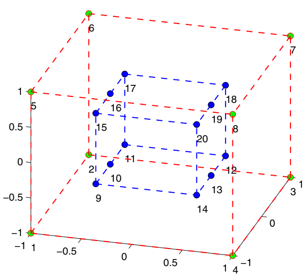

Fig. 1. A target formation for a leader-follower network with 8 leaders and 12 followers.

Denote the set of leaders by $\mathcal{V}_a = \{1, 2, \ldots, 8\}$ and denote the set of followers by $\mathcal{V}_b = \{9, 10, \ldots, 20\}$. The followers can select its neighbors by verifying the convexity assumption at any time t. But for simplicity of simulation, we consider fixed topologies as shown in Fig. 2. It can be checked that every follower is uniformly jointly reachable from $\mathcal{V}_a$ for the time-varying graph $\mathcal{G}(t)$ by taking $T = 3$. A simulation result is given in Fig. 3 where several snapshots at different time are presented. From the simulation we see that the followers can start at any initial positions and are asymptotically merged with the leaders to form a target formation.

5. CONCLUSION

This paper studies the formation merging control problem under directed and switching topologies for a leader-

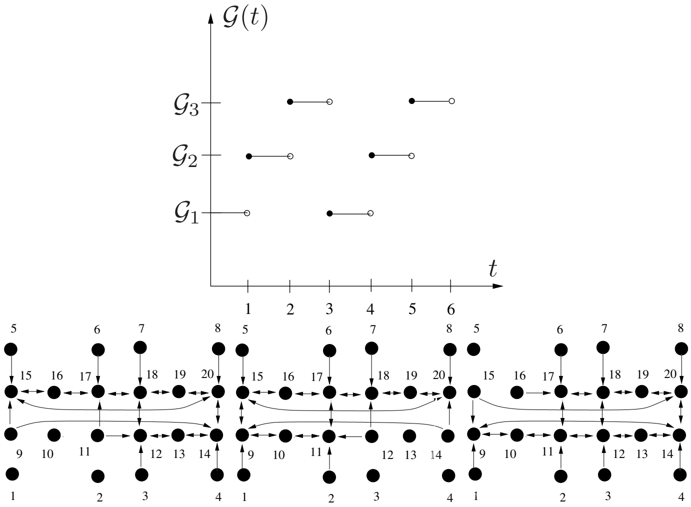

Fig. 2. A periodic switching graph $\mathcal{G}(t)$ that switches among three different topologies $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

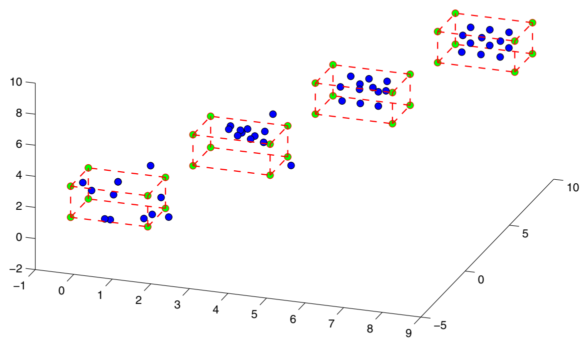

Fig. 3. The followers are asymptotically merged with a formation of leaders to form a larger target formation.

follower network. A distributed control law is proposed for this purpose with the control parameters designed based on the specific target formation of the whole network. We introduce a rule for the selection of neighbors also based on the target formation to meet a convexity assumption. With the introduction of this convexity assumption, we present a necessary and sufficient condition for asymptotically merging a group of followers with a group of leaders to form a globally rigid formation. One possible future research is how to deal with the formation merging control problem when the convexity assumption is relaxed.

REFERENCES

Anderson, B.D.O. and Yu, C. (2011). Range-only sensing for formation shape control and easy sensor network localization. In *Proceedings of the 23rd Chinese Control and Decision Conference*, 3310–3315. Mianyang, China.

Beineke, L.W. and Wilson, R.J. (1997). *Graph Connections: Relationships Between Graph Theory and Other Areas of Mathematics*. Clarendon Press.

Cao, M., Anderson, B.D.O., Morse, A.S., and Yu, C. (2008). Control of acyclic formations of mobile autonomous agents. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 1187–1192. Cancun, Mexico.

Cao, M., Yu, C., and Anderson, B.D.O. (2011). Formation control using range-only measurements. *Automatica*, 47(4), 776–781.

Eren, T., Anderson, B.D.O., Whiteley, W., Morse, A.S., and Belhumeur, P.N. (2004). Merging globally rigid formations of mobile autonomous agents. In *Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems*, 1260–1261. Washington, DC, USA.

Guo, J., Lin, Z., Cao, M., and Yan, G. (2010). Adaptive control schemes for mobile robot formations with triangularized structures. *IET Control Theory & Applications*, 4(9), 1817–1827.

Han, Z., Wang, L., and Lin, Z. (2013). Local formation control strategies with undetermined and determined formation scales for co-leader vehicle networks. In *Proceedings of the 52th IEEE Conference on Decission and Control*, 7339–7344. Florence, Italy.

Hendrickx, J.M., Yu, C., Fidan, B., and Anderson, B.D.O. (2008). Rigidity and persistence for ensuring shape maintenance of multi-agent meta-formations. *Asian Journal of Control*, 10(2), 131–143.

Leonard, N.E., Paley, D.A., Lekien, F., Sepulchre, R., Fratantoni, D.M., and Davis, R.E. (2007). Collective motion, sensor networks, and ocean sampling. *Proceedings of the IEEE*, 95(1), 48–74.

Lin, Z. (2008). *Distributed Control and Analysis of Coupled Cell Systems*. VDM-Verlag, Germany.

Lin, Z., Broucke, M.E., and Francis, B.A. (2004). Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control*, 49(4), 622–629.

Lin, Z., Chen, Z., and Fu, M. (2013a). A linear control approach to distributed multi-agent formations in d-dimensional space. In *Proceedings of the 52th IEEE Conferene on Decisioin and Control*, 6049–6054. Florence, Italy.

Lin, Z., Ding, W., Yan, G., Yu, C., and Giua, A. (2013b). Leader-follower formation via complex laplacian. *Automatica*, 49(6), 1900–1906.

Murray, R.M. (2007). Recent research in cooperative control of multivehicle systems. *Journal of Dynamic Systems, Measurement, and Control*, 129(5), 571–583.

Wieland, P. and Allgower, F. (2009). An internal model principle for consensus in heterogeneous linear multi-agent systems. In *Estimation and Control of Networked Systems*, volume 1, 7–12.

Yu, C., Fidan, B., and Anderson, B.D.O. (2006a). Principles to control autonomous formation merging. In *Proceedings of the 2006 American Control Conference*, 762–768. Minneapolis, MN, USA.

Yu, C., Fidan, B., Hendrickx, J.M., and Anderson, B.D.O. (2006b). Merging multiple formations: a meta-formation prospective. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 4657–4663. San Diego, CA, USA.