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Controller Design for Networked Control Systems Affected by Correlated Packet Losses

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Abstract: We consider the controller design for a linear time invariant system where the communication between the controller and the actuators is affected by correlated random packet losses. These packet losses are modelled as a finite length Markov chain. We present a method that takes advantage of the structure of the problem in order to design the optimal controller. The storage requirements for the optimal control laws however increase exponentially when longer packet loss histories have to be taken into account. We therefore present two sub-optimal methods that allow one to trade storage requirements for control performance. Finally, the performance in regard to storage and control optimality of the proposed designs are illustrated using simulations.

Keywords: Optimal control, Wireless networks, Jump linear systems, Heuristics, Stochastic Systems, Networked Systems, Dropouts.

1. INTRODUCTION

In this work, we consider the controller design for dynamic systems where the communication between the controller and the actuators is affected by random packet dropouts. Here, the probability that the current packet will arrive depends on the history of past packet dropouts. The packet arrivals can be modeled as a finite state Markov process. Then, the optimal controller is designed as a function of the packet loss history and the system state, such that a linear quadratic cost function is minimized. Also, at the time when a control input is computed, the controller does not know whether this input will actually reach the actuator. This problem can fit into the Markov jump linear system framework by assuming that the packet transmission at the current time will be successful (Costa et al., 2006; Fragoso, 1989; Mo et al., 2013; Casiello and Loparo, 1989; Val and Başar, 1997; Ji et al., 1991). The difference in our approach is that we take advantage of the structure of the Markov transition matrix, which results in simpler expressions. Designing the controller in this manner leads to control laws that depend on the system state and on the history of packet dropouts. This results in high complexity controllers (meaning: many control laws), one for each possible observation of packet dropouts over a finite history. The complexity increases exponentially for longer packet dropout histories. It also will require significant amounts of memory to be stored in a lookup table e.g. in microcontrollers. Here one might ask, whether it is necessary to take the entire finite packet arrival history into account when designing the controller. To tackle this question, we design two sub-optimal reduced complexity controllers, that feature fewer control laws, and illustrate the loss of performance compared to a full complexity controller through simulation studies.

Controller design for networked control systems that are affected by random packet dropouts have received increasing attention in recent years. One approach is to model the network effects as an independent and identically distributed (i.i.d.) random process (Quevedo et al., 2008, 2011; Imer et al., 2006; Wu and Chen, 2007). This model is also extended to include transmission delays (Zhang et al., 2013, 2005; Quevedo et al., 2013). For many network setups, the network effects can however not be captured by an i.i.d. model. Therefore a 2-state Markov model, which captures that packet losses occur in bursts (Gilbert, 1960; Sadeghi et al., 2008), is frequently used (Peters et al., 2016; Mo et al., 2013; Song et al., 2016). While there exists an extensive literature on performance and complexity tradeoffs in terms of minimizing a cost function for estimation over networks with correlated packet dropouts (Dolz et al., 2014; Smith and Seiler, 2003), the controller design has rarely been considered in this setting. One main difference between the estimator and controller design is that the network is part of a closed loop system. This makes the controller design more difficult.

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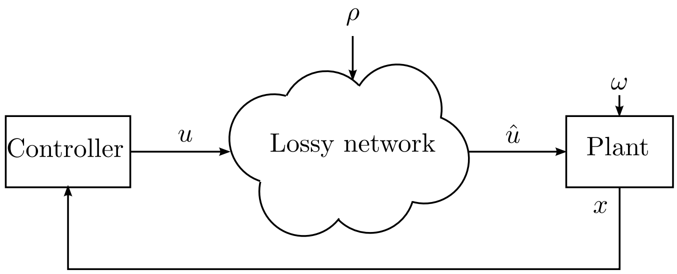

Fig. 1. Networked system where the controller and the actuators are connected through a link that is affected by packet dropouts.

The remainder of the paper is organized as follows: In Section 2 we present the network setup and the optimal controller design. In Section 3, we present two approaches that tradeoff storage complexity at the cost of control performance. The proof of the main result is shown in Section 4. Simulations studies are presented in Section 5. Section 6 draws conclusions

2. CONTROLLER DESIGN

We consider a linear time invariant system where the controller is connected to the actuators through a network that is affected by correlated packet losses. The system is illustrated in Fig. 1 and is of the form

$$x_{k+1} = Ax_k + \rho_k B u_k + \omega_k, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\rho_k \in \{0,1\}$ and $\omega_k \sim \mathcal{N}(0,\Sigma_\omega)$ is zero-mean white Gaussian noise with covariance Σ_ω. We set the input u_k to zero when no packet is received. Note that the previous input can be held by augmenting the A-matrix to contain the integrators that are located at the actuators of the system.

2.1 Network model

In (1), the variable ρ_k is a binary variable, where $\rho_k = 1$ indicates a successful transmission and is described by:

$$\mathbf{Pr}\{\rho_k = 1|\rho_{k-1},\cdots,\rho_0\} = \mathbf{Pr}\{\rho_k = 1|\rho_{k-1},\cdots,\rho_{k-d}\},$$

where $0 \le d < \infty$ is the length of the relevant history and $\mathbf{Pr}\{x|y\}$ denotes the probability of x knowing y. We accumulate the history of packet dropouts in the variable $\Theta_k \in \Xi = \{1,\cdots,2^d\}$. Here $\Theta_k = i$ denotes a certain realization of $(\rho_{k-1},\cdots,\rho_{k-d})$. It is easy to see that Θ_k is first order Markov, since

$$\begin{aligned}&\mathbf{Pr}\{\Theta_{k+1} = j|\Theta_k = i,\Theta_{k-1},\cdots,\Theta_0\}\\ &\quad = \mathbf{Pr}\{\Theta_{k+1} = j|\Theta_k = i\}\end{aligned}$$

Denote $p_{ij} \triangleq \mathbf{Pr}\{\Theta_{k+1} = j|\Theta_k = i\}$ and $\mathbf{Pr}\{\Theta_k = i\} \triangleq \pi_i$. We assign the outcomes of ρ_k to states Θ_k such that

$$\Theta_k = 1 + \sum_{\ell=1}^{d} \rho_{k-\ell} 2^{d-\ell}. \tag{2}$$

Let $r = 2^{d-1}$, then it is easy to see from (2) that $\rho_k = 1$ if $\Theta_{k+1} > r$. As seen in (2), for each Θ_k there are only two possible outcomes of Θ_{k+1}: one where the packet at time k arrives successfully and one where the packet at time k is lost. We define the variable

$$\phi_i = j \text{ if } p_{ij} > 0,\ r < j \le d. \tag{3}$$

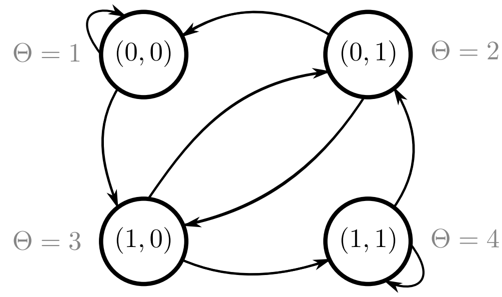

Fig. 2. Illustration of the grouping of the Markov chain that governs the packet dropouts. Here $(1,0)$ means that $\rho_{k-1} = 1$ and $\rho_{k-2} = 0$.

Likewise, knowing Θ_k the value of Θ_{k+1} where $\rho_k = 0$ is given by

$$\bar{\phi}_i = j \text{ if } p_{ij} > 0,\ 0 < j \le r, \tag{4}$$

and denote $p_{i,\phi_i} \triangleq \mathbf{Pr}\{\rho_k = 1|\Theta_k = i\}$ which is equivalent to $\mathbf{Pr}\{\Theta_{k+1} = \phi_i|\Theta_k = i\}$.

Example 1. Consider the case of a packet dropout distribution with $d = 2$. This means that the probability for a successful transmission at time k depends on the outcomes of ρ_{k-1} and ρ_{k-2}. This is illustrated in Fig. 2. Here we using (2) obtain a Markov chain where the transition matrix will be of the form

$$P = \begin{array}{c} \begin{array}{cccc} \Theta=1 & \Theta=2 & \Theta=3 & \Theta=4 \end{array} \\ \begin{bmatrix} 1-p_{00} & 0 & p_{00} & 0 \\ 1-p_{01} & 0 & p_{01} & 0 \\ 0 & 1-p_{10} & 0 & p_{10} \\ 0 & 1-p_{11} & 0 & p_{11} \end{bmatrix} \end{array} \begin{array}{c} \Theta=1 \\ \Theta=2 \\ \Theta=3 \\ \Theta=4 \end{array}, \tag{5}$$

where $p_{10} = \mathbf{Pr}\{\rho_k = 1|\rho_{k-1} = 1, \rho_{k-2} = 0\}$ which is identical to $p_{34} \triangleq \mathbf{Pr}\{\Theta_{k+1} = 4|\Theta_k = 3\}$. Here we have that $\phi_1 = 3$ and thus $p_{1\phi_1} = p_{13}$ and $p_{1\bar{\phi}_1} = p_{11}$. □

2.2 Optimal control

We want to design control laws that depend on the current system state x_k and network state Θ_k. Thus on the form

$$u_k = f_k(\Theta_k, x_k), \qquad k = 0,1,\cdots,N-1.$$

We want to design this controller to minimize the linear quadratic (LQ) cost function

$$\begin{aligned}&J_N(U_{0,N}, x_0, \Theta_0) = \\ &\mathbf{E}\left\{\sum_{k=0}^{N-1} \|x_k\|_Q + \rho_k\|u_k\|_R + \|x_N\|_{S_{N,\Theta_N}} \middle| x_0, \Theta_0\right\},\end{aligned} \tag{6}$$

where $\|x\|_Q \triangleq x^T Q x$ with $Q \ge 0$, $R \ge 0$ and $S_{N,i} \ge 0$, $\forall i \in \Xi$, $U_{k,N} \triangleq (u_k,\cdots,u_{N-1})$ and $\mathbf{E}\{x|y\}$ denoting the expectation of x given y. Note that (6) differs from the linear quadratic cost function used in classical linear quadratic regulator (LQR). The reason for this is, that the system (1) contains random variables ρ and ω. For this reason the expectation operator is needed to formulate a stochastic cost function.

Remark 1. The problem at hand can be fitted into the general Markov jump linear system (MJLS) framework, see e.g. Costa et al. (2006); Ji et al. (1991). The difference to the current setting is that we at time k do not have the outcome of ρ_k available. However, the results from

Costa et al. (2006); Ji et al. (1991) are identical if one assumes that $\rho_k = 1$ during the design phase. The result that we present in this section, will differ from the previous works on two points: (1) In the presented setting, it is a consequence of minimizing (6) that it is optimal to always assume $\rho_k = 1$. (2) The special structure of the transition matrix P results in simpler expressions, making it computationally more efficient.

Next, we present the controller design method.

Proposition 1. The optimal control at each time step is given by the linear function

$$u_k^\star = L_{k,\Theta_k}^\star x_k, \tag{7}$$

where for each $i \in \Xi$

$$L_{k,i}^\star = -\left(R + B^T S_{k+1,\phi_i} B\right)^{-1} B^T S_{k+1,\phi_i} A, \tag{8}$$

with

$$\begin{aligned} S_{k,i} &= Q + A^T \left(S_{k+1,\bar{\phi}_i} p_{i\bar{\phi}_i} + S_{k+1,\phi_i} p_{i\phi_i}\right) A \\ &- A^T S_{k+1,\phi_i} B \left(R + B^T S_{k+1,\phi_i} B\right)^{-1} B^T S_{k+1,\phi_i} A p_{i\phi_i}, \end{aligned} \tag{9}$$

and ϕ_i and $\bar{\phi}_i$ defined in (3) and (4). Here $p_{i\phi_i} = \mathbf{Pr}\left\{\Theta_{k+1} = \phi_i | \Theta_k = i\right\}$ and S_N is given. This results in the optimal cost

$$J_N^\star\left(x_0, \Theta_0\right) = x_0^T S_{0,\Theta_0} x_0 + c_{0,\Theta_0}, \tag{10}$$

where

$$c_{k,i} = \sum_{j \in \left\{\phi_i, \bar{\phi}_i\right\}} \left(\operatorname{trace}\left(\Sigma_\omega S_{k+1,j}\right) + c_{k+1,j}\right) p_{ij} \tag{11}$$

with $c_{N,i} = 0$ for all i.

Proof. The proof is straightforward by adapting the results from Ji et al. (1991); Costa et al. (2006) and is omitted.

Remark 2. It is worth to note that the controller at time k only depends on the history of the packet arrivals that are governed in Θ_k, which does not include ρ_k. Also, note that the controller is invariant of the oldest packet arrival that is captured in Θ_k. This follows directly from (3) to (5) where, e.g. $\phi_1 = \phi_2$. This means that while we obtain 2^d controllers, only 2^{d-1} of these controllers are unique and have to be stored in a lookup table.

Remark 3. Many network models such as the Gilbert Elliot model Gilbert (1960); Huang and Dey (2007) use a Markov chain with length $d = 1$. It is worth noting that using Proposition 1 only one controller gain is obtained. Thus, the controller does not have to observe the history of the packet dropouts to find the appropriate control gain.

To implement the controller design from Proposition 1, one would use a lookup table that contains the control gains, and then at each time step use Θ_k to select the corresponding control gain. This requires enough memory to store 2^{d-1} control gains or enough computational power to compute (8) and (9) online. One question that arises here is whether it is necessary to take the full packet dropout history into account in the controller design to guarantee a reasonable control performance. Omitting part of the history will result in fewer control gains, and thereby a smaller lookup table. In the next section, we will present a sub-optimal controller design that reduces the amount of unique control laws at the cost of performance.

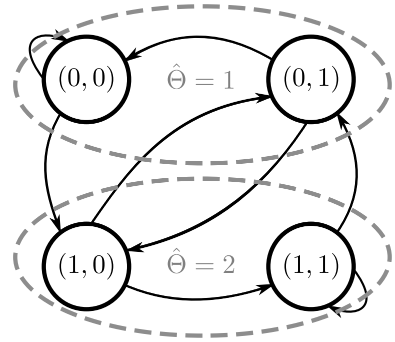

Fig. 3. An example of the grouping of the packet arrival sequences. The dashed circles indicate which packet arrival sequences are grouped together

3. REDUCED CONTROLLER COMPLEXITY

In this section, we will investigate whether it is necessary to design all 2^{d-1} control laws (as presented in Section 2) to attain a reasonable control performance.To be more precise, we will develop two approaches to design reduced complexity controllers. In both methods, we group the packet arrival sequences together and design a single control law for each of these groups. This leads to two problems: how to group the packet arrival sequences, and what should the control laws for these groups be?

3.1 Grouping the packet arrival sequences

Here we discuss the grouping method of the packet arrival sequences that is used for both of the reduced complexity controller designs. Our intuition is that the more recent history is the most relevant when designing the controller, while the older history is of lesser importance. We will therefore group the packet arrival sequences such that, for $0 < q \leq d$, the first 2^{d-q} packet arrivals are identical for all Θ in each group. We are thus interested in control laws of the form

$$\hat{u}_k = f_k\left(\hat{\Theta}_k, x_k\right), \tag{12}$$

where the variable $\hat{\Theta}_k \in \hat{\Xi}$, and each $\hat{\Theta}_k = i$ is linked to a given packet arrival sequence of the most recent $d - q$ packet arrivals. Here, the set $\hat{\Xi} = \left\{1, \cdots, 2^{d-q}\right\}$, and for each $v \in \hat{\Xi}$ we define

$$\hat{\Xi}_v \triangleq \left\{i \in \Xi : g\left(i\right) = v\right\},$$

where $g : \Xi \rightarrow \hat{\Xi}$. Thus each $\hat{\Xi}_v$ is a subset of Ξ, where the most recent $d - q$ packet arrivals in Θ_k are equal to $\hat{\Theta}_k$. Also note that $\hat{\Xi}_v$ are disjoint sets ($\hat{\Xi}_v \cap \hat{\Xi}_j = \emptyset$ if $v \neq j$).

Example 2. Consider the case from Example 1. Then, with $q = 1$, $\hat{\Theta}$ takes values in $\hat{\Xi} = \{1, 2\}$, where for $\hat{\Theta} = 1$ we have $\hat{\Xi}_1 = (1, 2)$ and for $\hat{\Theta} = 2$ we have $\hat{\Xi}_2 = (3, 4)$. This is depicted in Fig. 3. □

In the remainder of this section we propose controller designs, where the control laws depend on the groups $\hat{\Theta}$.

3.2 Group averaged controller

For the group averaged (GA) controller, we take the expectation of the control laws obtained in Proposition 1 over each group $\hat{\Theta}$. The controller is computed as

$$\hat{L}_{k,i} = \mathbf{E}\left\{ L_k \middle| \hat{\Theta}_k = i \right\}, \qquad \forall i \in \hat{\Xi}. \tag{13}$$

Computing the expectation leads to

$$\hat{L}_{k,i} = \sum_{j\in\Xi} L_{k,j}\, \mathbf{Pr}\left\{ \Theta_k = j \middle| \hat{\Theta}_k = i \right\},$$

where, using Bayes' law, one obtains the GA control law

$$\hat{L}_{k,i} = \sum_{j\in\hat{\Xi}_i} L_{k,j} \frac{\mathbf{Pr}\{\Theta_k = j\}}{\mathbf{Pr}\left\{\hat{\Theta}_k = i\right\}}, \quad \forall i \in \hat{\Xi}, \tag{14}$$

where $\mathbf{Pr}\left\{\hat{\Theta}_k\right\} = \sum_{j\in\hat{\Xi}_i} \pi_j$.

3.3 Optimal design approach

In this section, we aim at designing control laws in the form (12) to minimize the cost function

$$\bar{J}_K\left(\bar{U}_{0,N}, x_0\right) = \mathbf{E}\left\{ J\left(\bar{U}_{0,N}, x_0, \Theta_0\right) | x_0 \right\}, \tag{15}$$

where $\bar{U}_{0,N} = (\bar{u}_0, \cdots, \bar{u}_{N-1})$ is the set of reduced controllers, with $\bar{u}_k = \left(\hat{u}_{k,1}, \cdots, \hat{u}_{k,2^{d-q}}\right)$.

Ideally, as in Section 2.2, each control law would depend on the full length d history of the packet dropouts, that is covered by Θ_k and the state x_k. Here we instead constrain the input design to, at each time k, only depend on the reduced history that is covered by $\hat{\Theta}_k$ and the state x_k. Our goal is then to solve

$$\bar{U}^\star_{0,N} = \underset{\bar{U}_{0,N}}{\arg\min} \bar{J}_N\left(\bar{U}_{0,N}, x_0\right) \tag{16a}$$

$$\text{subject to } \bar{J}^\star_N(x_0) = \min_{\bar{U}_{0,N}} \bar{J}_N\left(\bar{U}_{0,N}, x_0\right). \tag{16b}$$

Here we obtain the following result.

Theorem 2. The optimal controls for (16) are given by

$$\hat{u}^\star_k = \bar{L}^\star_{k,v} x_k, \quad g(\Theta_k) = v \tag{17}$$

where

$$\bar{L}^\star_{k,v} = -\left(\bar{R}_v + B^T \bar{S}_{k+1,v} B\right)^{-1} B^T \bar{S}_{k+1,v} A \tag{18}$$

and for $i \in \Xi$

$$S_{k,i} = Q + A^T S_{k+1,\bar{\phi}_i} A p_{i\bar{\phi}_i} + \left(\bar{L}^\star_{k,g(i)}\right)^T R \bar{L}^\star_{k,g(i)} p_{i\phi_i} + \left(A + B\bar{L}^\star_{k,g(i)}\right)^T S_{k+1,\phi_i}\left(A + B\bar{L}^\star_{k,g(i)}\right) p_{i\phi_i}. \tag{19}$$

Here $S_{N,i} = S_N$ and

$$\bar{S}_{k,v} = \textstyle\sum_{i\in\hat{\Xi}_v} S_{k,\phi_i} p_{i\phi_i}\pi_i,\ \bar{R}_v = R\sum_{i\in\hat{\Xi}_v} p_{i\phi_i}\pi_i. \tag{20}$$

This results in the cost

$$\bar{J}^\star_N(x_0) = \sum_{i\in\Xi} x_0^T S_{0,i} x_0 \pi_i + c_{0,i}\pi_i, \tag{21}$$

where

$$c_{k,i} = \operatorname{trace}\left(\Sigma_\omega\left(S_{k+1,\phi_i} p_{i\phi_i} + S_{k+1,\bar{\phi}_i} p_{i\bar{\phi}_i}\right)\right) + c_{k+1,\phi_i} p_{i\phi_i} + c_{k+1,\bar{\phi}_i} p_{i\bar{\phi}_i}, \tag{22}$$

with $c_{N,i} = 0$, $i \in \Xi$. □

It is worth noting that without the complexity reduction, i.e. $\hat{\Xi} = \Xi$, the controller that is obtained in Theorem 2 is identical to the optimal controller given in Proposition 1.

4. PROOF OF THEOREM 2

Before we show the proof of Theorem 2 we need some definitions. Let $V_k \triangleq \|x_k\|_Q^2 + \rho_k \|\hat{u}_{k,v}\|_R^2$ and define the cost at stage $N-k$ depending on $\Theta_k = i$ as

$$\bar{J}_{N-k,i}\left(x_k, \bar{U}_{k+1,N}\right) \triangleq J_{N-k}\left(x_k, \bar{U}_{k,N}, i\right) = \mathbf{E}\left\{ \sum_{\ell=k}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_k, \Theta_k = i \right\}, \tag{23}$$

where $g(\Theta_\ell) = v$ with $g : \Xi \to \hat{\Xi}$. Also define the cost conditioned on $\hat{\Theta} = v$ by

$$F_{N-k}\left(x_k, \bar{U}_{k,N}, v\right) \triangleq \mathbf{E}\left\{ J_{N-k}\left(x_k, \bar{U}_{k,N}, \Theta_k\right) \middle| x_k, \hat{\Theta}_k = v \right\} \hat{\pi}_v, \tag{24}$$

where $\hat{\pi}_v \triangleq \mathbf{Pr}\left\{\hat{\Theta} = v\right\} = \sum_{i\in\hat{\Xi}_v} \pi_i$.

In the next lemma, we show that the optimal controls subject to the form (12) can be solved stage-wise.

Lemma 3. Let $\bar{J}_{N-\ell,i}(x_\ell)$ be defined as in (23). Then the cost (15) can at any stage $N-k$ be expressed as

$$\bar{J}_{N-k}\left(x_k, \bar{U}_{k,N}\right) = \sum_{v\in\hat{\Xi}} F_{N-k}\left(x_k, \left(\hat{u}_{k,v}, \bar{U}_{k+1,N}\right), v\right), \tag{25}$$

where

$$F_{N-k}\left(x_k, \left(\hat{u}_{k,v}, \bar{U}_{k+1,N}\right), v\right) = \mathbf{E}\left\{ \mathcal{L}_{N-k,\Theta_k}\left(x_k, \hat{u}_k\right) \middle| x_k, \hat{\Theta}_k = v \right\} \hat{\pi}_v,$$

and

$$\mathcal{L}_{N-k,\Theta_k}\left(x_k, \hat{u}_{k,v}\right) \triangleq x_k^T Q x_k + \rho_k \hat{u}_{k,v}^T R \hat{u}_{k,v} + \bar{J}_{N-k-1,\Theta_{k+1}}\left(Ax_k + \rho_k B\hat{u}_{k,v} + \omega_k, \bar{U}_{k+1,N}\right). \tag{26}$$

Proof. For the cost (15) we have at stage $N-k$

$$\begin{aligned} \bar{J}_{N-k}\left(x_k, \bar{U}_{k,N}\right) &= \mathbf{E}\left\{ \sum_{\ell=k}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_k \right\} \\ &= \sum_{v\in\hat{\Xi}} \mathbf{E}\left\{ \sum_{\ell=k}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_k, \hat{\Theta}_k = v \right\} \hat{\pi}_v \\ &= \sum_{v\in\hat{\Xi}} F_{N-k}\left(x_k, \left(\hat{u}_{k,v}, \bar{U}_{k+1,N}\right), v\right). \end{aligned} \tag{27}$$

Where the last step follows by the definition in (24) and by knowing $\hat{\Theta}_k = v$, we have $u_k = \hat{u}_{k,v}$.

Now for each $F_{N-k}\left(x_k, \left(\hat{u}_{k,v}, \bar{U}_{k+1,N}\right), v\right)$ we have

$$F_{N-k}\left(x_k, \left(\hat{u}_{k,v}, \bar{U}_{k+1,N}\right), v\right) = \mathbf{E}\left\{ V_k \middle| x_k, \hat{\Theta}_k = v \right\} + \underbrace{\mathbf{E}\left\{ \sum_{\ell=k+1}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_k, \hat{\Theta}_k = v \right\}}_{C}, \tag{28}$$

where

$$\begin{aligned} C &= \mathbf{E}\left\{ \mathbf{E}\left\{ \sum_{\ell=k+1}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_{k+1}, \Theta_{k+1} \right\} \middle| x_k, \hat{\Theta}_k = v \right\} \\ &= \int \mathbf{E}\left\{ \sum_{\ell=k+1}^{N-1} V_\ell + \|x_N\|^2_{S_0} \middle| x_{k+1}, \Theta_{k+1} \right\} \\ &\quad \times \mathbf{Pr}\left\{ x_{k+1}, \Theta_{k+1} \middle| x_k, \hat{\Theta}_k = v \right\} dx_{k+1} d\Theta_{k+1} \\ &= \int \bar{J}_{N-k-1,\Theta_{k+1}}\left(x_{k+1}, \bar{U}_{k+1,N}\right) \end{aligned}$$

$$\times \mathbf{Pr}\left\{x_{k+1}, \Theta_{k+1} \middle| x_k, \hat{\Theta}_k = v\right\} dx_{k+1} d\Theta_{k+1}$$
$$= \mathbf{E}\left\{\bar{J}_{N-k-1,\Theta_{k+1}}\left(x_{k+1}, \bar{U}_{k+1,N}\right) \middle| x_k, \hat{\Theta}_k = v\right\}.$$

The result follows by substituting this back into (28). □

We are now ready to state the proof of Theorem 2

Proof. [Proof of Theorem 2] In Lemma 3 we showed that the cost at each stage can be expressed by (25). Now we will show that when we only allow control laws of the form (12), the optimal control laws are given in the form (17). The proof follows by induction.

At stage 0 we have the cost

$$\bar{J}_0^\star(x_N) = \mathbf{E}\left\{x_N^T S_N x_N \middle| x_N\right\} = \sum_{i\in\Xi} \bar{J}_{0,i}(x_N)\,\pi_i,$$

with $\bar{J}_{0,i}(x_N)$ defined in (23). At stage $N-k-1$ we have with control laws of the form (12) the optimal cost

$$\bar{J}_{N-k-1}^\star(x_{k+1}) = \sum_{i\in\Xi} \bar{J}_{N-k-1,i}^\star(x_{k+1})\,\pi_i,$$

where $\bar{J}_{N-k-1,i}^\star(x_{k+1}) \triangleq \bar{J}_{N-k-1,i}\left(x_{k+1}, \bar{U}_{k+1,N}^\star\right)$ is given by

$$\bar{J}_{N-k-1,i}^\star(x_{k+1}) = x_{k+1}^T S_{k+1,i} x_{k+1} + c_{k+1,i}.$$

Then, by Lemma 3 we have at stage $N-k$ for $\hat{\Theta}_k = v$ the cost

$$F_{N-k}^\star(x_k, v) = \min_{\hat{u}_{k,v}} \mathbf{E}\left\{\mathcal{L}_{N-k,\Theta_k}(x_k, \hat{u}_{k,v}) \middle| x_k, \hat{\Theta}_k = v\right\}\hat{\pi}_v$$
$$= \min_{\hat{u}_{k,v}} \sum_{i\in\Xi} \mathbf{E}\left\{\mathcal{L}_{N-k,\Theta_k}(x_k, \hat{u}_{k,v}) | x_k, \Theta_k = i\right\}$$
$$\times \mathbf{Pr}\left\{\Theta_k = i \middle| \hat{\Theta}_k = v\right\}\hat{\pi}_v$$
$$\overset{(a)}{=} \min_{\hat{u}_{k,v}} \sum_{i\in\hat{\Xi}_v} \underbrace{\mathbf{E}\left\{\mathcal{L}_{N-k,\Theta_k}(x_k, \hat{u}_{k,v}) | x_k, \Theta_k = i\right\}}_{D_i}\pi_i, \quad (29)$$

where we in (a) used Bayes' law. Now

$$D_i = \mathbf{E}\left\{\mathbf{E}\left\{\mathcal{L}_{N-k,\Theta_k}(x_k, \hat{u}_{k,v}) | x_k, \Theta_{k+1}, \Theta_k\right\} | x_k, \Theta_k = i\right\}$$
$$= \sum_{j\in\Xi} \mathbf{E}\left\{\mathcal{L}_{N-k,i}(x_k, \hat{u}_{k,v}) | x_k, \Theta_{k+1} = j\right\} p_{ij}$$
$$\overset{(a)}{=} \sum_{j\in\{\bar{\phi}_i, \phi_i\}} \mathcal{M}_{N-k,j}(x_k, \hat{u}_{k,v})\, p_{ij}, \quad (30)$$

where (a) follows by (3) and (4) and

$$\mathcal{M}_{N-k,j}(x_k, \hat{u}_{k,v}) \triangleq \mathbf{E}\left\{\mathcal{L}_{N-k,i}(x_k, \hat{u}_{k,v}) | x_k, \Theta_{k+1} = j\right\}$$
$$= x_k^T\left(Q + A^T S_{k+1,j} A\right)x_k + \operatorname{trace}\left(\Sigma_\omega S_{k+1,j}\right) + c_{k+1,j}$$
$$+ \mathbb{1}_{(r+1,\cdots,d)}(j)\, \hat{u}_{k,v}^T\left(R + B^T S_{k+1,j} B\right)\hat{u}_{k,v}$$
$$+ 2\,\mathbb{1}_{(r+1,\cdots,d)}(j)\, \hat{u}_{k,v}^T B^T S_{k+1,j} A x_k.$$

Substituting (30) back into (29) yields

$$F_{N-k}^\star(x_k, v) = \sum_{i\in\hat{\Xi}_v} \mathcal{M}_{N-k,\bar{\phi}_i}(x_k, 0)\, p_{i\bar{\phi}_i}\pi_i$$
$$+ \min_{\hat{u}_{k,v}} \mathcal{M}_{N-k,\phi_i}(x_k, \hat{u}_{k,v})\, p_{i\phi_i}\pi_i.$$

Writing out $\mathcal{M}_{N-k,\bar{\phi}_i}$ and $\mathcal{M}_{N-k,\phi_i}$ yields

$$F_{N-k}^\star(x_k, v) = \sum_{i\in\hat{\Xi}_v} \sum_{j\in\{\bar{\phi}_i, \phi_i\}} \Big[x_{N-1}^T\left(Q + A^T S_{k+1,j} A\right)x_{N-1}$$
$$+ \operatorname{trace}\left(\Sigma_\omega S_{k,j}\right) + c_{k+1,j}\Big] p_{ij}\pi_i \quad (31)$$
$$+ \min_{\hat{u}_{k,v}} \Big[\sum_{i\in\hat{\Xi}_v} \Big(\hat{u}_{k,v}^T\left(R + B^T S_{k+1,\phi_i} B\right)\hat{u}_{k,v}$$
$$+ 2\hat{u}_{k,v}^T B^T S_{k,\phi_i} A x_k\Big) p_{i\phi_i}\pi_i\Big].$$

Here the last part can, using (20), be rewritten to

$$\min_{\hat{u}_{k,v}} \hat{u}_{k,v}^T\left(\bar{R}_v + B^T \bar{S}_{k+1,v} B\right)\hat{u}_{k,v} + 2\hat{u}_k^T B^T \bar{S}_{k+1,v} A x_k.$$

Taking the gradient of (31) with respect to $\hat{u}_{k,v}$ and setting this equal to 0 then results in (17) and (18). Inserting the result into D_i in (30) and rewriting this results in

$$D_i = x_k^T S_{k,i} x_k + \sum_{j\in\{\phi_i, \bar{\phi}_i\}} \left(\operatorname{trace}\left(\Sigma_\omega S_{k,j}\right) + c_{k+1,j}\right) p_{ij}$$
$$= x_k^T S_{k,i} x_k + c_{k,i} = \bar{J}_{N-k,i}^\star(x_k), \quad (32)$$

where $S_{k,i}$ is given in (19) and $c_{k,i}$ in (22). Combining (27), (29) and (32) then results in (21). □

Θ	$\mathbf{Pr}\{\rho_k = 1 \vert \Theta\}$	$\mathbf{Pr}\{\rho_k = 0 \vert \Theta\}$
16	$\mathbf{Pr}\{1 \vert 1111\} = 0.5$	$\mathbf{Pr}\{0 \vert 1111\} = 0.5$
15	$\mathbf{Pr}\{1 \vert 1110\} = 0.35$	$\mathbf{Pr}\{0 \vert 1110\} = 0.65$
14	$\mathbf{Pr}\{1 \vert 1101\} = 0.22$	$\mathbf{Pr}\{0 \vert 1101\} = 0.78$
13	$\mathbf{Pr}\{1 \vert 1100\} = 0.8$	$\mathbf{Pr}\{0 \vert 1100\} = 0.2$
12	$\mathbf{Pr}\{1 \vert 1011\} = 0.15$	$\mathbf{Pr}\{0 \vert 1011\} = 0.85$
11	$\mathbf{Pr}\{1 \vert 1010\} = 0.7$	$\mathbf{Pr}\{0 \vert 1010\} = 0.3$
10	$\mathbf{Pr}\{1 \vert 1001\} = 0.25$	$\mathbf{Pr}\{0 \vert 1001\} = 0.75$
9	$\mathbf{Pr}\{1 \vert 1000\} = 0.8$	$\mathbf{Pr}\{0 \vert 1000\} = 0.2$
8	$\mathbf{Pr}\{1 \vert 0111\} = 0.4$	$\mathbf{Pr}\{0 \vert 0111\} = 0.6$
7	$\mathbf{Pr}\{1 \vert 0110\} = 0.2$	$\mathbf{Pr}\{0 \vert 0110\} = 0.8$
6	$\mathbf{Pr}\{1 \vert 0101\} = 0.5$	$\mathbf{Pr}\{0 \vert 0101\} = 0.5$
5	$\mathbf{Pr}\{1 \vert 0100\} = 0.7$	$\mathbf{Pr}\{0 \vert 0100\} = 0.3$
4	$\mathbf{Pr}\{1 \vert 0011\} = 0.5$	$\mathbf{Pr}\{0 \vert 0011\} = 0.5$
3	$\mathbf{Pr}\{1 \vert 0010\} = 0.9$	$\mathbf{Pr}\{0 \vert 0010\} = 0.1$
2	$\mathbf{Pr}\{1 \vert 0001\} = 0.6$	$\mathbf{Pr}\{0 \vert 0001\} = 0.4$
1	$\mathbf{Pr}\{1 \vert 0000\} = 0.3$	$\mathbf{Pr}\{0 \vert 0000\} = 0.7$

Table 1. The probabilities for packet arrivals and dropouts used for the simulations.

5. SIMULATIONS

We compare the performance of the proposed controller designs using Monte Carlo simulation studies. We design the controllers for a cost function with horizon length $N = 500$. The controllers are implemented as model predictive control (Borrelli et al., 2017), that is we set $u_k = L_{0,\Theta_k}^\star x_k$ for all k. Here we consider the stable system

$$A = \begin{bmatrix} 0.819 & 0 \\ 0.906 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 9.063 \\ 4.683 \end{bmatrix},$$

and $\Sigma_\omega = I_2$. The probabilities for a successful transmission at time k depends on the packet dropout history of length $d = 4$. These are, using (2), written into a transition matrix as illustrated in (5), where the probabilities are stated in Table 1. In (6), $S_N = Q = I_2$ and $R = 1$.

For the simulations we average over 200 runs each of length 10 000 time steps with $x_0 \sim \mathcal{N}(0, I_2)$. The network is assumed to be in steady state with $\mathbf{Pr}\{\Theta_0 = i\} = \pi_i$. The averaged results are shown in Fig. 4. Here the numbers indicate how many control laws are needed to be stored in a lookup table. Note that for the optimal controllers that

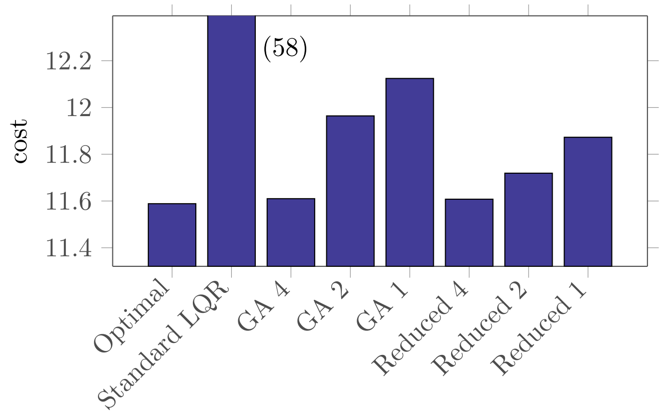

Fig. 4. The Optimal controller designed using Proposition 1, the GA controller, designed using (14) with 1,2 and 4 control laws and the reduced controller designed using Theorem 2 with 1,2 and 4 control laws.

are designed using Proposition 1, 8 unique control laws are used at every time step. We use a standard LQR that does not take the network into account to benchmark.

The results show that all controllers significantly outperform the standards LQR, which achieves a cost of 58. Also, there is no significant performance degradation for the reduced controllers designed by Theorem 2 and the GA controllers from Section 3.2 compared to the optimal control. When only one or two control laws are desired, the ("reduced") design presented in Theorem 2 outperforms the more simple GA design from Section 3.2.

6. CONCLUSIONS

We presented a method to design controllers for systems that are affected by random packet dropouts between the controller and actuators. The probability for a successful transmission depends on a finite history of packet transmission outcomes. This results in a special structure for and underlying Markov transition matrix, which we take advantage of to obtain simple expressions for the controller design. An implementation of this controller will however result in a lookup table, often of large size..

To reduce the size of the lookup table used for control, we presented two sub-optimal methods. Both of these methods show an only minor performance degradation when the number of controllers is reduced, which allows for a good tradeoff between performance and controller complexity.

Future work would involve performance and stability analysis for the presented controllers.

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