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Modified Mikhailov plots for robust absolute stability with non-parametric perturbations and uncertain nonlinearity

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This paper generalizes the classical Mikhailov stability result for testing robust absolute stability of linear systems with both non-parametric perturbations and uncertain nonlinearity. Three types of non-parametric perturbations are considered: additive, multiplicative and stable-factor perturbations. Modified Mikhailov plots are developed for simultaneously testing the stability of the 'nominal' system and computing the maximum robust absolute stability margin.

1. Introduction

The classical Mikhailov stability criterion (Mikhailov 1938), also known as the Cremer–Leonhard criterion (Cremer 1947, Leonhard 1944), provides both deep understanding of the Hurwitzness of polynomials and a simple graphical means for testing it. However, this result has been regarded for theoretical interest only (Gantmacher 1959) because the Routh–Hurwitz stability criterion requires much less computation. Interest in the Mikhailov stability criterion has recently re-arisen for its potential application in robust stability analysis where the Routh–Hurwitz criterion performs poorly.

The first hint that the Mikhailov criterion might be used for robust stability was perhaps indicated in the nominal paper of Kharitonov (1978) where the Hermite–Bieler theorem, a 'cousin' version of the Mikhailov criterion, was used to prove the famous Kharitonov theorem. Since then, other versions of the criterion—although the name of Mikhailov is usually not mentioned—have been used to derive various robust stability results (see, for example, Barmish 1989, Minnichelli *et al*. 1989, Fu 1989).

The first direct use of the Mikhailov criterion for robust stability analysis was in recent papers by Tsypkin and Polyak (1991, 1992). Tsypkin and Polyak (1992) considered the robust stability of 'generalized' interval polynomials where the parametric perturbations in polynomial coefficients are bounded by a weighted l_p norm. They provided an elegant solution to the problem, which involves drawing only a single modified Mikhailov plot to determine both the stability of the nominal (i.e. centre) polynomial and the maximum robust stability margin (i.e. the maximum l_p norm for robust stability). This approach they later extended to solve the robust stability problem for polytopes of polynomials (Tsypkin and Polyak 1991). Once again, they showed that both the stability of the nominal polynomial and the maximum robust stability margin could be obtained by using a single modified Mikhailov plot.

For robust stability analysis of systems with non-parametric perturbations

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and/or uncertain nonlinearity, Nyquist-like methods are often used. For example, the circle criterion and Popov criterion are used for testing absolute stability of systems with uncertain nonlinearity (see, e.g. Popov 1973, Narendra and Taylor 1973). For linear systems with additive or multiplicative perturbations, a robust stability test can be carried out using the circle criterion (Chen and Desoer 1982). The results of a recent paper by Tsypkin and Polyak (1993) provide a new circle criterion for robust absolute stability of systems with both non-parametric perturbations and uncertain nonlinearity by using a modified Nyquist plot.

This paper is motivated by the need for a common framework to analyse robust stability. More specifically, we show that robust absolute stability of systems with non-parametric perturbations and uncertain nonlinearity can simply be tested by using a modified Mikhailov plot. Three types of non-parametric perturbations are considered: in addition to additive and multiplicative ones, we also consider the so-called stable-factor perturbations. The latter have been studied by Kwakernaak (1983) and Vidyasagar (1985). The class of stable-factor perturbations is free from the restrictive assumptions that the perturbations must preserve the number of unstable poles of the plant and that the plant is void of poles on the jω axis. In all these cases, we show how to use modified Mikhailov plots for both checking the stability of the nominal plant and computing the robust absolute stability margin (i.e. the maximum size of the perturbations defined in a certain sense for preserving stability).

2. Mikhailov stability criterion and its modification

This section restates some well-known results about the Mikhailov stability criterion.

Lemma 1—Mikhailov stability criterion (Mikhailov 1938): *An nth order real polynomial $p(s)$ is strictly Hurwitz stable (or stable, for short) if and only if the plot of $p(\mathrm{j}\omega)$ rotates through n quadrants in the complex plane in the counter-clockwise direction without crossing the origin when ω increases from 0 to ∞.*

Remark 1: The plot of $p(\mathrm{j}\omega)$ is commonly referred to as the *Mikhailov plot*. □

For complex polynomials, negative ω values also need be used and the number of quadrants that the Mikhailov plot needs to pass through increases to $2n$. The Mikhailov criterion is a natural consequence of the fact that a Hurwitz polynomial must have n zeros in the open left half-plane, and each of them contributes 180° of phase lead when ω varies from $-\infty$ to ∞. In fact, the Mikhailov plot of a stable polynomial has monotonic phase increase. The Mikhailov plot can also be used to count the number of stable zeros. Indeed, supposing the Mikhailov plot does not cross the origin, then the number of stable zeros equals half of the number of quadrants the plot passes through when ω varies from $-\infty$ to ∞. It is interesting to note that this criterion is mathematically equivalent to the Routh–Hurwitz criterion and the Hermite–Bieler theorem, it can also be used to prove the Nyquist criterion without the help of the Principle of the Argument. Furthermore, the Mikhailov criterion also generalizes to most other stability regions.

The standard Mikhailov plots are somehow inconvenient to use because of their unboundedness. It is often necessary to use a modified Mikhailov plot, as shown below.

Lemma 2—Modified Mikhailov stability criterion (Tsypkin and Polyak 1992): *Given an nth order real polynomial $p(s)$, define*

$$\hat{p}(j\omega) = \frac{\mathrm{Re}\,[p(j\omega)]}{S(\omega)} + j\,\frac{\mathrm{Im}\,[p(j\omega)]}{T(\omega)} \tag{1}$$

where $S(\omega) > 0$ and $T(\omega) > 0$ are arbitrarily chosen continuous scaling functions. Then, $p(s)$ is strictly Hurwitz stable (or stable, for short) if and only if the plot of $\hat{p}(j\omega)$ rotates through n quadrants in the complex plane in the counterclockwise direction without crossing the origin when ω increases from 0 to ∞.

Remark 2: The plot of $\hat{p}(j\omega)$ is called the modified Mikhailov plot. Since the functions $S(\omega)$ and $T(\omega)$ are arbitrary, they can be chosen for additional purposes. For example, one can choose $S(\omega)$ and $T(\omega)$ to make the modified Mikhailov plot bounded. Tsypkin and Polyak (1991, 1992) used specially modified Mikhailov plots to obtain robust stability margins of the polynomials with parametric perturbations. The objective of this paper is to obtain modified Mikhailov plots for robust stability analysis for systems with non-parametric perturbations and/or uncertain nonlinearity.

3. Problem formulation

Consider a single-input single-output (SISO) feedback system as in Fig. 1, where $G(s)$ is the transfer function of a linear and time-invariant (LTI) plant and $f(\cdot)$ is a static nonlinear function. The transfer function $G(s)$ consists of a nominal model and non-parametric perturbations, with the nominal model being expressed by

$$G^0(s) = \frac{N^0(s)}{D^0(s)} \tag{2}$$

where $N^0(s)$ and $D^0(s)$ are polynomials. Three types of perturbations are considered for $G(s)$.

3.1. *Additive perturbations*

$$G(s) = G^0(s) + \Delta G(s) \tag{3}$$

with the constraints that $G(s)$ and $G^0(s)$ have the same number of unstable

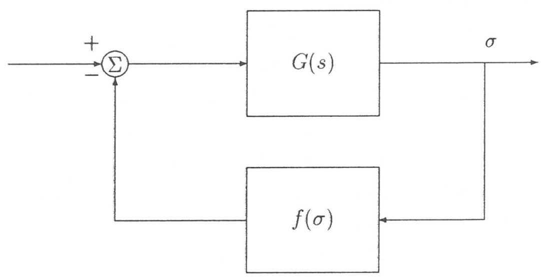

Figure 1. Closed-loop uncertain system.

poles in the right half-plane, that $G^0(s)$ is void of poles on the $j\omega$ axis, and that

$$|\Delta G(j\omega)| \leq \gamma\alpha(\omega), \quad \alpha(\omega) > 0, \quad \forall\omega \in \boldsymbol{R} \tag{4}$$

3.2. *Multiplicative perturbations*

$$G(s) = G^0(s)[1 + L(s)] \tag{5}$$

with the constraints that $G(s)$ and $G^0(s)$ have the same number of unstable poles in the right half-plane, that $G^0(s)$ is void of poles on the $j\omega$ axis, and that

$$|L(j\omega)| \leq \gamma\beta(\omega), \quad \beta(\omega) > 0, \quad \forall\omega \in \boldsymbol{R} \tag{6}$$

3.3 *Stable-factor perturbations* (Vidyasagar 1985)

$$G(s) = \frac{N^0(s) + \Delta N(s)}{D^0(s) + \Delta D(s)} \tag{7}$$

with

$$\left|\frac{\Delta N(j\omega)}{\tau_1(\omega)}\right|^2 + \left|\frac{\Delta D(j\omega)}{\tau_2(\omega)}\right|^2 \leq \gamma^2, \quad \tau_i(\omega) > 0, \quad i = 1, 2, \quad \forall\omega \in \boldsymbol{R} \tag{8}$$

In the above, the parameter $\gamma \geq 0$ represents the 'size' of the perturbations, and $\alpha(\omega)$, $\beta(\omega)$ and $\tau_i(\omega)$, $i = 1, 2$ are continuous frequency-weighting functions. The terminology of stable-factor perturbations comes from Vidyasagar (1985) where transfer functions are expressed in the coprime factorized form. We do not use coprime factorization here because of the flexible weighting functions $\tau_i(\omega)$.

Furthermore, the nonlinear function, which is uncertain, belongs to a sector as follows

$$K_0(1 + c\gamma)^{-1} \leq \frac{f(\sigma)}{\sigma} \leq K_0(1 - c\gamma)^{-1}, \quad \forall\sigma \in \boldsymbol{R} \tag{9}$$

where $K_0 > 0$ is the nominal feedback gain, $c > 0$ is a weighting constant and $\gamma \geq 0$ is the same scaling parameter as for non-parametric perturbations.

Our objective is to derive a modified Mikhailov plot which can be used for testing both stability of the nominal plant (when $\gamma = 0$) and estimating a γ bound, calling it γ_{max}, such that the stability of the feedback system in Fig. 1 is preserved for all perturbations with $0 \leq \gamma < \gamma_{max}$. This γ_{max} will be called the *robust absolute stability margin*.

4. Modified Mikhailov plots

Two main results are provided in this section. The first deals with robust absolute stability with either additive or multiplicative perturbations, and a single modified Mikhailov plot is given for both checking the stability of the nominal system and providing a bound for γ_{max}, the robust absolute stability margin. The second result is for robust absolute stability with stable-factor perturbations.

Theorem 1: *Consider the robust absolute stability problem associated with the closed-loop system in Fig.* 1 *with additive perturbations* (3)–(4). *Then, the* *nominal closed-loop system is stable if and only if the hodograph of the following modified function*

$$p_a(j\omega) = \frac{K_0 N_0(j\omega) + D_0(j\omega)}{(K_0\alpha(\omega) + c)|D_0(j\omega)|} \tag{10}$$

encircles the origin n quadrants in the counter-clockwise direction without crossing the origin when ω increases from 0 to ∞. If so, the robust absolute stability margin γ_{max} is no less than the minimum distance of the plot from the origin, i.e.

$$\gamma_{max} \geqslant \min\{|p_a(j\omega)|: 0 \leqslant \omega \leqslant \infty\} \tag{11}$$

Similarly, the same result holds for multiplicative perturbations in (5)–(6) *with the hodograph of the following modified function*

$$p_m(j\omega) = \frac{K_0 N_0(j\omega) + D_0(j\omega)}{K_0|N_0(j\omega)|\beta(\omega) + |D_0(j\omega)|c} \tag{12}$$

For illustrative purposes, a modified Mikhailov plot is shown in Fig. 2 which corresponds to the following nominal plant and multiplicative perturbations and uncertain nonlinearity described by

$$G_0(s) = \frac{s+2}{(0{\cdot}2s+1)^2(s-1)} \tag{13}$$

$$\beta(\omega) = \frac{0{\cdot}1\omega}{(\omega^2 + 0{\cdot}5)^{1/2}} \tag{14}$$

$$K_0 = 1, \quad c = 0{\cdot}1 \tag{15}$$

Since the modified Mikhailov plot rotates through three quadrants, the nominal closed-loop system is stable. It is obtained from Fig. 2 that $\gamma_{max} \geqslant 5{\cdot}04$ which is obtained at $\omega = 0{\cdot}75$.

Proof of Theorem 1: The reason for the modified Mikhailov plots to determine the stability of the nominal system is obvious because the modification does not

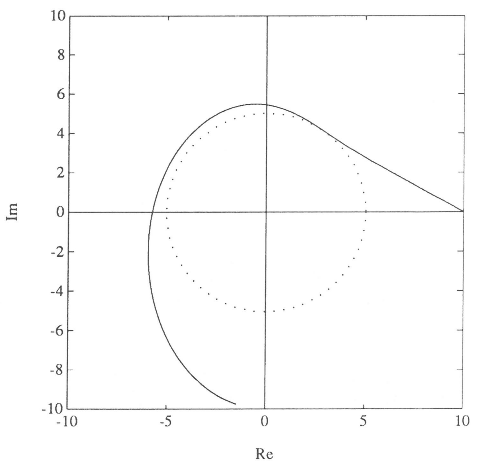

Figure 2. Modified Mikhailov plot for multiplicative perturbations and uncertain nonlinearity.

change the encirclement, see Lemma 2. The fact that $\gamma_{\max}$ is greater than or equal to the minimum modulus of the modified Mikhailov plot results from the circle criterion and the well-known zero exclusion principle. For additive perturbations, the circle criterion and zero exclusion principle yield the following necessary and sufficient condition for robust absolute stability (Tsypkin and Polyak 1993):

$$\left| \frac{G^0(\mathrm{j}\omega) + K_0^{-1}}{\alpha(\omega) + cK_0^{-1}} \right| > \gamma, \quad 0 \leqslant 0 < \infty \tag{16}$$

which leads to (11). The proof for multiplicative perturbations is similar, and therefore omitted. □

Theorem 2: *Consider the robust absolute stability problem associated with the closed-loop in Fig.* 1 *with stable-factor perturbations* (7)–(8). *Then, the system achieves robust absolute stability if the hodograph of the following function*

$$p_{\mathrm{sf}}(\mathrm{j}\omega, \gamma) = \frac{K_0 N_0(\mathrm{j}\omega) + D_0(\mathrm{j}\omega)}{|D_0(\mathrm{j}\omega)|c + (K_0^2\tau_1^2(\omega) + (1 + c\gamma)^2\tau_2^2(\omega))^{1/2}} \tag{17}$$

rotates outside of the γ*-circle* (*circle of radius* γ *centred at the origin*) *for n quadrants in the counter-clockwise direction when* ω *increases from* 0 *to* ∞.

Figure 3 shows a plot of $p_{\mathrm{sf}}(\mathrm{j}\omega, \gamma)$ for the same system as in (13) and (15) but with stable-factor perturbations described by

$$\tau_1(\omega) = 0{\cdot}1(\omega^2 + 0{\cdot}2)^{1/2} \tag{18}$$

and

$$\tau_2(\omega) = 0{\cdot}1(\omega^2 + 0{\cdot}5)^{1/2} \tag{19}$$

The plot is drawn for $\gamma = 4{\cdot}4$. It is seen from Fig. 3 that robust absolute stability of the closed-loop system is guaranteed for $\gamma < 4{\cdot}4$.

Proof of Theorem 2: The encirclement requirement is for the stability of the nominal system, see Lemma 2. The fact that $|p_{\mathrm{sf}}(\mathrm{j}\omega)| > \gamma$ guarantees the robust

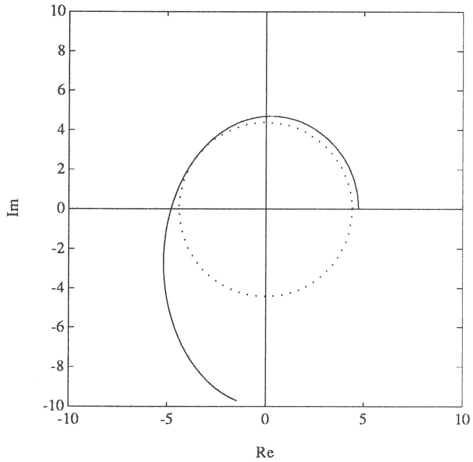

Figure 3. Modified Mikhailov plot for stable-factor perturbations and uncertain nonlinearity.

absolute stability also follows from the circle criterion and zero exclusion principle. In fact, supposing the nominal system is stable, then the application of the circle criterion and zero exclusion principle yields that robust absolute stability of the perturbed system is preserved if and only if the following condition

$$K_0^{-1}(1 + \gamma c\rho) + \frac{N_0(j\omega) + \Delta N(j\omega)}{D_0(j\omega) + \Delta D(j\omega)} \neq 0 \qquad (20)$$

holds for all complex numbers ρ with $|\rho| \leqslant 1$ and $\Delta N(s)$ and $\Delta D(s)$ satisfying (8) at all $\omega \geqslant 0$. Equation (20) above is equivalent to

$$K_0 N_0(j\omega) + D_0(j\omega) + \gamma D_0(j\omega)c\rho + K_0 \Delta N(j\omega) + (1 + \gamma c\rho)\Delta D(j\omega) \neq 0 \qquad (21)$$

This condition is guaranteed by

$$|K_0 N_0(j\omega) + D_0(j\omega)| - \gamma|D_0(j\omega)|c > K_0|\Delta N(j\omega)| + |1 + \gamma c||\Delta D(j\omega)| \qquad (22)$$

Since the maximum of the right-hand side above over the constraint (8) is given by $\gamma[K_0^2\tau_1^2(\omega) + (1 + c\gamma)^2\tau_2^2(\omega)]^{1/2}$, the sufficient condition in Theorem 2 is derived. □

Remark 3: The function $p_{sf}(j\omega, \gamma)$ above depends on γ. This dependency vanishes when either $\tau_2(\omega)$ (i.e. $\Delta D(s)$) or the uncertain nonlinearity disappears. In either of these cases, the condition in Theorem 2 actually becomes necessary and sufficient. The case of $\tau_2(\omega) = 0$ is the same as additive perturbations, and the case without uncertain nonlinearity is shown below. □

Theorem 3: *Consider the robust stability problem associated with the closed-loop system in Fig. 1 with stable-factor perturbations in (7)–(8) and constant feedback gain K_0. Then, the system is robustly stable if and only if the hodograph of the following function*

$$p_{sf}(j\omega) = \frac{K_0 N_0(j\omega) + D_0(j\omega)}{[K_0^2\tau_1^2(\omega) + \tau_2^2(\omega)]^{1/2}} \qquad (23)$$

rotates outside the γ-circle for n quadrants in the counter-clockwise direction when ω increases from 0 to ∞, i.e.

$$\gamma_{max} = \min\{|p_{sf}(j\omega)|: 0 \leqslant \omega \leqslant \infty\} \qquad (24)$$

The proof of this result is almost identical to Theorem 2, and is therefore omitted.

5. Conclusions

Modified Mikhailov plots have been developed for testing robust absolute stabililty for systems with both non-parametric perturbations and uncertain nonlinearity. These plots are also used to compute a lower bound for the robust absolute stability margin γ_{max}. It should be noted that the exact margin γ_{max} is difficult to compute because the frequency domain approach only yields sufficient conditions for robust stability (see Narendra and Taylor 1973, for example). Furthermore, the lower bound for γ_{max} for stable-factored perturbations has extra sufficiency due to the overbounding of the crossing effect of non-parametric perturbations and uncertain nonlinearity; see the proof of Theorem 2. Nevertheless, the lower bound for γ_{max} in all cases becomes the exact margin when the uncertain nonlinearity diminishes.

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