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Finite test of robust strict positive realness

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The paper is concerned with the problem of testing the robust strict positive realness (SPRness) of a family of rational functions with both the numerator and the denominator dependent on the same set of parameters. We show that this problem can be solved by using a series of Routh tables. In other words, the robust SPRness of the whole family can be tested by performing only a finite number of elementary operations (arithmetic operations, logical operations and sign tests).

1. Introduction

It is well known that the problem of strict positive realness (SPRness) plays an important role in many system analysis and design problems. Examples range from absolute stability analysis for linear systems with uncertain/nonlinear perturbations in Siljak (1969), to the convergence study of adaptive controllers in Goodwin and Sin (1984). There are already several papers on the SPRness of rational functions with uncertain parameters. A family of rational functions called interval plant was considered in Dasgupta (1987) and it is shown that the robust SPR of this family of rational functions is equivalent to the SPR of 16 special members. This result is extended by Vicino and Tesi (1991) to the SPR problem of a real shifted family of interval transfer functions. Chapellat *et al.* (1989) strengthened the condition of Dasgupta (1987) such that the number of functions needing to be checked is reduced to eight. Dasgupta *et al.* (1991) considered a family of rational functions with the denominator and the numerator multilinearly or linearly dependent on two independent sets of parameters. It is shown that the whole family is robustly SPR if and only if the rational functions associated with the extreme values of the parameters are SPR. A more general case is considered by Fu (1992), where the transfer function is allowed to have both independent multilinear parameters and dependent linear parameters in numerator and denominator. It is shown that only certain extreme points and edges of the parameter set need to be tested for the SPRness of the whole family. However, the remaining problem is to find an efficient method for checking the SPRness of a family of rational functions involving a single uncertain parameter.

This paper considers the same problem as did Fu (1992) as mentioned above. Following some recent papers on finite decidability of stability and other related problems, we show that the SPRness of the whole family can be determined by using a series of Routh tables which involve only a finite number of elementary operations (i.e. arithmetic operations, logical operations and sign tests). The structure of this paper is as follows: in §2 we formulate the robust SPR problem and recapture the main result of Fu (1992) and results for testing the positivity of polynomials. The main result is given in §3 and its computational aspects are discussed in §4.

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2. Problem formulation and preliminaries

Definition (Narendra and Taylor 1973): A rational function

$$G(s) = \frac{N(s)}{D(s)} \tag{1}$$

is called *positive real* (PR) if: $G(s)$ is real for real s, and $\operatorname{Re} G(s) \geqslant 0$, $\forall \operatorname{Re}[s] > 0$.

Suppose that $G(s)$ is not identically zero. Then $G(s)$ is called strictly positive real (SPR) if $G(s-\varepsilon)$ is PR for some $\varepsilon > 0$. Further, a family of rational functions is said to be robustly SPR if every member of the family is SPR.

Consider the following parametrized rational function:

$$G(s, \boldsymbol{q}_n, \boldsymbol{q}_d, \boldsymbol{q}_b) = \frac{N(s, \boldsymbol{q}_n, \boldsymbol{q}_b)}{D(s, \boldsymbol{q}_d, \boldsymbol{q}_b)}$$

$$= \frac{\sum_{i=0}^{m} b_i(\boldsymbol{q}_n, \boldsymbol{q}_b)\, s^i}{\sum_{i=0}^{n} a_i(\boldsymbol{q}_d, \boldsymbol{q}_b)\, s^i} = \frac{\sum_{i=0}^{m} [b_{i0} + b_{i1}(\boldsymbol{q}_n, \boldsymbol{q}_b)]\, s^i}{\sum_{i=0}^{n} [a_{i0} + a_{i1}(\boldsymbol{q}_d, \boldsymbol{q}_b)]\, s^i} \tag{2}$$

where a_{i0} and b_{i0} are the coefficients of the nominal parts of the denominator and numerator, respectively; $a_{i1}(\cdot)$ and $b_{i1}(\cdot)$ represent uncertainties in the coefficients.

$$\boldsymbol{q}_n \in Q_n \subset \mathbb{R}^{p_n}, \boldsymbol{q}_d \in Q_d \in \mathbb{R}^{p_d}, \boldsymbol{q}_b \in Q_b \subset \mathbb{R}^{p_b}; Q_n, Q_d, Q_b$$

are given bounding sets. It is assumed that:

(*a*) $a_{i1}(\boldsymbol{q}_d, \boldsymbol{q}_b)$ are multilinear functions of $\boldsymbol{q}_d$ and linear functions of $\boldsymbol{q}_b$
(*b*) $b_{i1}(\boldsymbol{q}_n, \boldsymbol{q}_b)$ are multilinear functions of $\boldsymbol{q}_n$ and linear functions of $\boldsymbol{q}_b$
(*c*) Q_n, Q_d and Q_b are hyperrectangles, all containing the origin
(*d*) the leading coefficients $a_{n0} + a_{n1}(\boldsymbol{q}_d, \boldsymbol{q}_b)$ and $b_{m0} + b_{m1}(\boldsymbol{q}_n, \boldsymbol{q}_b)$ do not vanish for any $\boldsymbol{q}_n \in Q_n, \boldsymbol{q}_d \in Q_d, \boldsymbol{q}_b \in Q_b$.

Below, let

$$\boldsymbol{q} = (\boldsymbol{q}_n, \boldsymbol{q}_d, \boldsymbol{q}_b), \quad Q = Q_n \oplus Q_d \oplus Q_b \tag{3}$$

and denote the set of vertices and the set of edges of a hyper-rectangle $\mathbf{H}$ by $V(\mathbf{H})$ and $E(\mathbf{H})$, respectively. We also define two subsets of Q as follows:

$$Q_{\text{edge}} = V(Q_n) \oplus V(Q_d) \oplus E(Q_b) \tag{4}$$

$$Q_v = V(Q_n) \oplus V(Q_d) \oplus V(Q_b) \tag{5}$$

and define the set of admissible rational functions as

$$\mathscr{G} := \{G(s, \boldsymbol{q}_n, \boldsymbol{q}_d, \boldsymbol{q}_b) : \boldsymbol{q}_n \in Q_n, \boldsymbol{q}_d \in Q_d, \boldsymbol{q}_b \in Q_b\} \tag{6}$$

or in short

$$\mathscr{G} := \{G(s, \boldsymbol{q}) : \boldsymbol{q} \in Q\} \tag{7}$$

Then we have the following result.

Lemma 1 (Fu 1992): *Given the family of rational functions* $\mathscr{G}$ *in* (6) *satisfying assumptions* (*a*)–(*d*), $\mathscr{G}$ *is robustly SPR if and only if* $G(s, \boldsymbol{q})$ *is SPR for every* $\boldsymbol{q} \in Q_{\text{edge}}$.

This result shows that it is sufficient to test a certain number of edges of Q. To check the SPRness of an individual rational function, we have the following result.

Lemma 2 (Ioannou and Tao 1987): *Assume that $G(s)$ is a real rational function, not identically zero for all s. Then $G(s)$ is SPR if and only if:*

(*a*) *$G(s)$ is analytic in $Re[s] \geqslant 0$, i.e. $D(s)$ is strictly Hurwitz*

(*b*) $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \in [0, \infty)$

(*c*) (i) $\lim_{|\omega|\to\infty} \mathrm{Re}[G(j\omega)] > 0$ *when* $r^* = 0$

(ii) $\lim_{|\omega|\to\infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0$ *when* $r^* = 1$, or

(iii) $\lim_{|\omega|\to\infty} \mathrm{Re}[G(j\omega)] > 0, \lim_{|\omega|\to\infty} \dfrac{G(j\omega)}{j\omega} > 0$ *when* $r^* = -1$,

where r^* is the relative degree of $G(s)$.

Remark 1: Condition (ci) above is actually implied by (b). We list it here simply for convenience.

In order to produce a finite test, we need to use the Cauchy index.

Definition 1 (Gantmacher 1960): The Cauchy index of a real rational function $R(x)$ in a real interval (a, b) is denoted by $I_a^b R(x)$ and defined by the difference between the number of jumps of $R(x)$ from $-\infty$ to $+\infty$ and that of jumps from $+\infty$ to $-\infty$, where a and b are real numbers or $\pm\infty$.

The Cauchy index can be computed by using Sturm's theorem (Gantmacher 1960) which involves constructing a Routh table (Gantmacher 1960). We will only be interested in $I_{-\infty}^{\infty} R(x)$ in this paper. We also point out that the test of $\mathrm{Re}[G(j\omega)] > 0$, $\forall \omega \in [0, \infty)$ can be converted into a Cauchy index problem, then be carried out by constructing a Routh table (Jury 1970).

3. Main results

First we need a robust version of Lemma 2. For the robust SPRness of a family of rational functions, we give the following result.

Theorem 1: *Consider the family of rational functions* (7); *then $\mathcal{G}$ is robustly SPR if and only if the following conditions hold*:

(*a*) *$D(s, \boldsymbol{q}^0)$ is strictly Hurwitz, for some $\boldsymbol{q}^0 \in Q_{\text{edge}}$*

(*b*) $\mathrm{Re}[N(j\omega, \boldsymbol{q})]\mathrm{Re}[D(j\omega, \boldsymbol{q})] + \mathrm{Im}[N(j\omega, \boldsymbol{q})]\mathrm{Im}[D(j\omega, \boldsymbol{q})] > 0, \quad \forall \omega \in [0, \infty), \quad \forall \boldsymbol{q} \in Q_{\text{edge}}$

(*c*) (i) $\lim_{|\omega|\to\infty} \mathrm{Re}[G(j\omega, \boldsymbol{q})] > 0, \forall \boldsymbol{q} \in Q_{\text{edge}}$ *if $r^* = 0$, or*

(ii) $\lim_{|\omega|\to\infty} \omega^2 \mathrm{Re}[G(j\omega, \boldsymbol{q})] > 0, \forall \boldsymbol{q} \in Q_{\text{edge}}$ *if $r^* = 1$, or*

(iii) $\lim_{|\omega|\to\infty} \mathrm{Re}[G(j\omega, \boldsymbol{q})] > 0$ *and* $\lim_{|\omega|\to\infty} \dfrac{G(j\omega, \boldsymbol{q})}{j\omega} > 0, \forall \boldsymbol{q} \in Q_{\text{edge}}$ *if $r^* = -1$*

where r^* is the relative degree of $G(j\omega, \boldsymbol{q})$.

Proof:

Necessity. This follows directly from lemma 2, by noting that

$$\mathrm{Re}\,[G(\mathrm{j}\omega, \boldsymbol{q})] = \frac{\mathrm{Re}\,[N(\mathrm{j}\omega, \boldsymbol{q})]\,\mathrm{Re}\,[D(\mathrm{j}\omega, \boldsymbol{q})] + \mathrm{Im}\,[N(\mathrm{j}\omega, \boldsymbol{q})]\,\mathrm{Im}\,[D(\mathrm{j}\omega, \boldsymbol{q})]}{(\mathrm{Re}\,[D(\mathrm{j}\omega, \boldsymbol{q})])^2 + (\mathrm{Im}\,[D(\mathrm{j}\omega, \boldsymbol{q})])^2} \tag{8}$$

Sufficiency. Assuming that conditions (*a*)–(*c*) hold, we need to show that $\mathcal{G}$ is robustly SPR. From Lemma 1, we need only to show that $G(s, \boldsymbol{q})$ is robustly SPR for every $\boldsymbol{q} \in Q_{\mathrm{edge}}$. Suppose that there exists some $\tilde{\boldsymbol{q}} \in Q_{\mathrm{edge}}$ such that $G(s, \tilde{\boldsymbol{q}})$ is not SPR. Then, from (8) and Lemma 2, the only possibility is that $G(s, \tilde{\boldsymbol{q}})$ is not analytic in $\mathrm{Re}\,[s] \geqslant 0$, i.e. $D(s, \tilde{\boldsymbol{q}})$ has roots in the closed right half plane (CRHP). Note that the set Q_{edge} is a connected set. Then from the continuity of the roots of $D(s, \boldsymbol{q})$ with respect to $\boldsymbol{q}$, there must be another $\hat{\boldsymbol{q}} \in Q_{\mathrm{edge}}$ and $\hat{\omega} \in \mathbb{R}$, such that $D(\mathrm{j}\hat{\omega}, \hat{\boldsymbol{q}}) = 0$, which means that

$$\mathrm{Re}\,[N(\mathrm{j}\hat{\omega}, \hat{\boldsymbol{q}})]\,\mathrm{Re}\,[D(\mathrm{j}\hat{\omega}, \hat{\boldsymbol{q}})] + \mathrm{Im}\,[N(\mathrm{j}\hat{\omega}, \hat{\boldsymbol{q}})]\,\mathrm{Im}\,[D(\mathrm{j}\hat{\omega}, \hat{\boldsymbol{q}})] = 0$$

Clearly this contradicts (*b*). So $D(s, \boldsymbol{q})$ is analytic in $\mathrm{Re}\,[s] \geqslant 0$, $\forall \boldsymbol{q} \in Q_{\mathrm{edge}}$.

Consequently, $G(s, \boldsymbol{q})$ is SPR for all $\boldsymbol{q} \in Q_{\mathrm{edge}}$ and, by Lemma 1, this is equivalent to that $G(s, \boldsymbol{q})$ is SPR for all $\boldsymbol{q} \in Q$. □

Now we concentrate on checking condition (*b*) in Theorem 1, i.e. whether the image of an edge stays in the open right half plane for all ω. For each edge in Q_{edge}, the corresponding transfer function (2) is typically expressed as (by fixing all the parameters in $\boldsymbol{q}_n$ and $\boldsymbol{q}_d$ to their vertices)

$$g(s, \lambda) := \frac{N(s, \lambda)}{D(s, \lambda)} = \frac{N_0(s) + \lambda N_1(s)}{D_0(s) + \lambda D_1(s)}$$
$$= \frac{\sum_{i=0}^{m} c_i s^i}{\sum_{i=0}^{n} d_i s^i} = \frac{\sum_{i=0}^{m} [c_{i0} + \lambda c_{i1}]\, s^i}{\sum_{i=0}^{n} [d_{i0} + \lambda d_{i1}]\, s^i}, \quad \underline{\lambda} \leqslant \lambda \leqslant \bar{\lambda} \tag{9}$$

where λ represent the free parameter in Q_b. From (9) it follows that

$$\mathrm{Re}\,[g(\mathrm{j}\omega, \lambda)] > 0$$

if and only if

$$P(\lambda, \omega) := \lambda^2\, \mathrm{Re}\,(N_1 D_1^*) + \lambda\, \mathrm{Re}\,(N_0 D_1^* + N_1 D_0^*) + \mathrm{Re}\,(N_0 D_0^*) > 0, \quad \forall \omega \in [0, \infty), \lambda \in [\underline{\lambda}, \bar{\lambda}] \tag{10}$$

Below, we will derive a finite algorithm for checking (10); see the Appendix for proof.

Lemma 3—Key lemma: *Denote*

$$P(\lambda, \omega) = a_1(\omega)\,\lambda^2 + a_2(\omega)\,\lambda + a_3(\omega) \tag{11}$$

where $a_i(\omega)$, $i = 1, 2, 3$, *are real polynomials in* ω. *Then* $P(\lambda, \omega) > 0$, $\forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$ *and* $\forall \omega \in [0, +\infty)$ *if and only if the following conditions hold*:

(*a*) $a_1(+\infty)\,\lambda^2 + a_2(+\infty)\,\lambda + a_3(+\infty) > 0, \quad \forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$

(*b*) $a_1(\omega)\,\bar{\lambda}^2 + a_2(\omega)\,\bar{\lambda} + a_3(\omega) > 0$

$a_1(\omega)\,\underline{\lambda}^2 + a_2(\omega)\,\underline{\lambda} + a_3(\omega) > 0, \quad \forall \omega \in [0, +\infty)$

(*c*) $2a_3(\omega) + a_2(\omega)\,(\bar{\lambda} + \underline{\lambda}) + 2a_1(\omega)\,\underline{\lambda}\bar{\lambda} > 0$, *whenever* $a_2^2(\omega) - 4a_1(\omega)\,a_3(\omega) = 0$

From Lemma 3 and the following result, it will become clear that the SPRness of an edge can be tested with a finite number of elementary operations.

Lemma 4 (Fu 1993): *Given two real polynomials $r_1(\omega)$ and $r_2(\omega)$, the following two statements are equivalent:*

(*a*) $r_2(\omega) > 0$, *whenever* $r_1(\omega) = 0$, *for every* $\omega \in (-\infty, +\infty)$

(*b*) $I_{-\infty}^{+\infty} \dfrac{\dot{r}_1(\omega)}{r_1(\omega)} = I_{-\infty}^{+\infty} \dfrac{\dot{r}_1(\omega)\, r_2(\omega)}{r_1(\omega)}$

where $\dot{r}_1(\omega)$ is the derivative of $r_1(\omega)$ with respect to ω.

The application of Theorem 1, Lemma 3 and Lemma 4 leads us to the following two theorems.

Theorem 2: *Define*

$$f_1 = a_2^2 - 4a_1 a_3, \quad f_2 = 2a_3 + a_2(\bar{\lambda} + \underline{\lambda}) + 2a_1 \underline{\lambda}\bar{\lambda} \tag{12}$$

Then the transfer function in (9) *is SPR* $\forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$ *and* $\forall \omega \in \mathbb{R}$ *if and only if the following conditions hold:*

(*a*) $D_0(s)$ *is strictly Hurwitz*

(*b*) *condition* (*b*) *in Lemma* 3 *is satisfied*

(*c*) $I_{-\infty}^{+\infty} \dfrac{\dot{f}_1(\omega)}{f_1(\omega)} = I_{-\infty}^{+\infty} \dfrac{\dot{f}_1(\omega) f_2(\omega)}{f_1(\omega)}$

(*d*) (i) $\dfrac{c_m}{d_n} > 0$, *if* $r^* = 0$ (*i.e.* $m = n$)

(ii) $c_{n-1} d_{n-1} - c_{n-2} d_n > 0$, *if* $r^* = 1$ (*i.e.* $m = n-1$)

(iii) $\left.\begin{array}{l} c_n d_n - c_{n+1} d_{n-1} > 0 \\ \dfrac{c_{n+1}}{d_n} > 0 \end{array}\right\}$, *if* $r^* = -1$ (*i.e.* $m = n+1$)

Proof: It is obvious that condition (*a*) is the same in both Theorems 1 and 2. From the definition of $P(\lambda, \omega)$ in (10), it is clear that condition (*b*) in Theorem 1 holds if and only if conditions (*a*)–(*c*) in Lemma 3 do. Note that condition (*c*) in Lemma 3 is the same as condition (*c*) in Theorem 2 (due to Lemma 4). Also, condition (*a*) in lemma 3 is implied by condition (*c*) in Theorem 1. Therefore, it suffices to show that condition (*c*) in Theorem 1 is equivalent to condition (*d*) in Theorem 2. To this end, we obtain from (9) that

$$g(\mathrm{j}\omega, \lambda) = \frac{(-1)^n c_m d_n(\mathrm{j}\omega)^{m+n} + (-1)^{n-1} c_m d_{n-1}(\mathrm{j}\omega)^{m+n-1} + (-1)^n c_{m-1} d_n(\mathrm{j}\omega)^{m+n-1} + \dots}{(-1)^n d_n^2(\mathrm{j}\omega)^{2n} + \dots} \tag{13}$$

and analyse three cases:

(*a*) $r^* = 0$. In this case

$$\lim_{|\omega|\to\infty} \mathrm{Re}[g(\mathrm{j}\omega, \lambda)] = \frac{c_n}{d_n}$$

(*b*) $r^* = 1$. This implies that

$$\lim_{|\omega|\to\infty} \omega^2 \mathrm{Re}[g(\mathrm{j}\omega, \lambda)] = \frac{c_{n-1}d_{n-1} - c_{n-2}d_n}{d_n^2} > 0$$

(*c*) $r^* = -1$. This gives

$$\lim_{|\omega|\to\infty} \mathrm{Re}[g(\mathrm{j}\omega, \lambda)] = \frac{c_n d_n - c_{n+1}d_{n-1}}{d_n^2} > 0, \quad \lim_{|\omega|\to\infty} \frac{g(\mathrm{j}\omega, \lambda)}{\mathrm{j}\omega} = \frac{c_{n+1}}{d_n}$$

It is therefore clear that condition (*c*) in Theorem 1 is equivalent to condition (*d*) in Theorem 2. □

In view of Lemma 1 and Theorem 2, we have the following general result (the proof is omitted due to its triviality).

Theorem 3: *Similarly to that of* (9)–(12), *we define*

$$f_{1j} = a_{2j}^2 - 4a_{1j}a_{3j}, \quad f_{2j} = 2a_{3j} + a_{2j}(\bar{\lambda} + \underline{\lambda}) + 2a_{1j}\underline{\lambda}\bar{\lambda} \tag{14}$$

where j denotes the jth edge of Q_{edge}. *Then* $\mathscr{G}$ *in* (7) *is robustly SPR if and only if the following conditions hold*:

(*a*) $\sum_{i=0}^{n} a_i(\boldsymbol{q}^0)s^i$ *is strictly Hurwitz for some* $\boldsymbol{q}^0 \in V(Q_d) \oplus E(Q_b)$

(*b*) $\mathrm{Re}\, G(\mathrm{j}\omega, \boldsymbol{q}) > 0, \quad \forall \boldsymbol{q} \in Q_v, \quad \omega \in [0, +\infty)$

(*c*) $I_{-\infty}^{+\infty} \dfrac{\dot{f}_{1j}(\omega)}{f_{1j}(\omega)} = I_{-\infty}^{+\infty} \dfrac{\dot{f}_{1j}(\omega) f_{2j}(\omega)}{f_{1j}(\omega)}, j = 1, \ldots, n_e$

where $n_e = p_b 2^{(p_n + p_a + p_b - 1)}$ *is the number of edges in* Q_{edge}

(*d*) (i) $\dfrac{b_n(\boldsymbol{q})}{a_n(\boldsymbol{q})} > 0, \quad \forall \boldsymbol{q} \in Q_{\text{edge}}$, *if* $r^* = 0$, *or*

(ii) $b_{n-1}(\boldsymbol{q})a_{n-1}(\boldsymbol{q}) - b_{n-2}(\boldsymbol{q})a_n(\boldsymbol{q}) > 0, \quad \forall \boldsymbol{q} \in Q_{\text{edge}}$, *if* $r^* = 1$, *or*

(iii) $\left.\begin{array}{l} b_n(\boldsymbol{q})a_n(\boldsymbol{q}) - b_{n+1}(\boldsymbol{q})a_{n-1}(\boldsymbol{q}) > 0 \\ \dfrac{b_{n+1}(\boldsymbol{q})}{a_n(\boldsymbol{q})} > 0 \end{array}\right\}, \quad \forall \boldsymbol{q} \in Q_{\text{edge}}$, *if* $r^* = -1$.

4. Computational aspects

We now briefly examine the computational requirement of the algorithm in Theorem 3. Note that there are all together $2^{(p_n+p_a+p_b)}$ vertices and $p_b 2^{(p_n+p_a+p_b-1)}$ edges in $\mathscr{G}$. To check Theorem 3(*a*), one Routh table of degree n is needed. In Theorem 3(*b*) a Routh table of degree $2n$ is required for each vertex. As for Theorem 3(*c*), two Routh tables of degree $2n$ are needed for each edge, one for each Cauchy index. In addition, some simple computation is needed for Theorem 3(*d*), i.e. we have to determine whether, for each edge of Q_{edge}, a portion of a parabola stays positive.

Note that a Routh table of degree n requires $O(n^2)$ calculations; the complexity of the algorithm is $O(n^2)$ in terms of n.

5. Implementation and example

In this section we deal with the issue of implementation for our results. Firstly, we give an algorithmic form of Theorem 3.

Algorithm: Denote $n_e = 2^{(p_n+p_d+p_b)}$, $n_e = p_b 2^{(p_n+p_d+p_b-1)}$.

Step 1. Check if

$$\sum_{i=0}^{n} a_i(\boldsymbol{q}^0)\, s^i$$

is strictly Hurwitz, where $\boldsymbol{q}^0$ can be any convenient point in Q. If not, the family $\mathscr{G}$ in (7) is not robustly SPR.

Step 2. Construct a Routh table to check if $\operatorname{Re} G(j\omega, \boldsymbol{q}_i) > 0$, for all $\boldsymbol{q}_i \in Q_v$, $i = 1, 2, \ldots, n_v$. If not, $\mathscr{G}$ is not robustly SPR.

Step 3. For each edge, say the jth edge, of Q_{edge}, construct f_{1j} and f_{2j} and the corresponding Routh tables to check if the two Cauchy indices in condition (c) of Theorem 3 are the same, $i = 1, 2, \ldots, n_e$. If not, $\mathscr{G}$ is not robustly SPR.

Step 4. Find r^* and Check if condition (d) in Theorem 3 is true for all $\boldsymbol{q} \in Q_{\text{edge}}$. If all are true, then $\mathscr{G}$ is robust SPR; otherwise, $\mathscr{G}$ is not robust SPR.

To demonstrate the algorithm above, let us look at the following example, which is a typical one-port network.

Example: Assume that

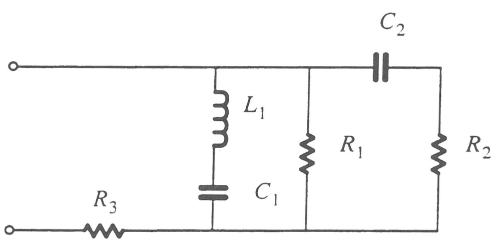
$$G(s, \boldsymbol{q}_n, \boldsymbol{q}_d, \boldsymbol{q}_b) = \frac{N(s, \boldsymbol{q}_n, \boldsymbol{q}_b)}{D(s, \boldsymbol{q}_b)}$$

represents the impedance of the circuit in the Figure, where $C_1 = 1$, $C_2 = \frac{1}{9}$, $R_1 = 3$, $R_2 = 6$, $L_1 = 0{\cdot}1q_2 + 1$ and $R_3 = 2q_1 + 8$ in per unit values, $[q_1, q_2] \in Q$, $Q = [0, 1] \times [0, 2]$ represent some uncertain parameters in R_3 and L_1. It is tedious but straightforward to check that

$$N(s, \boldsymbol{q}_n, \boldsymbol{q}_b) = (0{\cdot}1q_2 + 1)(2q_1 + 10)\, s^3 + ((0{\cdot}1q_2 + 1)(2q_1 + 11) + 4q_1 + 16)\, s^2 + (8q_1 + 34)\, s + 2q_1 + 11 \quad (15)$$

$$D(s, \boldsymbol{q}_b) = (0{\cdot}1q_2 + 1)\, s^3 + (0{\cdot}1q_2 + 3)\, s^2 + 4s + 1 \quad (16)$$

No $\boldsymbol{q}_d$ is involved. As can be seen, the numerator is multilinear in q_1 and q_2, and the denominator is linear in q_2.



A one-port network.

Now, we want to test the robust SPRness of (15) and (16), following the algorithm above.

Step 1. Taking $\boldsymbol{q}^0 = (0, 0)$, we have

$$\sum_{i=0}^{n} a_i(\boldsymbol{q}^0)\, s^i = s^3 + 3s^2 + 4s + 1$$

which is tested to be Hurwitz by a Routh table.

Step 2. There are four vertex points for (q_1, q_2), given by

$$(0,0), \quad (1,0), \quad (0,2), \quad (1,2) \tag{17}$$

Since

$$\begin{aligned} &\operatorname{Re} G(\mathrm{j}\omega, \boldsymbol{q}_i) > 0 \\ &\Leftrightarrow g(\omega^2, \boldsymbol{q}_i) := \operatorname{Re} D \operatorname{Re} N + \operatorname{Im} D \operatorname{Im} N > 0 \end{aligned} \tag{18}$$

it follows that $\operatorname{Re} G(\mathrm{j}\omega, \boldsymbol{q}_i) > 0$ if and only if $g(\omega^2, \boldsymbol{q}_i)$ has no real zeros. The number of positive real zeros π_i of $g(\omega^2, \boldsymbol{q}_i)$ can be found by

$$\pi_i = \tfrac{1}{2} I_{-\infty}^{+\infty} \frac{\omega g'(\omega^2, \boldsymbol{q}_i)}{g(\omega^2, \boldsymbol{q}_i)} \tag{19}$$

Rewriting

$$I_{-\infty}^{+\infty} \frac{\omega g'(\omega^2, \boldsymbol{q}_i)}{g(\omega^2, \boldsymbol{q}_i)} = I_{-\infty}^{+\infty} \frac{g_{20}\,\omega^{n-1} - g_{21}\,\omega^{n-3} + \cdots}{g_{10}\,\omega^n - g_{11}\,\omega^{n-2} + \cdots} \tag{20}$$

It is known that (Gantmacher 1960)

$$I_{-\infty}^{+\infty} \frac{g_{20}\,\omega^{n-1} - g_{21}\,\omega^{n-3} + \cdots}{g_{10}\,\omega^n - g_{11}\,\omega^{n-2} + \cdots} = n - 2\mathfrak{K} \tag{21}$$

where $\mathfrak{K}$ is the sign variations of the first column of the Routh table below:

$$\left.\begin{matrix} g_{10}, & g_{11}, & g_{12}, & \cdots \\ g_{20}, & g_{21}, & g_{22}, & \cdots \\ g_{30}, & g_{31}, & g_{32}, & \cdots \end{matrix}\right\} \tag{22}$$

where $g_{30} = (g_{20}\,g_{11} - g_{10}\,g_{21})/g_{20}, \quad g_{31} = (g_{20}\,g_{12} - g_{10}\,g_{22})/g_{20}, \ldots$

For $q_1 = (0, 0)$, we have

$$g(\omega^2, (0,0)) = 10\omega^6 + 7\omega^4 + 76\omega^2 + 11 \tag{23}$$

The Routh table (Table 1) shows that $\pi_1 = 6 - 2 \times 3 = 0$; i.e. $\operatorname{Re}(G(\mathrm{j}\omega, q_1)) > 0$. Similarly, the Routh tables show that $\operatorname{Re}(G(\mathrm{j}\omega, q_i)) > 0$ at all other vertices.

Step 3. There are two edges in Q_{edge}, being $q_1 = 0,\ q_2 \in [0, 2]$ and $q_1 = 1,\ q_2 \in [0, 2]$. Accordingly, we have

$$f_{11}(\omega^2) = \omega^6(4\omega^6 - 1{\cdot}6\omega^4 + 56{\cdot}4799\omega^2 - 52{\cdot}8) \tag{24}$$

$$f_{21}(\omega^2) = 20\omega^6 + 24{\cdot}7999\omega^4 + 147{\cdot}6\omega^2 + 22 \tag{25}$$

$$f_{12}(\omega^2) = \omega^6(5{\cdot}76\omega^6 - 1{\cdot}9199\omega^4 + 84{\cdot}96\omega^2 - 78) \tag{26}$$

$$f_{22}(\omega^2) = 24\omega^6 + 31{\cdot}2\omega^4 + 186{\cdot}8\omega^2 + 26 \tag{27}$$

10·0000	−7·0000	76·0000	−11·0000
30·0000	−14·0000	76·0000	0
−2·3333	50·6667	−11·0000	0
637·4286	−65·4286	0	0
50·4272	−11·0000	0	0
73·6178	0	0	0
−11·0000	0	0	0

Table 1. Routh table for the vertex $\boldsymbol{q}_1 = (0, 0)$.

A_1	B_1	A_2	B_2
4	4	5·7600	5·7600
12	−467·7760	17·2800	−891·5200
0·5333	−158·8726	0·6400	−225·0211
−844·0000	$-1{\cdot}8819 \times 10^4$	$-1{\cdot}5254 \times 10^3$	$-3{\cdot}5250 \times 10^4$
36·9383	57·3011	55·7921	86·2924
74·9022	$1{\cdot}2679 \times 10^4$	111·5985	$2{\cdot}3269 \times 10^4$
52·8000	52·800	78·0000	78·0000

Table 2. First columns of four Routh tables.

We only need consider the zeros of $f_{11}(\omega^2)/\omega^6$ and $f_{12}(\omega^2)/\omega^6$. Denote

$$A_j = I_{-\infty}^{+\infty} \frac{(f_{1j}(\omega^2)/\omega^6)'}{f_{1j}(\omega^2)/\omega^6}, \quad B_j = I_{-\infty}^{+\infty} \frac{(f_{1j}(\omega^2)/\omega^6)' f_{2j}(\omega^2)}{f_{1j}(\omega^2)/\omega^6} \tag{28}$$

we readily construct Routh tables with the first columns listed (Table 2). It is apparent that the numbers of sign variations are all the same for A_j and $B_j, j = 1, 2$. The result shows that all the edges are SPR.

Step 4. Clearly, $r^* = 0$, and condition (*di*) is involved, which does not need to be checked; see Remark 1.

6. Conclusion

The results above address the SPR problem of a family of rational functions. We have provided a finite algorithm which can test the robust SPRness by using only $O(n^2)$ elementary operations, where n is the degree of the rational functions. Thus, the commonly used value set approach, or frequency sweeping, is obviated. Although our results are for the Hurwitz stability region, an extension of the results can be obtained for more general stability regions, such as the unit circle or other circular regions which are of importance to filter designs (Tesi *et al.* 1993). This can be done by using the bilinear transformation which converts the circular regions to the open left plane. For more general stability regions, similar results can also be developed, provided that the region can be converted into the open left half plane by using the so-called strongly admissible rational function (see Sondergeld 1983 for a definition).

Appendix

Proof of Lemma 3: For each $\omega \in \mathbb{R}$, consider two cases according to the value of $a_1(\omega)$.

Case 1: $a_1(\omega) \neq 0$. In this case $P(\lambda, \omega)$ is part of a parabola with two end points $\mathscr{E} = a_1 \bar{\lambda}^2 + a_2 \bar{\lambda} + a_3$ and $\mathscr{J} = a_1 \underline{\lambda}^2 + a_2 \underline{\lambda} + a_3$. This parabola attains its minimum or maximum value at $\lambda^0 = -a_2/2a_1$ with the corresponding

$$P(\lambda^0, \omega) = \frac{4a_1 a_3 - a_2^2}{4a_1}$$

So, we need to show that, under conditions (a) and (b)

$$P(\lambda, \omega) > 0, \quad \forall \lambda \in [\underline{\lambda}, \bar{\lambda}] \text{ and } \forall \omega \in [0, +\infty)$$

if and only if (c) is true. For this purpose we consider two subcases.

Subcase 1: $a_1(\omega) < 0$. In this case, λ^0 corresponds to the maximum of $P(\lambda, \omega)$. Therefore

$$\min_{\lambda} P(\lambda, \omega) = \min(\mathscr{J}, \mathscr{E}) > 0$$

will guarantee that $4a_1 a_3 - a_2^2 \neq 0$. That is, the condition (c) is void.

Subcase 2: $a_1(\omega) > 0$. Now

$$\min_{\lambda \in \mathbb{R}} P(\lambda, \omega) = P(\lambda^0, \omega) = \frac{4a_1 a_3 - a_2^2}{4a_1}$$

From (a) we know that $P(\lambda, +\infty) > 0$, $\forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$. Since $P(\lambda, \omega)$ is a continuous function of λ and ω, if there were $\tilde{\lambda}$ & $\tilde{\omega}$ such that $P(\tilde{\lambda}, \tilde{\omega}) < 0$, there would exist an $\hat{\omega} \in [0, +\infty)$ such that $P(\lambda^0, \hat{\omega}) = 0$. So it is enough to check that

$$\lambda^0 \notin [\underline{\lambda}, \bar{\lambda}], \quad \text{whenever } P(\lambda^0, \omega) = \frac{4a_1 a_3 - a_2^2}{4a_1} = 0 \tag{A 1}$$

To check (A 1) we observe that

$$\begin{aligned} \lambda^0 \notin [\underline{\lambda}, \bar{\lambda}] &\Leftrightarrow (\lambda^0 - \bar{\lambda})(\lambda^0 - \underline{\lambda}) > 0 \\ &\Leftrightarrow a_2^2 + 2a_1 a_2(\underline{\lambda} + \bar{\lambda}) + 4a_1^2 \underline{\lambda}\bar{\lambda} > 0 \end{aligned} \tag{A 2}$$

which is equivalent to the condition (c) after substituting in $a_2^2 - 4a_1 a_3 = 0$.

Case 2: $a_1(\omega) = 0$. Obviously, (b) is both necessary and sufficient for

$$P(\lambda, \omega) > 0, \quad \forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$$

We need to confirm that (c) hold automatically. Indeed, if $a_2^2 - 4a_1 a_3 = 0$, then $a_2 = 0$. Consequently, $2a_3 + a_2(\bar{\lambda} + \underline{\lambda}) + 2a_1 \underline{\lambda}\bar{\lambda} = 2a_3 > 0$, due to (b). □

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