INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL, VOL. 2, 87–105 (1992)

H_∞ ESTIMATION FOR UNCERTAIN SYSTEMS

MINYUE FU, CARLOS E. DE SOUZA AND LIHUA XIE

Department of Electrical and Computer Engineering, University of Newcastle, NSW 2308, Australia

SUMMARY

This paper deals with the problem of H_∞ estimation for linear systems with a certain type of time-varying norm-bounded parameter uncertainty in both the state and output matrices. We address the problem of designing an *asymptotically stable estimator* that guarantees a prescribed level of H_∞ noise attenuation for all admissible parameter uncertainties. Both an *interpolation theory approach* and a *Riccati equation approach* are proposed to solve the estimation problem, with each method having its own advantages. The first approach seems more numerically attractive whilst the second one provides a simple structure for the estimator with its solution given in terms of two algebraic Riccati equations and a parameterization of a class of suitable H_∞ estimators. The Riccati equation approach also pinpoints the 'worst-case' uncertainty.

KEY WORDS H_∞ estimation Uncertain systems Interpolation theory Scaled H_∞ control Algebraic Riccati equations

1. INTRODUCTION

Over the past three decades considerable interest has been devoted to the problem of optimal filtering. Most previous work focuses on the minimization of the variance of the estimation error[1] under the assumption that the noise sources are white processes with known statistics or coloured noise with known colouring filter. In many practical cases, the application of such estimators may not be appropriate because it would require excessive prior knowledge of the nature of the noise. Moreover, those estimators will not be suitable for applications which require the estimation error spectrum to be made uniformly small. This has led several researchers to develop optimal estimation in the minimum H_∞-norm sense; see, for example, References 2, 7–10, 14, 17, 19 and 21.

In H_∞ estimation, the process and measurement noises are arbitrary signals with bounded energy. The estimator is designed to minimize the H_∞-norm of the power spectral density matrix of the estimation error, or to maintain the H_∞-norm within a prescribed bound. A polynomial approach was initially proposed to solve this problem[8–10] and recently a *Riccati equation approach* has been developed for the H_∞ estimation problem; see, for example, References 2, 14, 17, 19 and 21. Note that the latter approach dualizes recent results on state feedback H_∞ control.[4] Very recently, Reference 7 presented a solution to the H_∞ estimation problem via the interpolation theory. It was also shown in Reference 7 that this problem is essentially identical to the so-called *optimal loop transfer recovery* problem.[6]

A common feature of the works referred to above is that the uncertainty in the system model

This paper was recommended for publication by editor M. J. Grimble

1049–8923/92/020087–19$14.50

Received 22 June 1990
Revised 8 November 1991

is only in the form of an exogenous signal, and hence they cannot handle systems with parameter uncertainties. In this paper we consider linear systems subject to both time-varying parameter uncertainty and noise signal with bounded energy. The parameter uncertainty allowed is norm-bounded and appears in both the state and output matrices. The problem addressed is the *design of an estimator such that the estimation error dynamics is quadratically stable and the induced operator norm of the mapping from the noise signal to the estimation error is kept within a prescribed bound for all admissible parameter uncertainties.* The above problem is referred to as *robust H_∞ estimation.*

Both an *interpolation theory approach* and a *Riccati equation approach* are provided for solving the robust H_∞ estimation problem with each method having its own advantages. The Riccati equation approach provides a simple structure for the estimator, a parameterization of a class of robust H_∞ estimators and good theoretical insights to the problem. It also pinpoints the 'worst case' uncertainty. The interpolation approach, however, has the following features: (i) frequency weightings on estimation error and noise signal can be easily handled; (ii) unnecessary high estimator gain can be avoided, which is usually the case with the Riccati equation approach when the desired norm bound approaches its minimum.

A major implication of our results is that the structure of the estimator has to take into account the effect of the uncertainty. This explains why estimators for uncertain systems designed based on 'nominal' models often give conservative or restrictive results. Also, the solution given by the Riccati equation approach is in terms of *two* algebraic Riccati equations. Considering that state feedback H_∞ control for systems with the same kind of parameter uncertainty can be solved in terms of only one algebraic Riccati equation,[18] it seems that there is no *duality* for these two problems. It is observed that when there is no parameter uncertainty in the system, the H_∞ estimator proposed in this paper will recover well-known results on H_∞ estimation. Finally, we point out that our results have potential applications in the areas both of control engineering and of signal processing.

2. PROBLEM FORMULATION

Throughout this paper we consider linear uncertain systems modelled by differential equations of the form

$$(\Sigma_1):\ \dot{x}(t) = [A + \Delta A(t)]\,x(t) + Bw(t) \tag{1a}$$

$$y(t) = [C + \Delta C(t)]\,x(t) + Dw(t) \tag{1b}$$

$$z(t) = Lx(t) \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is a noise signal which belongs to $L_2[0, \infty)$, $y(t) \in \mathbb{R}^r$ is the measured output, $z(t) \in \mathbb{R}^p$ is a linear combination of the state variables to be estimated, A, B, C, D and L are known real constant matrices that describe the nominal system and $\Delta A(\cdot)$ and $\Delta C(\cdot)$ represent the time-varying parameter uncertainties. These uncertainties are in the following structure

$$\begin{bmatrix} \Delta A(t) \\ \Delta C(t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)E \tag{2}$$

with $F(\cdot)$: $\mathbb{R} \to \mathbb{R}^{i \times j}$ being an unknown matrix function with Lebesgue measurable elements and satisfying

$$F^{\mathrm{T}}(t)F(t) \leqslant I, \qquad \forall t \tag{3}$$

where H_1, H_2, and E are known real constant matrices with appropriate dimensions. In the above, the superscript 'T' denotes the transpose and the notation $X \geqslant Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). When the matrix L is the identity, the state $x(t)$ is to be estimated. The noise signal $w(t)$ is generated by the following system

$$w(s) = W_1(s)\bar{w}(s) \tag{4}$$

where $W_1(s)$ is a given $m \times m$ proper and invertible rational matrix with both $W_1(s)$ and $W_1^{-1}(s)$ being stable, and $\bar{w}(t)$ is the exogenous noise which belongs to $L_2[0, \infty)$. $W_1(s)$ will be called the weighting matrix for the disturbance input, $\bar{w}(t)$. In the above, $f(s)$ denotes the Laplace transform of a time function $f(t)$.

Let a linear estimator of $z(t)$ be of the form

$$(\Sigma_e)\colon\ \dot{x}_e(t) = A_e x_e(t) + K_e y(t), \qquad x_e(0) = 0 \tag{5a}$$

$$z_e(t) = L_e x_e(t) \tag{5b}$$

where $x_e(t) \in \mathbb{R}^l$ is the estimator state, the matrices A_e, K_e and L_e and the dimension l are to be chosen. The estimation error is denoted by

$$e(t) \triangleq z(t) - z_e(t) = Lx(t) - L_e x_e(t) \tag{6}$$

while the weighted estimation error is given by

$$\bar{e}(s) = W_2(s)e(s) \tag{7}$$

where $W_2(s)$ is a given $p \times p$ proper and invertible rational matrix with both $W_2(s)$ and $W_2^{-1}(s)$ being stable. Similarly to $W_1(s)$, $W_2(s)$ will be called the weighting matrix for the estimation error.

Before stating the problem of robust H_∞ estimation, we recall the notion of quadratic stability. Consider the following uncertain system, with $\Delta A(\cdot)$ being the parameter uncertainty:

$$\dot{x}(t) = [A(t) + \Delta A(t)]x(t) \tag{8}$$

Definition 2.1[11]

The system (8) is said to be quadratically stable if there exists a symmetric positive definite matrix P such that for all admissible uncertainty $\Delta A(\cdot)$, we have

$$[A + \Delta A(t)]^{\mathrm{T}} P + P[A + \Delta A(t)] < 0 \tag{9}$$

□

It can be easily seen that the quadratic stability of (8) implies its uniform asymptotic stability.

The problem of *robust H_∞ estimation* is stated as follows. *Given a prescribed level of disturbance attenuation $\gamma > 0$, find an estimator of the form* (5) *for $z(t)$ such that the weighted estimation error dynamics is quadratically stable and with zero initial conditions for $x(t)$ and $x_e(t)$, $\|\bar{e}\|_2 < \gamma \|\bar{w}\|_2$ for any non-zero $\bar{w} \in L_2[0, \infty)$ and for all admissible $F(t)$ satisfying* (3).

In the above $\|\cdot\|_2$ denotes the usual $L_2[0, \infty)$ norm. See Figure 1 for illustration.

Note that if the parameter uncertainty disappears, i.e. $H_1 = 0$, $H_2 = 0$ and $E = 0$, then the above estimation problem becomes the standard H_∞ estimation problem for the nominal system, which has been studied by a number of researchers; see, for example, References 7, 14, 17 and 21.

The above estimation problem is quite general and encompasses a number of typical filtering problems which arise in the areas of control engineering and signal processing. For example, consider the filtering problem with a signal generating process as shown in Figure 2.[10] The signals $v(t)$ and $n(t)$ are energy bounded noise sources and $s(t)$ is the signal to be estimated using the noisy measurement, $y(t)$. Both the signal generator and measurement subsystems are described by linear state-space models with the signal generator model being strictly proper and the measurement system is assumed square. Also, consider that either the signal generator or the measurement system is subject to time-varying norm-bounded parameter uncertainty that can be expressed in the form of (2). In this situation, it is easy to see that this filtering problem can be recast as a robust H_∞ estimation problem similar to the one this paper analyses.

In connection with the robust H_∞ estimation problem for the system (1), we shall introduce an auxiliary H_∞ estimation problem. We define the following parameterized system

$$(\Sigma_2):\ \dot{x}(t) = Ax(t) + \begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \tilde{w}(t) \tag{10a}$$

$$y(t) = Cx(t) + \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \tilde{w}(t) \tag{10b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\tilde{w}(t) \in \mathbb{R}^{m+i}$ is a noise signal which belongs to $L_2[0, \infty)$, $y(t) \in \mathbb{R}^r$ is the measured output, A, B, C, D, H_1 and H_2 are the same as in the system (1), $\varepsilon > 0$ is a scaling parameter to be chosen and $\gamma > 0$ is the disturbance attenuation level we wish to achieve for the robust H_∞ estimation problem. Associated with the system (10), we define a linear combination of the state variables of (Σ_2) to be estimated as given by

$$\tilde{z}(t) = \begin{bmatrix} \tilde{z}_1(t) \\ \tilde{z}_2(t) \end{bmatrix} = \begin{bmatrix} L \\ \varepsilon E \end{bmatrix} x(t) \tag{10c}$$

Also, let us introduce the following estimate of $\tilde{z}(t)$ obtained using the estimator (5),

$$\tilde{z}_e(t) = \begin{bmatrix} z_e(t) \\ 0 \end{bmatrix} = \begin{bmatrix} L_e \\ 0 \end{bmatrix} x_e(t) \tag{10d}$$

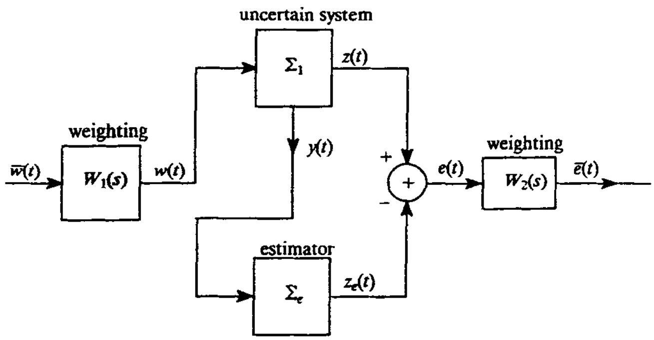

Figure 1. Block diagram of the robust estimation problem associated with system (1)

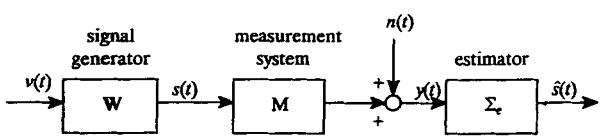

Figure 2. Signal generating process

Note that the lower entry of the matrix above is forced to be zero. (The reason for it will be clear shortly.) Hence, the corresponding estimation error reads

$$\tilde{e}(t) = \tilde{z}(t) - \tilde{z}_e(t) \tag{11}$$

$$= \begin{bmatrix} L & -L_e \\ \varepsilon E & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_e(t) \end{bmatrix} \in \mathbb{R}^{(p+j)} \tag{12}$$

We denote by $\tilde{T}(s)$ the transfer function from $\tilde{w}(t)$ to $\tilde{e}(t)$, i.e.

$$\tilde{e}(s) = \tilde{T}(s)\tilde{w}(s) \tag{13}$$

and define

$$\widetilde{W}_1(s) = \operatorname{diag}\{W_1(s),\ I_{i\times i}\} \tag{14a}$$

$$\widetilde{W}_2(s) = \operatorname{diag}\{W_2(s),\ I_{j\times j}\} \tag{14b}$$

where $I_{n\times n}$ denotes the $n \times n$ identity matrix. Moreover, we partition $\tilde{w}(t)$ accordingly to (10) as

$$\tilde{w}(t) = \begin{bmatrix} \tilde{w}_1(t) \\ \tilde{w}_2(t) \end{bmatrix} \tag{14c}$$

The following crucial theorem reveals a relationship between the robust H_∞ estimation problem associated with (1) and the H_∞ estimation problem associated with (10), estimation error (11) and weighting matrices (13).

Theorem 2.1

Given a prescribed level of disturbance attenuation $\gamma > 0$, suppose there exist some scaling parameter $\varepsilon > 0$ and an estimator in the form of (5) such that

$$\| \widetilde{W}_2(s)\tilde{T}(s)\widetilde{W}_1(s) \|_\infty < \gamma \tag{15}$$

Then, for the system (1) with the same estimator (5), the weighted estimation error dynamics $\bar{e}$ is quadratically stable and with zero initial conditions for $x(t)$ and $x_e(t)$, $\| \bar{e} \|_2 < \gamma \| \bar{w} \|_2$ for any non-zero $\bar{w} \in L_2[0, \infty)$ and all admissible $F(t)$ satisfying (3).

See Appendix A for proof and Figure 3 for illustration.

Remark 2.1. Although in the above theorem the estimator is assumed to be strictly proper, this does not cause any loss of generality. That is, if the robust H_∞ estimation problem is

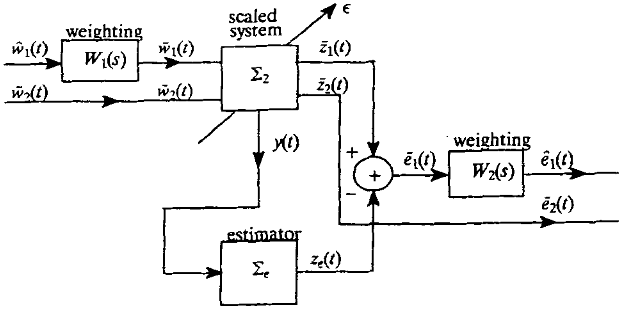

Figure 3. Block diagram of the scaled estimation problem associated with system (10)

solvable via a proper or even improper estimator, then it can be closely approximated via a strictly proper estimator. This can be achieved by cascading with the estimator a low-pass filter with sufficiently high cutoff frequency and relative degree. Hence, *the strict properness of the estimator is not essential for the result in Theorem 2.1.* This observation is important because the interpolation approach we use in Section 3 may lead to proper or even improper estimators.

Remark 2.2. Following Theorem 2.1, the robust H_∞ estimation problem for system (1) can be solved by finding a scaling parameter $\varepsilon > 0$ such that the H_∞ estimation problem for system (10) with estimation error (11) and weighting matrices (13) is solvable. Although the latter problem seems simpler, the zero entry in the matrix in (10d) makes it a non-standard one. That is, only very particular linear combinations of $x_e(t)$ can be used to 'match' $\tilde{z}(t)$. This restriction implies that the estimator has no influence on the second block of the estimation error, $\tilde{e}_2(t)$. Consequently, the choice of the matrices in (5) becomes non-trivial, as will be shown in Sections 3 and 4.

We end this section by introducing the following assumption which will be used to guarantee the stability of the estimation error dynamics:

Assumption A

The system matrix A is stable.

3. SOLUTION VIA INTERPOLATION APPROACH

In the section, we use the interpolation theory for solving the auxiliary H_∞ estimation problem associated with system (10), estimation error (11) and weighting matrices (13). A complete solution to this problem will be derived in the sequel.

Denote by $G_1(s)$ and $G_2(s)$ the transfer functions from $\tilde{w}(t)$ to $y(t)$ and to $\tilde{z}(t)$, respectively, i.e.

$$G_1(s) = \begin{bmatrix} D + C(sI - A)^{-1}B & \frac{\gamma}{\varepsilon}(H_2 + C(sI - A)^{-1}H_1) \end{bmatrix} \tag{16}$$

$$G_2(s) = \begin{bmatrix} L(sI - A)^{-1}B & \frac{\gamma}{\varepsilon} L(sI - A)^{-1}H_1 \\ \varepsilon E(sI - A)^{-1}B & \gamma E(sI - A)^{-1}H_1 \end{bmatrix} \tag{17}$$

Observe that $G_1(s)$ and $G_2(s)$ are parameterized by ε. Also, denote by $G_e(s)$ the transfer function of the estimator for z, i.e.

$$z_e(s) = G_e(s)y(s) \tag{18}$$

where $G_e(s)$ is allowed to be any $p \times r$ stable rational matrix either proper or improper. Note that concerning about the improperness of $G_e(s)$ is not necessary because, in view of Remark 2.1, $G_e(s)$ can always be approximated by a strictly proper estimator.

With the notation given above, the transfer function from $\tilde{w}(t)$ to $\tilde{e}(t)$ is given by

$$\tilde{T}(s) = G_2(s) - \begin{bmatrix} G_e(s) \\ 0 \end{bmatrix} G_1(s) \tag{19}$$

Define

$$T_1(s) = G_1(s)\tilde{W}_1(s) \tag{20a}$$

$$T_2(s) = \tilde{W}_2(s)\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{20b}$$

$$T_3(s) = \tilde{W}_2(s)G_2(s)\tilde{W}_1(s) \tag{20c}$$

Then, the H_∞ optimal estimation problem associated with the system (10) becomes the following: for any fixed $\varepsilon > 0$, which is the scaling parameter to be searched, compute

$$\gamma_\varepsilon = \min_{Q(s)} \| T_3(s) - T_2(s)Q(s)T_1(s) \|_\infty \tag{21}$$

subject to that $Q(s)$ is stable, and find all or some $Q(s)$ for which the minimum is achieved. Once $\gamma_\varepsilon < \gamma$ and $Q(s)$ are determined, we have

$$G_e(s) = [I \quad 0]\, Q(s) \tag{22}$$

To summarize, we provide the following theorem:

Theorem 3.1

Consider the uncertain system (1) satisfying Assumption A. Given $\gamma > 0$, the associated robust H_∞ estimation problem is solvable if for some $\varepsilon > 0$, the corresponding γ_ε given in (21) is less than γ. In this case, a suitable estimator is

$$z_e(s) = G_e(s)y(s)$$

where $G_e(s)$ is given by (22).

We now focus on the H_∞ optimization problem in (21). Note that this is a two-sided H_∞ optimization problem. Therefore, standard techniques such as the inner–outer factorization[5] and the interpolation theory[13] can be applied directly.

Owing to the special structure of $T_2(s)$, a little exercise can reduce the problem in (21) to a one-sided H_∞ optimization problem with lower dimensions. To achieve this, we first denote

$$T_3(s) = \begin{bmatrix} T_{3,1}(s) \\ T_{3,2}(s) \end{bmatrix} \tag{23}$$

where

$$T_{3,1}(s) = \left[W_2(s)L(sI - A)^{-1}BW_1(s) \qquad \frac{\gamma}{\varepsilon} W_2(s)L(sI - A)^{-1}H_1 \right] \tag{24a}$$

$$T_{3,2}(s) = [\varepsilon E(sI - A)^{-1}BW_1(s) \qquad \gamma E(sI - A)^{-1}H_1] \tag{24b}$$

Also note that both $T_{3,1}(s)$ and $T_{3,2}(s)$ depend on ε. Furthermore, $T_{3,1}(s)$ and $T_{3,2}(s)$ are stable. Now defining

$$T(s) = \tilde{W}_2(s)\tilde{T}(s)\tilde{W}_1(s) \tag{25}$$

we have

$$T(s) = \begin{bmatrix} T_{3,1}(s) - W_2(s)G_e(s)T_1(s) \\ T_{3,2}(s) \end{bmatrix} \tag{26}$$

In order to have $\| T(s) \|_\infty < \gamma$, it is necessary to have $\| T_{3,2}(s) \|_\infty < \gamma$. With this condition, $[I - \gamma^{-2} T_{3,2}^{\mathrm{T}}(-s) T_{3,2}(s)]^{-1}$ can be decomposed into the following co-spectral factorization form:[5]

$$[I - \gamma^{-2} T_{3,2}^{\mathrm{T}}(-s) T_{3,2}(s)]^{-1} = V(s) V^{\mathrm{T}}(-s) \tag{27}$$

where $V(s)$ is a $(m+i) \times (m+i)$ invertible rational matrix with both $V(s)$ and $V^{-1}(s)$ being stable. Therefore, $\| T(s) \|_\infty < \gamma$ if and only if

$$\begin{aligned} &\text{(i)} \quad \| T_{3,2}(s) \|_\infty < \gamma \qquad \text{and} \\ &\text{(ii)} \quad \| T_{3,1}(s) V(s) - W_2(s) G_e(s) T_1(s) V(s) \|_\infty < \gamma \end{aligned} \tag{28}$$

Remark 3.1. Note that it can be easily observed from (27) and (24b) that the zeros of $V(s)$ are the eigenvalues of A which are also poles of $T_{3,1}(s)$ and $T_1(s)$. Therefore, there are n pairs of stable zero-pole cancellation in $T_{3,1}(s)V(s) - W_2(s)G_e(s)T_1(s)V(s)$.

In the case when $W_1(s) = I$, a suitable co-spectral factor $V(s)$ is provided in the following lemma.

Lemma 3.1

Consider the transfer matrix $T_{3,2}(s)$ satisfying Assumption A and $\| T_{3,2}(s) \|_\infty < \gamma$. Then, there exists an invertible stable transfer matrix $V(s)$ with $V^{-1}(s)$ stable and having state-space realization

$$V(s) = \tilde{C}(sI - \tilde{A})^{-1} \tilde{B} + I \tag{29}$$

such that

$$[I - \gamma^{-2} T_{3,2}^{\mathrm{T}}(-s) T_{3,2}(s)]^{-1} = V(s) V^{\mathrm{T}}(-s) \tag{30}$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are given by

$$\tilde{B} = \gamma^{-1} [B \qquad \gamma \epsilon^{-1} H_1] \tag{31}$$

$$\tilde{A} = A + \tilde{B} \tilde{B}^{\mathrm{T}} P \tag{32}$$

$$\tilde{C} = \tilde{B}^{\mathrm{T}} P \tag{33}$$

and $P \geqslant 0$ is the stabilizing solution* to the following algebraic Riccati equation (ARE):

$$A^{\mathrm{T}} P + PA + P(\gamma^{-2} B B^{\mathrm{T}} + \varepsilon^{-2} H_1 H_1^{\mathrm{T}}) P + \varepsilon^2 E^{\mathrm{T}} E = 0 \tag{34}$$

Proof. See Appendix B.

Remark 3.2. Observe that the existence of the matrix P in Lemma 3.1 is guaranteed by the given assumptions. Indeed, it follows from well known H_∞ control results[4] that the existence of such matrix P is equivalent to having A stable and $\| T_{3,2}(s) \|_\infty < \gamma$.

---

* A solution $P = P^{\mathrm{T}}$ of the ARE, $A^{\mathrm{T}} P + PA + PMP + Q = 0$, where A, M and Q are $n \times n$ matrices with M and Q symmetric, is said to be a stabilizing solution if $A + MP$ is stable.

Now, defining

$$\hat{T}_3(s) = T_{3,1}(s)V(s) \tag{35a}$$

$$\hat{T}_1(s) = T_1(s)V(s) \tag{35b}$$

$$\hat{Q}(s) = W_2(s)G_e(s) \tag{35c}$$

the problem in (21) now becomes the following one-sided H_∞ optimization problem:

$$\gamma_\varepsilon = \min_{\hat{Q}(s)} \| \hat{T}_3(s) - \hat{Q}(s)\hat{T}_1(s) \|_\infty \tag{36}$$

subject to that $\hat{Q}(s)$ is stable. See Figure 4 for illustration. This result is summarized in the theorem below.

Theorem 3.2

Consider the uncertain system (1) satisfying Assumption A. Given $\gamma > 0$, the associated robust H_∞ estimation problem is solvable if for some $\varepsilon > 0$, the following conditions are satisfied:

(i) $\| T_{3,2}(s) \|_\infty < \gamma$;
(ii) the corresponding γ_ε given in (36) is less than γ.

If conditions (i) and (ii) hold, then a suitable estimator is given by

$$z_e(s) = G_e(s)y(s) \tag{37}$$

where

$$G_e(s) = W_2^{-1}(s)\hat{Q}(s) \tag{38}$$

and $\hat{Q}(s)$ is a minimizer for (36).

A complete solution to the H_∞ optimization problem in (36) can be found in Reference 7.

Remark 3.3. It should be noted that in view of Remark 3.1, with identity weighting matrices for the estimation error and the noise signal, i.e. $W_1(s) = I_{m\times m}$ and $W_2(s) = I_{p\times p}$, the transfer matrices $\hat{T}_1(s)$ and $\hat{T}_3(s)$ in the H_∞ optimization problem (36) are nth order stable rational matrices.

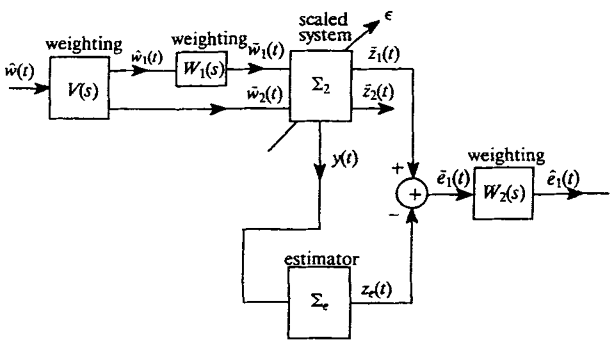

Figure 4. Block diagram of the scaled estimation problem associated with the interpolation approach

4. SOLUTION VIA RICCATI EQUATION APPROACH

In this section, we first show that the robust H_∞ estimation problem for the uncertain system (1) can be converted into a scaled H_∞ control problem. Thus, existing techniques on H_∞ control, including the J-factorization approach and the Riccati equation approach, can be applied to obtain a solution to the robust H_∞ estimation problem. To provide rich insights into the estimator structure and for the simplification of presentation, we choose to derive a solution via the well-known Riccati equation approach. Throughout this section we adopt the following assumptions:

Assumptions B

(i) $[D \quad H_2]$ is of full row rank;

(ii) $\operatorname{rank}\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = n + r, \qquad \forall \omega \geqslant 0$

Furthermore, we assume that the weighting matrices for the noise input and for the estimation error have been absorbed in the system description.

Assumption B(i) implies that all components of the measured output of the auxiliary system (10) are noisy. Note that if the parameter uncertainty in the output matrix disappears, i.e. $H_2 = 0$, Assumption B(i) reduces to $DD^{\mathrm{T}} > 0$, which is a standard assumption in H_∞ estimation for the nominal system.

In connection with the robust H_∞ estimation problem for the system (1), we introduce the following parameterized linear time-invariant system:

$$(\Sigma_3){:}\ \dot{x}(t) = Ax(t) + \begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} w_c(t) \tag{39a}$$

$$z_c(t) = \begin{bmatrix} L \\ \varepsilon E \end{bmatrix} x(t) + \begin{bmatrix} -I \\ 0 \end{bmatrix} u_c(t) \tag{39b}$$

$$y_c(t) = Cx(t) + \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} w_c(t) \tag{39c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_c(t) \in \mathbb{R}^p$ is the control input, $w_c(t) \in \mathbb{R}^{m+i}$ is the disturbance input which belongs to $L_2[0, \infty)$, $z_c(t) \in \mathbb{R}^{p+j}$ is the controlled output, $y_c(t) \in \mathbb{R}^r$ is the measured output, A, B, C, D, H_1, H_2, E and L are as in (1), $\varepsilon > 0$ is a scaling parameter to be chosen and $\gamma > 0$ is the desired level of noise attenuation for the robust H_∞ estimation problem.

Before presenting the main result of this section, we first recall the notion of stability with disturbance attenuation. Consider the following linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{40a}$$

$$z(t) = Cx(t) \tag{40b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the input, and $z(t) \in \mathbb{R}^p$ is the output.

Definition 4.1 [15]

Given a constant $\gamma > 0$, the system (40) is said to be stable with disturbance attenuation γ if A is stable and $\| C(sI - A)^{-1}B \|_\infty < \gamma$.

The next theorem reveals the relationship among the robust H_∞ estimation problem for the system (1), the scaled H_∞ estimation problem associated with the system (10) and estimation error (11), and the scaled H_∞ control problem for the system (39).

Theorem 4.1

Consider the system (1) satisfying Assumptions A and B. Let $\gamma > 0$ be a prescribed level of noise attenuation for the associated robust H_∞ estimation problem and $G_e(s)$ denote a given linear estimator. Then, $G_e(s)$ solves the scaled H_∞ estimation problem associated with the system (10) and estimation error (11) for some $\varepsilon > 0$ if and only if for the same $\varepsilon > 0$, the closed-loop system of (39) with the feedback control law $u_c = G_e(s)y_c$ is stable with disturbance attenuation γ. In this case, $G_e(s)$ is also a suitable estimator for the robust H_∞ estimation problem for the system (1).

Proof: See Appendix C.

The result of Theorem 4.1 shows that the problem of robust H_∞ estimation problem can now be solved by existing techniques of H_∞ control. It is our further interest to characterize the robust H_∞ estimators and to analyse the effect of the parameter uncertainty on the structure of the estimators.

First, note that from Assumptions A and B it follows that for any $\varepsilon > 0$,

$$\hat{R} = DD^{\mathrm{T}} + \frac{\gamma^2}{\varepsilon^2} H_2 H_2^{\mathrm{T}} > 0$$

$$\operatorname{rank}\begin{bmatrix} A - \mathrm{j}\omega I & 0 \\ \begin{bmatrix} \varepsilon E \\ L \end{bmatrix} & \begin{bmatrix} 0 \\ -I \end{bmatrix} \end{bmatrix} = n + p, \qquad \forall \omega \geqslant 0$$

and

$$\operatorname{rank}\begin{bmatrix} A - \mathrm{j}\omega I & \begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \\ C & \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \end{bmatrix} = n + r, \qquad \forall \omega \geqslant 0$$

In view of Theorem 4.1, the robust H_∞ estimation problem can now be solved by using the results in Reference 16. A complete solution is provided below.

Theorem 4.2

Consider the system (1) satisfying Assumptions A and B. Given a prescribed level of noise attenuation $\gamma > 0$, the robust H_∞ estimation problem for the system (1) is solvable if for some $\varepsilon > 0$ the following conditions are satisfied:

(a) There exists a stabilizing solution $P = P^{\mathrm{T}} \geqslant 0$ to the ARE:

$$A^{\mathrm{T}}P + PA + P(\gamma^{-2}BB^{\mathrm{T}} + \varepsilon^{-2}H_1H_1^{\mathrm{T}})P + \varepsilon^2 E^{\mathrm{T}}E = 0 \tag{41}$$

(b) There exists a stabilizing solution $Q = Q^{\mathrm{T}} \geqslant 0$ to the ARE:

$$\bar{A}(\varepsilon)Q + Q\bar{A}^{\mathrm{T}}(\varepsilon) + Q[\gamma^{-2}(L^{\mathrm{T}}L + \varepsilon^2 E^{\mathrm{T}}E) - C^{\mathrm{T}}\hat{R}^{-1}C]Q + \bar{B}(\varepsilon)\bar{B}^{\mathrm{T}}(\varepsilon) = 0 \tag{42}$$

where

$$\bar{A}(\varepsilon) = A - \hat{B}\hat{D}^{\mathrm{T}}\hat{R}^{-1}C, \qquad \bar{B}(\varepsilon) = \hat{B}(I - \hat{D}^{\mathrm{T}}\hat{R}^{-1}\hat{D})^{1/2} \tag{43}$$

$$\hat{B} = \begin{bmatrix} B & \frac{\gamma}{\varepsilon}H_1 \end{bmatrix}, \qquad \hat{D} = \begin{bmatrix} D & \frac{\gamma}{\varepsilon}H_2 \end{bmatrix}, \qquad \hat{R} = \hat{D}\hat{D}^{\mathrm{T}} \tag{44}$$

(c) $I - \gamma^{-2}PQ > 0$ (45)

When conditions (a)–(c) are satisfied, we have that

(i) A suitable estimator is given by:

$$\dot{x}_e(t) = \hat{A}x_e(t) + K_e[y(t) - \hat{C}x_e(t)] \tag{46a}$$

$$z_e(t) = Lx_e(t) \tag{46b}$$

where

$$\hat{A} = A + \gamma^{-2}\hat{B}\hat{B}^{\mathrm{T}}P, \qquad \hat{C} = C + \gamma^{-2}\hat{D}\hat{B}^{\mathrm{T}}P \tag{47}$$

$$K_e = (I - \gamma^{-2}QP)^{-1}[Q\hat{C}^{\mathrm{T}} + \hat{B}\hat{D}^{\mathrm{T}}]\hat{R}^{-1} \tag{48}$$

(ii) A class of strictly proper linear dynamic estimators which solve the robust H_∞ estimation problem for the system (1) is given by the following parametric characterization:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + K_e v(t) + Mr(t) \tag{49a}$$

$$\hat{z}(t) = L\hat{x}(t) + r(t) \tag{49b}$$

$$v(t) = y(t) - \hat{C}\hat{x}(t) \tag{49c}$$

$$r = W(s)v \tag{49d}$$

where

$$M = -\gamma^{-2}(I - \gamma^{-2}QP)^{-1}QL^{\mathrm{T}} \tag{49e}$$

and $W(s)$ is any strictly proper and stable transfer function matrix satisfying $\| W(s) \|_\infty < \gamma$.

Remark 4.1. Note that for sufficiently small $\varepsilon > 0$, condition (a) in Theorem 4.2 is equivalent to the quadratic stability of the system (1). Indeed, the quadratic stability of (1) implies the existence of a symmetric positive-definite matrix P_1 such that

$$[A + H_1F(t)E]^{\mathrm{T}}P_1 + P_1[A + H_1F(t)E] < 0 \tag{50}$$

for all $F(t)$ satisfying (3). By Theorem 2.7 of Reference 11, (50) is equivalent to having A stable and $\| E(sI - A)^{-1}H_1 \|_\infty < 1$. Obviously, for sufficiently small ε,

$$\| \varepsilon E(sI - A)^{-1}[\gamma^{-1}B \; \epsilon^{-1}H_1] \|_\infty < 1$$

which is equivalent to condition (a) in Theorem 4.2.

An alternative explanation of the above remark is provided by the following monotonicity result for the ARE(41).

Lemma 4.1

Given a scalar $\gamma > 0$, if for $\varepsilon = \bar{\varepsilon} > 0$ (41) has a stabilizing solution $\bar{P} \geqslant 0$, then for any

$\varepsilon \in (0, \bar{\varepsilon}]$ (41) has a stabilizing solution P as well. Moreover,

$$0 \leqslant P \leqslant \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^2 \bar{P}$$

Proof. Since $\bar{P}$ is a stabilizing solution to the ARE

$$A^{\mathrm{T}}P + PA + P(\gamma^{-2}BB^{\mathrm{T}} + \bar{\varepsilon}^{-2}H_1H_1^{\mathrm{T}})P + \bar{\varepsilon}^2E^{\mathrm{T}}E = 0$$

it follows from Reference 3 that for any $0 < \rho \leqslant 1$ the ARE

$$A^{\mathrm{T}}P + PA + P(\rho\gamma^{-2}BB^{\mathrm{T}} + \bar{\varepsilon}^{-2}H_1H_1^{\mathrm{T}})P + \bar{\varepsilon}^2E^{\mathrm{T}}E = 0$$

has a stabilizing solution P_ρ and $0 \leqslant P_\rho \leqslant \bar{P}$. Now, letting $P = \rho P_\rho$ and $\varepsilon^2 = \rho\bar{\varepsilon}^2$, then P is a stabilizing solution to (41). Note that $\varepsilon \in (0, \bar{\varepsilon}]$ since $0 < \rho \leqslant 1$. Moreover, we have that

$$0 \leqslant P = \rho P_\rho \leqslant \rho\bar{P} = \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^2 \bar{P}$$

Remark 4.2. Note that the estimator (46) can be rewritten in the following form

$$\dot{x}_e(t) = (A + \Delta A_{\text{worst}})x_e(t) + Bw_{\text{worst}}(t) + K_e[y - (C + \Delta C_{\text{worst}})x_e(t) - Dw_{\text{worst}}(t)]$$

where

$$\Delta A_{\text{worst}} = \varepsilon^{-2}H_1H_1^{\mathrm{T}}P$$
$$\Delta C_{\text{worst}} = \varepsilon^{-2}H_2H_1^{\mathrm{T}}\mathrm{P}$$
$$w_{\text{worst}}(t) = \gamma^{-2}B^{\mathrm{T}}Px_e(t)$$

The above estimator can be interpreted as a modified Luenberger observer with ΔA_{worst} and ΔC_{worst} being interpreted as the worst-case parameter uncertainty in the state and output matrices, respectively, and $w_{\text{worst}}(t)$ is the estimated worst-case contribution of the noise. Observe that the estimator gain matrix, K_e, in (48) also depends on the structural matrices H_1, H_2 and E of the parameter uncertainty. When the parameter uncertainty disappears, then $P = 0$ and condition (a) in Theorem 4.2 will become superfluous. In this situation, the estimator (46) will reduce to well known results on H_∞ estimation; see, for example, References 14 and 21.

The results in this section can be easily extended to allow for unknown initial state in the robust H_∞ estimation problem by using the results of Reference 12. In this situation, the following performance measure as introduced in Reference 14 is used:

$$J_e = \sup_{0 \neq (x_0, w) \in \mathbb{R}^n \times L_2[0,\infty)} \frac{\| e \|_2^2}{\| w \|_2^2 + x_0^{\mathrm{T}}Rx_0}; \qquad R = R^{\mathrm{T}} > 0$$

with $x(0) = x_0$ being the unknown initial state of (1) and zero initial condition for the estimator is assumed. The weighting matrix R is a measure of the confidence in the *a priori* knowledge of the initial state. A 'large' value of R reflects that the initial condition is very certain to be very close to zero. Now, the problem of robust H_∞ estimation with unknown initial state is stated as follows:

Given $\gamma > 0$, find an estimator (Σ_e) such that:

(a) *the augmented system of* (Σ_1) and (Σ_e) is quadratically stable;
(b) $J_e < \gamma^2$ for all admissible $F(t)$ satisfying (3).

The above performance measure (b) can be interpreted as an 'L_2-induced gain' between (x_0, w) and e.

In view of results of Reference 12, it is relatively easy to extend Theorems 2.1 and 4.1 to the case of unknown initial state of (1). These results are combined in the next theorem (the proof is straightforward and therefore omitted).

Theorem 4.3

Consider the system (1) with unknown initial state and satisfying Assumptions A and B. Let $\gamma > 0$ be a prescribed level of noise attenuation for the problem of robust H_∞ estimation with unknown initial state. If for some $\varepsilon > 0$, there exists a linear, proper time-varyng feedback control law $u_c = G_e y_c$ for the system (39) such that the closed-loop system is stable and satisfies

$$\sup_{0 \neq (\hat{x}_0, w_c) \in \mathbb{R}^n \times L_2[0,\infty)} \frac{\| z_c \|_2^2}{\| w_c \|_2^2 + \hat{x}_0^T R \hat{x}_0} < \gamma^2$$

where $\hat{x}_0$ is the unknown initial state of the system (39) and zero initial condition for the controller is assumed, then the problem of robust H_∞ estimation with unknown initial state is solvable with the estimator $z_e = Gey$.

Remark 4.3. The result of Theorem 4.3 shows that a solution to the problem of the robust H_∞ estimation with unknown initial state can be obtained by solving a scaled output feedback H_∞ control problem with unknown initial state for system (39). Note that the latter problem can be solved by existing techniques of H_∞ control; see, for example, Reference 12.

5. A NUMERICAL EXAMPLE

Consider the linear uncertain system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 + 0\cdot 5f(t) & -2 + 0\cdot 4f(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{51a}$$

$$y(t) = [-1 \quad 1]x(t) + 0\cdot 1w(t) \tag{51b}$$

where $f(t)$ is an uncertain parameter satisfying $| f(t) | \leqslant 1$. The linear combination of the state variable to be estimated is

$$z(t) = [1 \quad 0]x(t) \tag{52}$$

and identity weighting matrices are assumed.

Note that the above system is of the form (1) with

$$H_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad H_2 = 0, \qquad E = [0\cdot 5 \quad 0\cdot 4].$$

We now apply the Riccati equation approach proposed in the previous section to solve the associated robust H_∞ estimation problem. For a prescribed level of noise attenuation $\gamma = 0\cdot 52$, a suitable scaling parameter is $\varepsilon = 0\cdot 6561$. An estimator obtained by the Riccati equation approach is given by the following state-space realization

$$A_e = \begin{bmatrix} -135\cdot4895 & 142\cdot8206 \\ -91\cdot7539 & 93\cdot7555 \end{bmatrix}$$

$$K_e = \begin{bmatrix} -138\cdot9606 \\ -93\cdot4961 \end{bmatrix}$$

$$L_e = [1 \quad 0]$$

Alternatively, the transfer function of the estimator is as follows:

$$G_e(s) = -5\cdot2056 \frac{(s + 2\cdot3377)}{(s + 15\cdot0396)(0\cdot0375s + 1)}$$

The above estimator can be used to estimate $z(t)$ for the system (51) and will guarantee that for any parameter uncertainty $f(t)$ such that $|f(t)| \leqslant 1$ and for any non-zero $w \in L_2[0, \infty)$, the estimation error dynamics is quadratically stable and $\| z - z_e \|_2 < 0\cdot52 \| w_2 \|_2$. Figure 5 shows the spectrum of the estimation error for a time-invariant uncertainty $f(t) \equiv 1$ obtained by using the robust H_∞ estimator $G_e(s)$ (the bold line) and that by using the optimal H_∞ estimator for the nominal system of (51) (the dotted line). It can be observed that the robust

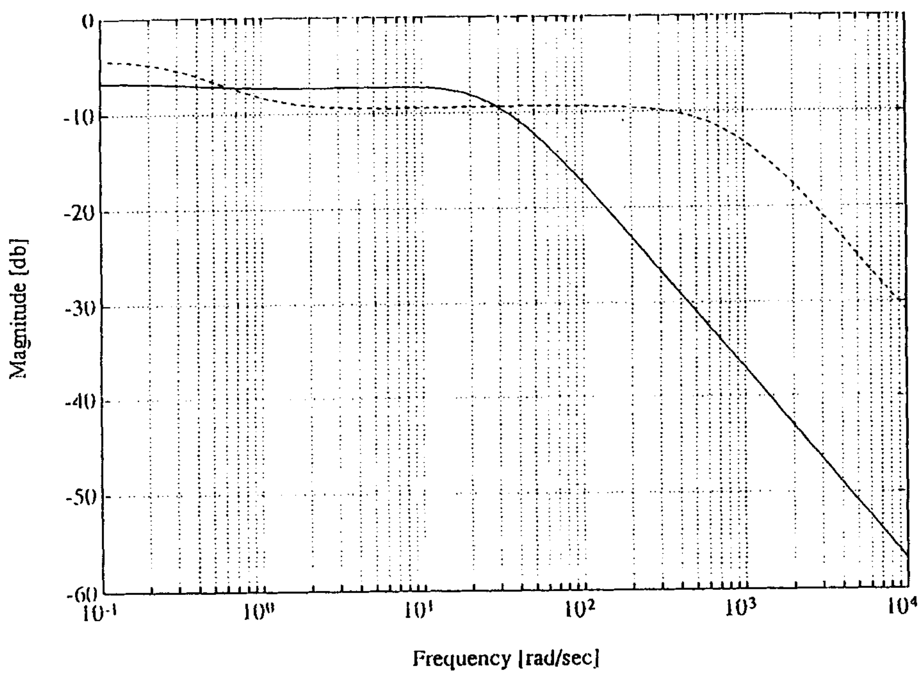

Figure 5. The estimation error spectra of the robust H_∞ estimator and the optimal H_∞ estimator based on the nominal system of (51) when $f(t) \equiv 1$

H_∞ estimator $G_e(s)$ achieves more preferable noise attenuation and performance than the standard optimal H_∞ estimator which is based on the nominal system.

6. CONCLUSION

This paper has analysed the H_∞ estimation problem for systems with parameter uncertainty. Two approaches have been proposed to solve this problem, one based on the interpolation theory and another on the Riccati equation technique. The first method seems more numerically attractive, especially when the minimum H_∞ norm of the estimation error dynamics needs to be achieved. The second approach, on the other hand, provides a simple structure for the estimator and an interpretation of the 'worst-case' uncertainty.

When this paper was revised, we came across a recent paper by Yaesh and Shaked[22] which treats a similar robust H_∞ estimation problem and converts it into a scaled H_∞ control problem by using a linear quadratic game approach. This approach can be viewed as a dual version of ours.

ACKNOWLEDGEMENTS

This work is supported by the Australian Research Council and the University of Newcastle Research Management Committee, Australia.

APPENDIX A: PROOF OF THEOREM 2.1

To derive the transfer function $\tilde{W}_2(s)\tilde{T}(s)\tilde{W}_1(s)$, we assume the following state-space realizations for the weighting matrices:

$$W_i(s)=\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \qquad i=1,2 \tag{A.1}$$

where the matrix notation for a state-space realization of a transfer function is used, i.e.,

$$C(sI-A)^{-1}B+D \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Defining

$$\hat{e}_1 = W_2(s)\tilde{e}_1 \tag{A.2a}$$

$$\tilde{w}_1 = W_1(s)\hat{w}_1 \tag{A.2b}$$

it is straightforward to verify that the augmented system associated with (5), (10)–(13), (A.1) and (A.2) (see Figure 2) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix} = A_c\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix} + \begin{bmatrix} B_c & \frac{\gamma}{\varepsilon}H_c\end{bmatrix}\begin{bmatrix}\hat{w}_1\\ \hat{w}_2\end{bmatrix} \tag{A.3a}$$

$$\begin{bmatrix}\hat{e}_1\\ \tilde{e}_2\end{bmatrix} = \begin{bmatrix}C_c\\ \varepsilon E_c\end{bmatrix}\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix} \tag{A.3b}$$

where

$$A_c=\begin{bmatrix} A_1 & 0 & 0 & 0\\ BC_1 & A & 0 & 0\\ K_eDC_1 & K_eC & A_e & 0\\ 0 & B_2L & -B_2L_e & A_2\end{bmatrix}$$

$$B_c=\begin{bmatrix} B_1\\ BD_1\\ K_eDD_1\\ 0\end{bmatrix} \tag{A.3d}$$

$$H_c=\begin{bmatrix} 0\\ H_1\\ K_eH_2\\ 0\end{bmatrix} \tag{A.3e}$$

$$C_c=[0\ \ D_2L\ \ -D_2L_e\ \ C_2] \tag{A.3f}$$

$$E_c=[0\ \ E\ \ 0\ \ 0] \tag{A.3g}$$

and ξ_1 and ξ_2 are states of (A.1) for $i=1, 2$, respectively. Furthermore, the transfer function from

$$\begin{bmatrix}\hat{w}_1\\ \tilde{w}_2\end{bmatrix} \text{ to } \begin{bmatrix}\bar{e}_1\\ \bar{e}_2\end{bmatrix}$$

gives $\widetilde{W}_2(s)\tilde{T}(s)\widetilde{W}_1(s)$.

Suppose condition (14) holds for some $\varepsilon>0$ and the estimator (5). From Theorem 3.1 of Reference 15, there exists a positive-definite and symmetric matrix P such that

$$A_c^{\mathrm{T}}P+PA_c+\gamma^{-2}P\begin{bmatrix}B_c & \frac{\gamma}{\varepsilon}H_c\end{bmatrix}\begin{bmatrix}B_c & \frac{\gamma}{\varepsilon}H_c\end{bmatrix}^{\mathrm{T}}P+\begin{bmatrix}C_c\\ \varepsilon E_c\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}C_c\\ \varepsilon E_c\end{bmatrix}<0$$

i.e.

$$A_c^{\mathrm{T}}P+PA_c+\gamma^{-2}PB_cB_c^{\mathrm{T}}P+\frac{1}{\varepsilon^2}PH_cH_c^{\mathrm{T}}P+C_c^{\mathrm{T}}C_c+\varepsilon^2E_c^{\mathrm{T}}E_c<0 \tag{A.4}$$

Using Lemma 3.1 of Reference 20, (A.4) implies

$$[A_c+H_cF(t)E_c]^{\mathrm{T}}P+P[A_c+H_cF(t)E_c]+\gamma^{-2}PB_cB_c^{\mathrm{T}}P+C_c^{\mathrm{T}}C_c<0 \tag{A.5}$$

for all $F(t)\in\mathbb{R}^{i\times j}$ satisfying (3).

On the other hand, it is straightforward to show that the augmented system associated with (1), (5), (A.1) and (A.2) (see Figure 1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix}=[A_c+H_cF(t)E_c]\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix}+B_c\bar{w} \tag{A.6a}$$

$$\bar{e}=C_c\begin{bmatrix}\xi_1\\ x\\ x_e\\ \xi_2\end{bmatrix} \tag{A.6b}$$

Finally, using Lemma 2.2 of Reference 20, (A.5) implies that (A.6) is quadratically stable and $\|\bar{e}\|_2<\gamma\|\bar{w}\|_2$ for all non-zero $\bar{w}\in L_2[0,\infty)$. This completes the proof.

APPENDIX B: PROOF OF LEMMA 3.1

Initially note that the existence of a stabilizing solution $P=P^{\mathrm{T}}\geqslant 0$ to the ARE (34) follows immediately from well known results on H_∞ control[4] and the fact that A is stable and $\|T_{3,2}(s)\|_\infty<\gamma$. Moreover, from Reference 5 we have

$$[I-\gamma^{-2}T_{3,2}^{\mathrm{T}}(-s)T_{3,2}(s)]^{-1}=\left[\begin{array}{cc|c} A & \tilde{B}\tilde{B}^{\mathrm{T}} & \tilde{B} \\ -\varepsilon^2E^{\mathrm{T}}E & -A^{\mathrm{T}} & 0 \\ \hline 0 & \tilde{B}^{\mathrm{T}} & I \end{array}\right]$$

Consider the transformation matrix

$$U=\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

Using (31)–(34), it is easy to show that

$$U^{-1}\begin{bmatrix} A & \tilde{B}\tilde{B}^{\mathrm{T}} \\ -\varepsilon^2E^{\mathrm{T}}E & -A^{\mathrm{T}} \end{bmatrix}U=\begin{bmatrix} \tilde{A} & \tilde{B}\tilde{B}^{\mathrm{T}} \\ 0 & -\tilde{A}^{\mathrm{T}} \end{bmatrix}$$

$$U^{-1}\begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}=\begin{bmatrix} \tilde{B} \\ -\tilde{C}^{\mathrm{T}} \end{bmatrix}$$

and

$$[0 \quad \tilde{B}^{\mathrm{T}}]\,U=[\tilde{C} \quad \tilde{B}^{\mathrm{T}}]$$

Therefore,

$$[I-\gamma^{-2}T_{3,2}^{\mathrm{T}}(-s)T_{3,2}(s)]^{-1}=\left[\begin{array}{cc|c} \tilde{A} & \tilde{B}\tilde{B}^{\mathrm{T}} & \tilde{B} \\ 0 & -\tilde{A}^{\mathrm{T}} & -\tilde{C}^{\mathrm{T}} \\ \hline \tilde{C} & \tilde{B}^{\mathrm{T}} & I \end{array}\right]$$

which is the state-space realization of $V(s)V^{\mathrm{T}}(-s)$ given in (29) and (31)–(33). Note that P bei stabilizing solution of the ARE (34) guarantees the stability of $\tilde{A}$. Finally, it can be easily seen from that $V(s)$ is invertible and the poles of $V^{-1}(s)$ are the eigenvalues of A which are stable.

APPENDIX C: PROOF OF THEOREM 4.1

In view of Remark 2.1, without loss of generality let a state-space realization of estimator $G_e(s)$ l in (5). Then, the control law $u_c=G_e(s)y_c$ for (39) is given by

$$(\Sigma_c)\colon\ \dot{x}_c(t)=A_ex_c(t)+K_ey_c(t), \qquad x_c(0)=0 \qquad (($$

$$u_c(t)=L_ex_c(t) \qquad (($$

Now, letting $\eta=[x^{\mathrm{T}}\ x_c^{\mathrm{T}}]^{\mathrm{T}}$ the closed-loop system associated with (39) and (C.1) is of the form:

$$\dot{\eta}(t)=\tilde{A}_c\eta+\begin{bmatrix} \tilde{B}_c & \frac{\gamma}{\varepsilon}\tilde{H}_c \end{bmatrix}w_c \qquad (($$

$$z_c(t)=\begin{bmatrix} \tilde{L}_c \\ \varepsilon\tilde{E}_c \end{bmatrix}\eta \qquad (($$

where

$$\tilde{A}_c=\begin{bmatrix} A & 0 \\ K_eC & A_e \end{bmatrix}, \qquad \tilde{H}_c=\begin{bmatrix} H_1 \\ K_eH_2 \end{bmatrix}, \qquad \tilde{B}_c=\begin{bmatrix} B \\ K_eD \end{bmatrix}$$

$$\tilde{E}_c=[E \quad 0], \quad \tilde{L}_c=[L \quad -L_e]$$

On the other hand, with the assumption of identity weighting matrices for noise and estimation eı follows from the proof of Theorem 2.1 that the augmented system associated with (5), (10) and (also given by (C.2). Therefore, by considering Theorem 2.1 the desired result follows immediatel

REFERENCES

1. Anderson, B. D. O., and J. B. Moore, *Optimal Filtering*, Prentice Hall, Englewood Cliffs, NJ, 1979.

2. Bernstein, D. S., and W. M. Haddad, 'Steady-state Kalman filtering with an H_∞ error bound', *System & Control Lett.*, **12**, 9–16 (1989).
3. de Souza, C. E., and L. Xie, 'On the properties of indefinite algebraic Riccati equations', Tech. Report EE8952, Dept. Elec. & Comp. Eng., University of Newcastle, NSW, Australia, August 1989.
4. Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis, 'State space solutions to the standard H_2 and H_∞ control problems', *IEEE Trans. Automat. Control,* **39**, 831–847 (1989).
5. Francis, B. A., *A Course in H_∞ Control Theory*, Springer-Verlag, New York, 1987.
6. Fu, M., 'Exact, optimal and partial loop transfer recovery', *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, Hawaii, December 1990, pp. 1841–1846.
7. Fu, M., 'Interpolation approach to H_∞ estimation and its interconnection to loop transfer recovery', *Systems & Control Letters,* **17**, 29–36 (1991).
8. Grimble, M. J., 'H_∞ design of optimal linear filters', in *Linear Circuits, Systems and Signal Processing: Theory and Application*, C. I. Byrnes, C. F. Martin and R. E. Saeks (Eds), North-Holland, Amsterdam, 1988, pp. 533–540.
9. Grimble, M. J., and A. El Sayed, 'Solution of the H_∞ optimal linear filtering problem for discrete-time systems', *IEEE Trans. Acoust., Speech and Signal Processing,* **38**, 1092–1104 (1990).
10. Grimble, M. J., D. Ho, and A. El Sayed, 'H_∞ robust linear estimator', *Proc. IFAC Symp. Adaptive Syst. in Control and Signal Processing*, Glasgow, April 1989, pp. 159–164.
11. Khargonekar, P. P., I. R. Petersen, and K. Zhou, 'Robust stabilization of uncertain linear system: quadratic stabilizability and H_∞ control theory', *IEEE Trans. Automat. Control,* **35**, 356–361 (1990).
12. Khargonekar, P. P., K. M. Nagpal, and K. R. Poola, 'H_∞ control with transients', *SIAM J. Contr. and Optimiz.*, **29**, 1373–1393 (1991).
13. Limebeer, D. J. N., and B. D. O. Anderson, 'An interpolation theory approach to H_∞ controller degree bounds', *Linear Algebra and its Applications*, **98**, 347–386 (1988).
14. Nagpal, K. M., and P. P. Khargonekar, 'Filtering and smoothing in an H_∞ setting', *IEEE Trans. Automat. Control*, **36**, 152–166 (1991).
15. Petersen, I. R., 'Disturbance attenuation and H_∞ optimization: a design method based on the algebraic Riccati equation', *IEEE Trans. Automat. Control,* **32**, 427–429 (1987).
16. Safonov, M. G., D. J. N. Limebeer, and R. Y. Chiang, 'Simplifying the H_∞ theory via loop-shifting, matrix pencil and descriptor concepts', *Int. J. Control*, **50**, 2467–2488 (1989).
17. Shaked, U., 'H_∞-minimum error state estimation of linear stationary processes', *IEEE Trans. Automat. Control,* **35**, 554–558 (1990).
18. Xie, L., and C. E. de Souza, 'Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty', *IEEE Trans. Automat. Control*, 1992, to appear.
19. Xie, L. and C. E. de Souza, 'H_∞ state estimation for linear periodic systems', *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, Hawaii, December 1990, pp. 3188–3193.
20. Xie, L., M. Fu, and C. E. de Souza, 'H_∞ control and quadratic stabilization of systems with parameter uncertainty via output feedback', *IEEE Trans. Automat. Control,* 1992, to appear.
21. Yaesh, I., and U. Shaked, 'Game theory approach to optimal linear estimation in the minimum H_∞-norm sense', *Proc. 28th IEEE Conf. Decision and Control*, Tampa, FL, December 1989, pp. 415–420.
22. Yaesh, I., and U. Shaked, 'Optimal control and estimation of uncertain linear time-varying systems', preprint.