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ACHIEVING VECTOR RELATIVE DEGREE FOR NONLINEAR SYSTEMS WITH PARAMETRIC UNCERTAINTIES

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SUMMARY

Several algorithms for adaptive control, as well as for static state feedback decoupling, feedback linearization, or inversion of nonlinear multivariable systems require that the systems have full vector relative degree, in order to be applied. In this paper, we provide a parameter-independent method of achieving full vector relative degree for nonlinear multivariable systems which do not have it. We determine conditions under which a diagonal dynamic precompensation is sufficient to achieve vector relative degree for multivariable nonlinear systems, and describe a simple algorithm which determines such compensation.

KEY WORDS nonlinear systems; dynamic precompensation; vector relative degree

1. INTRODUCTION

There has been a considerable amount of work published in the area of adaptive control, feedback linearisation, stabilization, inversion and decoupling of multivariable nonlinear systems. A small sample of the literature in these areas is given by References 1–9, 12–17, and 19–22. The vector relative degree of a multivariable system serves as a generalization of relative degree for single-input/single-output systems (see Section 2 for definition). Just as is the case for linear multivariable systems, multivariable nonlinear systems which do not have vector relative degree require *dynamic* state feedback compensation in order to achieve it. The basic premise in the stabilization,[1] feedback linearization[2] and adaptive control literature cited[3,14–17,21] is that the nonlinear multivariable control systems in question already have vector relative degree, so that the techniques described therein could not be applied directly to systems without full vector relative degree.

The problem we study in this paper may be described as follows. Given a multivariable nonlinear system which does not have vector relative degree, construct a diagonal linear precompensator such that the resulting system will achieve vector relative degree. We seek diagonal linear dynamic precompensators in order to achieve vector relative degree because of their simplicity, and because exact knowledge of system parameters is then unnecessary.

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Originally, this work was motivated by the decoupling problem. Multivariable systems for which dynamic precompensation is required for decoupling, do not have full vector relative degree. When dynamic feedback is necessary for decoupling, the algorithms available in the literature* require exact knowledge of the system parameters in order to be implemented. The step of achieving full vector relative degree may be viewed as an intermediary step in the decoupling process. The lack of vector relative degree is, in fact, why dynamic (versus static) feedback is necessary in order to decouple.

Through this study, we provide a method of achieving vector relative degree using only linear diagonal dynamic precompensation. Our solution is both state and parameter independent, and depends on the differential structure of the nonlinear systems. This makes this work particularly applicable to implementing adaptive control algorithms for multivariable nonlinear systems which do not initially have vector relative degree, and whose parameters are unknown.

In what follows, we provide a *necessary and sufficient* condition for the existence of diagonal precompensation which achieves vector relative degree. Based on this, a simple algorithm for finding the diagonal precompensator is given. These results are an extension of those in References 11 and 18 for linear systems. We show that the solution to this problem is essentially associated with the nonsingularity or (non)generic singularity of certain matrix related to the system. Furthermore, our algorithm formulates a 'minimum' number of integrators required for the inputs, thus avoiding 'over-compensation'.

The paper is organized as follows. Section 2 deals with some preliminaries relevant to vector relative degree. The main result on dynamic compensation, an algorithm and some illustrating examples are presented in Section 3, and we conclude with some observations in Section 4.

2. DEFINITIONS AND PRELIMINARIES

In this paper, we consider the class of multivariable nonlinear systems modelled by

$$T\colon \begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) + k(x(t))u(t) \end{aligned} \tag{1}$$

where the state $x(t) \in M$, with M being an n-dimensional submanifold of $\mathbb{R}^q$, $q \geqslant n$, which we will assume can be treated as a connected open subset of $\mathbb{R}^n$; the control $u(t) \in \mathbb{R}^m$, the output $y(t) \in \mathbb{R}^m$, $f(\cdot)$ and $g_1(\cdot), \dots, g_m(\cdot)$ are smooth vector fields on M (where $g_i(\cdot)$ forms the ith column of $g(\cdot)$), and $h(\cdot)$ and $k(\cdot)$, are smooth (matrix) functions of appropriate dimensions. The matrix $k(x)$ is called the direct transmission matrix. In the sequel, we will denote the system above by $y = T(u)$ or simply T, the components of the local coordinate descriptions of f and h by f_i and h_i respectively, and the differential operator $\mathrm{d}/\mathrm{d}t$ by ρ.

Definition 2.1.

Given $x_0 \in M$, the system T in (1) is called *bicausal* at x_0 if there exists a neighbourhood N of x_0 in M such that $k(x)$ is nonsingular for all $x \in N$. The system T is called *bicausal* on M if it is bicausal at every point $x \in M$.

Remark 1. If the system T in (1) is bicausal (at a point or on M), then its inverse system

*See, for example, References 8, 13, 4, 12, and 19, on dynamic decoupling of nonlinear systems.

exists and can be expressed by

$$
\begin{aligned}
T_1: \dot{\hat{x}}(t) &= f(\hat{x}) - g(\hat{x})k^{-1}(\hat{x})h(\hat{x}) + g(\hat{x})k^{-1}(\hat{x})y(t) \\
u(t) &= -k^{-1}(\hat{x})h(\hat{x}) + k^{-1}(\hat{x})y(t)
\end{aligned}
\tag{2}
$$

where $\hat{x}$ is the state of the inverse system. Obviously, the inverse system is also bicausal.

The following definition, essentially duplicated from Reference 13, provides the concept of relative degree for a class of multivariable nonlinear systems. In the definition, $L_f\lambda(x)$ denotes the Lie derivative of a function $\lambda(x)$ in the direction of the vector field given in local coordinates by $f(x)$, i.e.,

$$
L_f\lambda(x) = \sum_{i=1}^{n} \frac{\partial\lambda}{\partial x_i} f_i(x) = \frac{\partial\lambda}{\partial x} f(x) \tag{3}
$$

Similarly,

$$
L_gL_f\lambda(x) = \frac{\partial L_f\lambda}{\partial x} g(x) \tag{4}
$$

and

$$
L_f^q\lambda(x) = \frac{\partial L_f^{q-1}\lambda}{\partial x} f(x) \tag{5}
$$

with $L_f^q\lambda(x)$ denoting the q'th derivative of $\lambda(x)$ along f, and $L_f^0\lambda(x) = \lambda(x)$.

Definition 2.2[13]

Given a neighbourhood, N, of $x_0 \in M$, $N \subset M$, the nonlinear system T in (1) with $k(x) \equiv 0$ has vector relative degree $\{r_1, r_2 \ldots, r_m\}$, $r_i \geqslant 1$, $i = 1, \ldots, m$, at x_0, if the following conditions are satisfied:

(a)
$$
L_{g_j}L_f^q h_i(x) \equiv 0 \tag{6}
$$

for all $1 \leqslant i, j \leqslant m$, $0 \leqslant q < r_i - 1$ and all $x \in N$.

(b) the $m \times m$ matrix

$$
A(x) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \ldots & L_{g_m}L_f^{r_1-1}h_1(x) \\ \cdots & \cdots & \cdots \\ L_{g_1}L_f^{r_m-1}h_m(x) & \ldots & L_{g_m}L_f^{r_m-1}h_m(x) \end{bmatrix} \tag{7}
$$

is nonsingular for all $x \in N$.

Moreover, T has (invariant) vector relative degree $\{r_1, r_2, \ldots, r_m\}$ on M if it has the same vector relative degree at all points in M.

Remark 2. When the system (1) has vector relative degree $\{r_1, r_2, \ldots, r_m\}$, we have the following input–output relationship:

$$
z(t) = b(x) + A(x)u(t) \tag{8}
$$

where

$$
z(t) = R(\rho)y(t) \tag{9}
$$

with

$$R(\rho) = \operatorname{diag}\{\rho^{r_1}, \ldots, \rho^{r_m}\}, \qquad \rho = \frac{\mathrm{d}}{\mathrm{d}t} \tag{10}$$

and

$$b(x) = [L_g^{r_1} h_1(x) \;\; L_g^{r_2} h_2(x) \cdots L_g^{r_m} h_m(x)]^{\mathrm{T}} \tag{11}$$

Remark 3. By construction, this definition applies only to systems of the form, (1), for which $k(x) \equiv 0$, (note, $r_i \geqslant 1$). When $k(x)$ is not zero, the system may still have vector relative degree if there exists a diagonal operator of the form, (10), with $r_i \geqslant 0$, such that when applied to $y(t)$, as in (9), $z(t)$ satisfies an output equation such as (8), with nonsingular $A(x)$, as in (7), but with the ith row replaced by the ith row of $k(x)$ whenever $r_i = 0$.

Remark 4. The existence of vector relative degree can be interpreted as that the resulting system:

$$\begin{aligned} T_z\colon \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ z(t) &= b(x(t)) + A(x(t))u(t) \end{aligned} \tag{12}$$

is bicausal, i.e., the $A(x(t))$ matrix is nonsingular.[13] Thus, systems with full vector relative degree can be decoupled by static state feedback of the form $u(t) = c(x) + d(x)v(t)$, which is applied after applying a state transformation on x, which brings it to a (partially) linearizable Brunovsky canonical form.[13] Note that only a limited class of multivariable systems have vector relative degree, even in the linear case. In fact, in the linear case, the vector relative degree corresponds to a diagonal interactor matrix.[11] The interactor matrix, which has been used for study of linear systems for a number of years,[10] can also be defined for nonlinear systems to represent the concept of relative degree. It is, however, sufficient to use the vector relative degree for the results of this paper.

We now give a definition which is a prerequisite to the results established in Section 3. It concerns the concepts of generic and nongeneric singularity of transfer matrices as discussed by Singh and Narendra;[18] however, as we find the discussion and definition therein to be somewhat unclear, we outline these concepts from a different and more precise viewpoint.

Definition 2.3

A singular square matrix is called *generically* singular if by eliminating a certain number (or none) of its rows and all the zero columns in the remaining rows, we either vacuum the matrix or form a tall* submatrix. A singular square matrix is called *nongenerically* singular if it is not generically singular.

Remark 5. A result of the nongeneric (resp. generic) character of a singular matrix is that if the nonzero constants in the nongenerically (resp. generically) dependent rows or columns of a singular constant matrix are perturbed slightly, the matrix can no longer (resp. will still) be singular.

Remark 6. For illustration, let us consider the following two singular matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{13}$$

* A tall matrix has more rows than columns.

The first matrix is generically singular because by eliminating the third row the second and third columns (which are zero in the first two rows), we end up with a tall matrix. The second matrix is nongenerically singular because a tall matrix cannot be formed by such an elimination.

Remark 7. The rows and columns can be interchanged in deciding the (non)generic singularity of a matrix. It can be easily proved that a singular matrix is generically singular if and only if by eliminating a certain number (or none) of its columns and eliminating the zero rows in the remaining columns, we either vacuum the matrix or form a wide submatrix. This follows from the following observation: any generic square matrix A can be transformed into the following form by permuting its rows and columns:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{14}$$

where A_{11} is tall which also means A_{22} is wide. Hence, by eliminating the columns involving A_{11} and A_{21} and then the rows associated with the zero block, we get a wide submatrix unless A_{22} is void.

In order to extend the notion of (non)generic singularity to matrix functions, we define the *index function* for any given matrix function $F(x) = \{F_{ij}(x)\}$ on M as follows:

$$I(F) = \{I_{ij}\}\colon\ I_{ij} = \begin{cases} 0 & \text{if } F_{ij}(x) = 0\ \forall x \in M \\ 1 & \text{otherwise} \end{cases} \tag{15}$$

Definition 2.4

A square matrix function $F(x)$ on M is called ***nonsingular on M*** if it is nonsingular for all $x \in M$. It is called ***generically singular on M*** if it is not nonsingular on M and its index matrix is generically singular. Further, $F(x)$ is called ***nongenerically singular on M*** if it is neither nonsingular nor generically singular on M.

Remark 8. If a matrix function $F(x)$ is generically singular on M, then it is generically singular for every $x \in M$. However, $F(x)$ being nongenerically singular on M does not necessarily mean that $F(x)$ is nongenerically singular for every $x \in M$. In fact, $F(x)$ can be nonsingular when evaluated at some x and generically singular at some other x. The following example exhibits this important point to an extreme: *that a nongenerically singular matrix function on M can even be a generically singular matrix when evaluated at each $x \in M$.* Take

$$\begin{bmatrix} r(x) & r(-x) & 0 \\ r(x) & r(-x) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad M = \mathbb{R} \tag{16}$$

where $r(x)$ is a ramp function defined by

$$r(x) = \begin{cases} x & x \geqslant 0 \\ 0 & x < 0 \end{cases} \tag{17}$$

To check that the above matrix function is generically singular when evaluated at each $x \in M$, one forms the required submatrix by eliminating the third row, third column and either the first or second column or both, depending on x. However, $F(x)$ is not generically singular on M because its index function is nongenerically singular.

The reason we categorize matrix functions like (16) as nongenerically singular is because the singularity structure of the matrix changes with state changes and/or parametric perturbations of its entries. Furthermore, examples of nongenerically singular matrix functions, such as (16), do not arise often in practice, since that requires functions which are constant on a set of nonzero measure, but not constant everywhere. Often, for example, no analytic functions have this property. For functions which do not satisfy this 'nonconstant on a set of nonzero measure' property, $F(x)$ nongenerically singular on M means $F(x)$ can be generically singular at most on a set of zero measure, i.e., the matrix $F(x)$ must be nongenerically singular at some $x \in M$.

3. MAIN RESULT

Given the system T in (1) and M, we wish to define conditions under which we can find a linear diagonal dynamic precompensator of the form

$$D(\rho^{-1}) = \operatorname{diag}\{\rho^{-d_j}\}, \qquad d_j \geqslant 0, \qquad 1 \leqslant j \leqslant m \tag{18}$$

which ensures that the composite system $T \circ D(\rho^{-1})$ has vector relative degree. Such precompensation would, in principle, attach d_j integrators to the jth input, to define

$$l = \sum_{j=1}^{m} d_j$$

new states with linear dynamics.

Assuming the precompensator (18) is applied, let $z(t)$ and $R(\rho)$ be defined as in (9) and (10) and let

$$u(t) = D(\rho^{-1})v(t) \tag{19}$$

we define a new system $K = R(\rho) \circ T \circ D(\rho^{-1})$ and express it by

$$\begin{aligned} K\colon \dot{\bar{x}}(t) &= \bar{f}(\bar{x}(t)) + \bar{g}(\bar{x}(t))v(t) \\ z(t) &= \bar{h}(\bar{x}(t)) + \bar{k}(\bar{x}(t))v(t) \end{aligned} \tag{20}$$

where $\bar{x} \in \bar{M}$, which is an $l + n$ dimensional manifold, and $\bar{x}$ is the state of the new system which contains the state of T and new internal state variables introduced by $D(\rho^{-1})$. When $l = 0$, we will refer to $\bar{x}$ and $\bar{M}$ in (20) simply as x and M.

Assumption 3.1 (well posedness)

The nonlinear system (1) is called well-posed if the following 'reachability' conditions are satisfied:

(i) For every $1 \leqslant i \leqslant m$ there exists some $u \in \mathbb{R}^m$ and integer $r_i \geqslant 0$ such that the direct transmission from u to $\rho^{r_i} y_i$ is nonzero at some $x \in M$.
(ii) For every nonzero $u \in \mathbb{R}^m$, there exists $1 \leqslant i \leqslant m$ and $\hat{r}_i$ such that the direct transmission from u to $\rho^{\hat{r}_i} y_i$ is nonzero at some $x \in M$.

Remark 9. Condition (i) is equivalent to saying that there exists $\{r_1, \ldots, r_m\}$, $r_i \geqslant 0$, such that every row of the matrix $A(x)$ in (8) is nonzero for all $x \in M$ (but $A(x)$ is not necessarily nonsingular). That is, each output, y_i should be 'influenced' by *some* u directly (through $\rho^{r_i} y_i$). Condition (ii) implies that every u should have 'influence' on some output y_i directly through $\rho^{\bar{r}_i} y_i$. These two conditions are satisfied by most nonlinear systems arising in engineering applications and all linear systems with nonsingular transfer matrices.

We now have the main result of the paper.

Theorem 3.1

Given the system $y = T(u)$ in (1) satisfying Assumption 3.1, one of the following two cases must occur and they are mutually exclusive:

(a) There exists a pair $D(\rho^{-1})$ and $R(\rho)$ in the form of (18) and (10) such that the direct transmission matrix $\bar{k}(\bar{x})$ of the resulting system K in (20) is nonsingular on $\bar{M}$. In this case, $T \circ D(\rho^{-1})$ has vector relative degree described by $R(\rho)$.

(b) There exists $D(\rho^{-1})$ and $R(\rho)$ in the form of (18) and (10) such that the direct transmission matrix $\bar{k}(\bar{x})$ is nongenerically singular on $\bar{M}$. In this case, there does not exist any other diagonal precompensator $D(\rho^{-1})$ of the form (18) such that $T \circ D(\rho^{-1})$ has vector relative degree on $\bar{M}$.

The theorem above indicates that we only need to find $R(\rho)$ and $D(\rho^{-1})$ which make $\bar{k}(\bar{x})$ either nonsingular or nongenerically singular on $\bar{M}$ in order to determine whether a suitable precompensator exists or not. The means for finding such a precompensator is given in the following algorithm. The proof for Theorem 3.1 and Algorithm 3.2 is given in Appendix A.

Algorithm 3.2

Initially let $D(\rho^{-1}) = I$.

Step 1. Find $R(\rho) = \operatorname{diag}\{\rho^{r_i}\}$, with $r_i \geqslant 0$, as small as possible, such that each row of $\bar{k}(\bar{x})$ is nonzero at some $\bar{x} \in \bar{M}$ (guaranteed by condition (i) of Assumption 3.1).

There are three possibilities:

1. $\bar{k}(\bar{x})$ is nonsingular on $\bar{M}$: $D(\rho^{-1})$ is a diagonal precompensator for T and $R(\rho)$ gives the associated vector relative degree.
2. $\bar{k}(\bar{x})$ is nongenerically singular on $\bar{M}$: no diagonal compensator exists which will give a vector relative degree.
3. $\bar{k}(\bar{x})$ is generically singular on $\bar{M}$: proceed to Step 2.

Step 2. Form the *tallest* submatrix of the index matrix $I(\bar{k})$ used to indicate the generic singularity (the submatrix cannot be vacuous because of the choice of $R(\rho)$). Then for each column index j of the submatrix, increment d_j by 1. Return to step 1.

The algorithm is complete when either of cases (i) or (ii) is achieved.

Remark 10. It is clear that this algorithm provides a minimal degree diagonal precompensation when such precompensation is possible because only enough integrators are added to remove the generic singularity in the direct transmission matrix at every step.

Remark 11. This algorithm relies only on the relative degree structure to construct $D(\rho^{-1})$. Also, $D(\rho^{-1})$ could just as well contain arbitrary linear rational diagonal entries, so long as the relative degree of the ith term is d_i, as determined by this algorithm (see (18)). In particular, one may require $D(\rho^{-1})$ to be stable.

We illustrate the algorithm and Theorem 3.1 with two examples. The first example will lead to the existence of a diagonal compensator and second one will fail to do so.

Example 1. The system T with $M = R^5$ is given by

$$\dot{x}(t) = \begin{bmatrix} 0 \\ x_4 \\ \lambda x_3 + x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = [x_1 \;\; x_2 \;\; x_5]^{\mathrm{T}} \tag{21}$$

This system satisfies Condition (i) of Assumption 3.1 because for any $i = 1, 2, 3$, $u(t) = (1, 0, 1)^{\mathrm{T}}$ and $r_i = 1$ suffice. To check that condition (ii) also holds, let $u = [u_1, u_2, u_3]'$ be any nonzero vector, if u_1 (or u_3) is nonzero, we take $i = 1$ (or 3) and $\hat{r}_i = 1$ is what is required. Otherwise, u_2 is nonzero and we take $i = 2$ and $\hat{r}_i = 2$.

Applying Step 1 of Algorithm 3.2, we let $D(\rho^{-1}) = I$ and easily verify that $R(\rho)\mathrm{p} = \mathrm{diag}\{\rho, \rho, \rho\}$. The resulting $z(t)$ is given by

$$z(t) = R(\rho)y(t) = \begin{bmatrix} 0 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v(t) \tag{22}$$

Since the resulting matrix $k(x)$ is generically singular on M, we go to Step 2. The index matrix is obtained simply by replacing x_3 by 1 in $k(x)$. The required tall submatrix indicating the generic singularity consists of rows 1 and 2 and column 1. Therefore, we update $D(\rho^{-1})$ to be $\mathrm{diag}\{\rho^{-1}, 1, 1\}$ which modifies $R(\rho)$ to $\mathrm{diag}\{\rho^2, \rho^2, \rho\}$ and we have

$$z(t) = R(\rho)y(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ x_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} v(t) \tag{23}$$

for which $\bar{k}(\bar{x})$ is nonsingular on $\bar{M}$. The compensated system is given by

$$\dot{\bar{x}}(t) = \begin{bmatrix} \xi_1 \\ x_4 + x_3\xi_1 \\ \lambda x_3 + x_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} v(t) \tag{24}$$

where $\xi_1 \in R$, $\bar{x} = [x \;\; \xi_1]^{\mathrm{T}}$, and, along with (23), its vector relative degree is $\{2, 2, 1\}$.

Example 2. The system T, with $M = \mathbb{R}^6$, satisfying Assumption 3.1, is given by

$$\dot{x}(t) = \begin{bmatrix} x_3 - 1 \\ x_4 + x_5 - 1 \\ x_5 \\ 0 \\ x_6 - 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ x_3 & -x_4 \\ x_3 & -x_4 \\ 0 & 0 \\ x_2 & x_6 \end{bmatrix} u(t)$$

$$y(t) = [x_1 \;\; x_2]^{\mathrm{T}} \tag{25}$$

Similary, with the initial setting of $D(\rho^{-1}) = I$, we have $R(\rho) = \mathrm{diag}\{\rho^2, \rho^2\}$ and

$$z(t) = R(\rho)y(t) = \begin{bmatrix} x_5 \\ x_6 - 1 \end{bmatrix} + \begin{bmatrix} x_3 & -x_4 \\ x_3 & -x_4 \end{bmatrix} v(t) \tag{26}$$

Since the resulting direct transmission matrix is $\bar{k}(\bar{x})$ is nongenerically singular (for all nonzero $x \in M$), we stop the algorithm and conclude that no diagonal precompensator exists which can establish a vector relative degree.

4. CONCLUDING REMARKS

A necessary and sufficient condition has been provided which determines whether a diagonal dynamic precompensation can give vector relative degree to a multivariable nonlinear system. In addition, a simple algorithm is given to determine a 'minimal' dynamic compensator necessary to achieve vector relative degree. These results are an extension to similar ones given by the authors[11] for treating linear systems. As in the linear case, we conclude that nonlinear systems for which diagonal precompensation leads to nongenerically singular direct transmission matrices can only achieve vector relative degree through nondiagonal dynamic precompensation.

Thus, with these results, we provide a robust method for implementing dynamic compensation in order to achieve vector relative degree: when the algorithm provides positive results, the diagonal dynamic precompensation constructed is independent of both the original state and small variations of the system parameters. It is the choice of diagonal linear precompensation which provides a solution which is parameter and state independent. This opens up the possibility of implementing parameter adaptive control schemes for some multivariable nonlinear systems which do not originally have full vector relative degree, to which, previously, none of those adaptive schemes could have been applied.

The notion of (non)generic singularity plays an important role in achieving vector relative degree. In particular, systems with nongenerically singular direct transmission matrix functions, ($\bar{k}(\bar{x})$), must be excluded from the class of systems which may be dynamically precompensated to achieve vector relative degree using the methods we propose, because of their sensitivity to structural changes in the system due to state variations and/or small parameter perturbations. To design for large parameter variations in the system, i.e., when the vector relative degree must be invariant on a large set of parameters, our algorithm can be easily modified for this purpose. One possible remedy is to augment the parameter vector, say, p, to the state, and add the following fictitious state equation: $\dot{p} = 0$. In this way, the resulting direct transmission matrix will be guaranteed to be nonsingular on the augmented state manifold.

We have a final remark concerning systems whose direct transmission matrix, $\bar{k}(\bar{x})$, cannot be made nonsingular on $\bar{M}$ by the introduction of a diagonal precompensation. If the system in question is controllable, then for some control purposes, such as tracking, it may be viable to restrict the flow of this system to an open submanifold of $\bar{M}$ which does not include the singularity points of $\bar{k}(\bar{x})$. With this restriction, it may be possible to achieve full vector relative degree on that submanifold.

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APPENDIX; PROOF OF THEOREM 3.1 AND ALGORITHM 3.2

To aid in the proof of Theorem 3.1, we require the following lemma, and for simplicity we denote the direct transmission matrix of a system T as $\mathrm{D}[T]$, which will be a function on the internal state manifold, M, of T.

Lemma A.1

Given any system T, the direct transmission matrices $\mathrm{D}[T]$ of T and $\mathrm{D}[\rho \circ T \circ \rho^{-1}]$ of the composite system $\rho \circ T \circ \rho^{-1}$, are identical.

Proof. Expressing T as

$$\begin{aligned} T\colon \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) + k(x(t))u(t) \end{aligned} \tag{27}$$

the system $\rho \circ T \circ \rho^{-1}$ can be written as

$$\begin{aligned} \rho \circ T \circ \rho^{-1}\colon \dot{\bar{u}}(t) &= u(t) \\ \dot{x}(t) &= f(x(t)) + g(x(t))\bar{u}(t) \\ \bar{z}(t) &= h(x(t)) + k(x(t))\bar{u}(t) \\ z(t) &= \dot{\bar{z}} \end{aligned} \tag{28}$$

Differentiating $\bar{z}$, we have the output

$$\begin{aligned} z = \dot{\bar{z}} &= \frac{\mathrm{d}}{\mathrm{d}t} h(x(t)) + \frac{\mathrm{d}}{\mathrm{d}t} k(x(t))\bar{u}(t) + k(x(t)) \frac{\mathrm{d}\bar{u}(t)}{\mathrm{d}t} \\ &= \frac{\mathrm{d}h(x(t))}{\mathrm{d}x(t)} \dot{x}(t) + \frac{\mathrm{d}k(x(t))}{\mathrm{d}x(t)} \dot{x}(t)\bar{u}(t) + k(x(t))u(t) \end{aligned} \tag{29}$$

Noticing that the first two terms above are functions of only $x(t)$ and $\bar{u}(t)$, which are internal states, then the direct transmission matrix for $\rho \circ T \circ \rho^{-1}$ is therefore $k(x(t))$ as for T. □

To prove Theorem 3.1, it is sufficient to show that possibilities 1 and 2 of Step 1 of Algorithm 3.2 are exclusive, and that given any system T with its direct transmission matrix function $\mathrm{D}[T]$ generically nonsingular on M, the application of Algorithm 3.2 will result in either case 1 or case 2, i.e., we will end up with a direct transmission matrix function which is either nonsingular or nongenerically nonsingular on $\bar{M}$.

In the following, Case (i) implies the exclusiveness of the possibilities 1 and 2, and Case (ii) proves that one or the other can always be achieved.

(i) Suppose $\mathrm{D}[R(\rho) \circ T \circ D(\rho^{-1})]$ is nongenerically singular on $\bar{M}$ for some pair $R(\rho)$ and $D(\rho^{-1})$ in the form of (10) and (18) (at some $\bar{x} \in \bar{M}$), then for any other pair $R_1(\rho)$ and $D_1(\rho^{-1})$ (in the same form), the matrix $\mathrm{D}[R_1(\rho) \circ T \circ D_1(\rho^{-1})]$ cannot be nonsingular on $\bar{M}$.

(ii) Given any system T, the application of Algorithm 3.2 will lead to another pair $R(\rho)$ and $D(\rho^{-1})$ such that $\mathrm{D}[R(\rho) \circ T \circ D(\rho^{-1})]$ is either nonsingular on $\bar{M}$ or nongenerically singular on $\bar{M}$.

For (i) it is sufficient to show the following:

(iii) Given a system K for which $\mathrm{D}[K]$ is nongenerically singular on $\bar{M}$, there do not exist $R_2(\rho)$ and $D_2(\rho^{-1})$ such that $\mathrm{D}[R_2(\rho) \circ K \circ D_2(\rho^{-1})]$ is nonsingular on $\bar{M}$.

To prove that (iii) implies (i), suppose (iii) holds and that $\mathsf{D}[K]$ is nongenerically singular on $\bar{M}$ for $K = R(\rho)\circ T\circ D(\rho^{-1})$. Then, for any given $R_1(\rho)$ and $D_1(\rho^{-1})$, define

$$D_2(\rho^{-1}) = D(\rho)\circ D_1(\rho^{-1})\circ\rho^{-\delta}$$
$$R_2(\rho) = \rho^{\delta}\circ R_1(\rho)\circ R(\rho^{-1})$$

where $\delta \geqslant 0$ is the minimum integer which keeps $D_2(\rho^{-1})$ and $R_2(\rho^{-1})$ causal. We have

$$R_2(\rho)\circ K\circ D_2(\rho^{-1}) = \rho^{\delta}\circ R_1(\rho)\circ T\circ D_1(\rho^{-1})\circ\rho^{-\delta}$$

using Lemma A.1,

$$\mathsf{D}[R_2(\rho)\circ K\circ D_2(\rho^{-1})] = \mathsf{D}[R_1(\rho)\circ T\circ D_1(\rho^{-1})]$$

Because $\mathsf{D}[R_2(\rho)\circ K\circ D_2(\rho^{-1})]$ cannot be nonsingular on $\bar{M}$ (implication of (iii)), then neither can $\mathsf{D}[R_1(\rho)\circ T\circ D_1(\rho^{-1})]$ for any given $R_1(\rho)$ and $D_1(\rho^{-1})$, i.e, (i) holds.

To show (iii) we suppose that $\mathsf{D}[K]$ is nongenerically singular on $\bar{M}$, and assume on the contrary that there exist matrices $R_2(p)$ and $D_2(\rho^{-1})$ such that $\mathsf{D}[R_2(\rho)\circ K\circ D_2(\rho^{-1})]$ is nonsingular on $\bar{M}$. We denote $R_2(\rho)\circ K\circ D_2(\rho^{-1})$ by $\bar{K}$. Without loss of generality we assume that we can choose

$$D_2(\rho^{-1}) = \begin{bmatrix} I_1 & 0 \\ 0 & D_{22}(\rho^{-1}) \end{bmatrix}, \qquad R_2(\rho) = \begin{bmatrix} I_2 & 0 \\ 0 & R_{22}(\rho) \end{bmatrix}$$

where I_1, I_2 are identity matrices, and $D_{22}(\rho^{-1})$ and $R_{22}(\rho)$ do not contain constant diagonal terms (the dimensions of I_1 and I_2 can be zero in general). This can be achieved by using row/column interchanges of inputs and outputs which do not effect the nonsingularity of $\mathsf{D}[K]$, but place K in the following form:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{30}$$

where the dimensions of the subsystems correspond to those of $R_{22}(\rho)$, $D_{22}(\rho^{-1})$, I_1 and I_2. We now argue that $R_2(\rho)$ and $D_2(\rho^{-1})$ can be chosen such that either I_1 or I_2 (or both) will have dimension at least one without loss of generality. This is easily accomplished by rewriting $R_2(\rho)$ and $D_2(\rho^{-1})$ as $\rho^{\varepsilon}\circ\tilde{R}_2(\rho)$ and $\tilde{D}_2(\rho^{-1})\circ\rho^{-\varepsilon}$ where ε is the minimal order of the diagonal terms of $R_2(\rho)$ and $D_2(\rho)$. According to Lemma A.1, $\mathsf{D}[\tilde{R}_2(\rho)\circ T\circ\tilde{D}_2(\rho^{-1})]$ must be nonsingular on $\bar{M}$. We could then replace $R_2(\rho)$ and $D_2(\rho^{-1})$ by $\tilde{R}_2(\rho)$ and $\tilde{D}_2(\rho^{-1})$ respectively and drop off the accent. Note that there must be an identity term in either the new $R_2(\rho)$ and/or the new $D_2(\rho^{-1})$.

With the given choice of $R_2(\rho)$ and $D_2(\rho^{-1})$, we have

$$\bar{K} = \begin{bmatrix} I_2 & 0 \\ 0 & R_{22}(\rho) \end{bmatrix}\circ\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\circ\begin{bmatrix} I_1 & 0 \\ 0 & D_{22}(\rho^{-1}) \end{bmatrix}$$
$$= \begin{bmatrix} K_{11} & K_{12}\circ D_{22}(\rho^{-1}) \\ R_{22}(\rho)\circ K_{21} & R_{22}(\rho)\circ K_{22}\circ D_{22}(\rho^{-1}) \end{bmatrix}$$

In order for $\mathsf{D}[\bar{K}]$ to be nonsingular on $\bar{M}$, $R_{22}(\rho)\circ K_{21}$ must be causal, which means that the relative degree of K_{21} is positive, that is, $\mathsf{D}[K_{21}] = 0$. This implies that K_{22} cannot be a tall matrix, otherwise $\mathsf{D}[K]$ would be generically singular on $\bar{M}$, contradicting the initial assumption. Similarly, $\mathsf{D}[K_{12}\circ D_{22}(\rho^{-1})] = 0$ because K_{12} is causal, which implies that K_{11} cannot be tall, otherwise $\mathsf{D}[\bar{K}]$ would be generically singular, thus contradicting the initial assumption that $\mathsf{D}[\bar{K}]$ is nonsingular. Therefore both K_{11} and K_{22} must be square matrices. This implies that I_1 and I_2 are of the same dimension.

Owing to the nonsingularity of $\mathsf{D}[\bar{K}]$ and that $\mathsf{D}[K_{12}\circ D_{22}(\rho^{-1})]$ is zero, we know that $\mathsf{D}[K_{11}]$ and $\mathsf{D}[R_{22}(\rho)\circ K_{22}\circ D_{22}(\rho^{-1})]$ must both be nonsingular on $\bar{M}$.

Using the nongeneric singularity of $\mathsf{D}[K]$ on $\bar{M}$, nonsingularity of $\mathsf{D}[K_{11}]$ and the fact that $\mathsf{D}[K_{21}] = 0$, we know that $\mathsf{D}[K_{22}]$ must be nongenerically singular on $\bar{M}$. In conclusion, our initial assumption leads to two observations:

(i) $\mathsf{D}[K_{22}]$ must be nongenerically singular $\bar{M}$.
(ii) $\exists\ R_{22}(\rho)$, $D_{22}(\rho^{-1})$ such that $\mathsf{D}[R_{22}(\rho)\circ K_{22}\circ D_{22}(\rho^{-1})]$ is nonsingular on $\bar{M}$.

Noticing however that the dimension of K_{22} is lower than that of K, the problem described in (iii) is repeated in a lower dimension, with K_{22}, $R_{22}(\rho)$, and $D_{22}(\rho^{-1})$ replacing K, $R_2(\rho)$, and $D_2(\rho^{-1})$. We can repeat the above argument until in the limit we must have a 1×1 nongenerically singular matrix function $\mathrm{D}[K_{22}]$ on $\bar{M}$, which is not possible. This conclusion contradicts the initial assumption and therefore $\mathrm{D}[\bar{K}(s)]$ cannot be made nonsingular on $\bar{M}$ by any choice of $R_2(\rho), D_2(\rho^{-1})$ if $\mathrm{D}[K]$ is nongenerically singular on $\bar{M}$.

To show (ii) we suppose that $D(\rho^{-1})$ is initialized to be I, $R(\rho)$ is chosen according to Step 1 of Algorithm 3.2 and that $\mathrm{D}[K]$ is generically singular on M, where K is given by (20). Note that the existence of the required $R(\rho)$ is guaranteed by condition (i) in Assumption 3.1 and Lemma A.1. Without loss of generality, let the index matrix $I(D[K])$ be given by

$$\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

where $I_{12}=0$ and I_{11} is tall. Further, take I_{11} to be as tall as possible. This implies that I_{22} is not generically singular.

Denote the dimensions of I_{11} by $i \times j$ and decompose K into $K_{11}, K_{12}, K_{21}, K_{22}$ according to the blocks of $I(D[K])$. Formulate $D_1(\rho^{-1}) = \mathrm{diag}\{I_1\rho^{-1}, I_2\}$ where I_1, I_2 are identity matrices of dimensions $j \times j$ and $(m-j)\times(m-j)$ respectively. The application of Step 1 of the algorithm and $R_1(\rho) = \mathrm{diag}\{\rho I_1, I_2\}$ gives

$$\bar{K} = R_1(\rho)K(\rho)D_1(\rho^{-1}) = \begin{bmatrix} \rho \circ K_{11} \circ \rho^{-1} & \rho \circ K_{12} \\ K_{21} \circ \rho^{-1} & K_{22} \end{bmatrix}$$

Here, the lower block of $R_1(\rho)$ is an identity because the every row of $D[K_{22}]$ is nonzero.

The resulting system $\bar{K}$, if its direct transmission matrix is still generically singular on $\bar{M}$, can again be reorganized in the form (30), i.e.,

$$\bar{K} = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix}$$

Obviously, the number of columns of $\bar{K}_{12}$ will be reduced if any column of $D[\rho \circ K_{12}]$ is nonzero at some $\bar{x} \in \bar{M}$. If every column of $\rho \circ K_{12}$ is identically zero, the number of columns of $\bar{K}_{12}$ remains the same. However, the latter case cannot continue to happen after a sufficient number of applications of the algorithm because every such iteration will force an additional differentiation on K_{12}. The fact that this increase in the number of differentiations will eventually force some column of $D[\rho \circ K_{12}]$ to become nonzero is guaranteed by condition (ii) of Assumption 3.1. To see this, one simply needs to consider condition (ii) with $u = [0 \;\; u_2]$ where $u_2 \neq 0$ is such that $D[K_{22}]u_2 = 0$, which is always possible because K_{22} must be wide.

We can apply the algorithm as many times as is required until the number of columns of $\bar{K}_{12}$ is reduced to zero, in which case the resulting $\mathrm{D}[\bar{K}]$ is either nonsingular or nongenerically singular on $\bar{M}$.

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