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ROBUST H_∞ TRACKING: A GAME THEORY APPROACH

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SUMMARY

This paper investigates the problem of finite-time-horizon robust H_∞ tracking, for linear continuous time-varying systems, from the game theory point of view. Three tracking problems are considered, depending on whether the reference signal to be tracked is perfectly known in advance, measured on line, or previewed in a fixed interval of time ahead. No *a priori* knowledge of a dynamic model for the reference signal is assumed, and the parameters of the system are not completely known. A game is defined where, given the specific information on the tracking signal, the controller plays against nature that can choose any initial condition, any bounded energy disturbance input and measurement noise, and any set of parameters in a prescribed bounded region. A standard quadratic pay-off function is defined where the energy of the tracking error signal is weighted against the energy of the disturbance, the noise signal, and the Euclidean norm of the initial condition.

Conditions for the existence of a saddle-point equilibrium in this zero-sum game are not easy to find. We, therefore, augment the state-space description of the system to convert the parameter uncertainty into exogenous bounded energy signals. An augmented game is then defined on the new perfectly known system, and it is shown that its saddle-point equilibrium solution, if it exists, guarantees a prescribed H_∞-norm performance of the tracker, in the original system, for all possible parameters.

Necessary and sufficient conditions for the existence of a saddle-point solution to the augmented game are determined. H_∞-tracking controllers, which guarantee the prescribed performance level for all possible parameters, are derived for both the state and the output feedback cases.

KEY WORDS robust tracking; uncertain systems; H_∞ control; game theory

1. INTRODUCTION

One of the main reasons for the vast attention that has been paid in the last decade to the methods of H_∞ control is the good robustness properties of the resulting control, in comparison to the 'conventional' L_2 design.[1] The linear H_∞ problem has been solved generally by

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transforming the problem to fit the model of the 'standard problem', and using the well-established solution of this problem to obtain the required output feedback.[2]

The H_∞ design problem is, in fact, a design that aims at minimizing the effect that the exogenous signals have on the system, at the worst-case situation. It has been recognized very quickly that this problem can be looked at as a zero-sum dynamic game, where the controller plays against nature that can pick the worst possible disturbance to increase the performance index the controller wants to minimize.[3,4]

The main practical disadvantages of the H_∞/game theory approach are:

(1) Although it provides a reasonable robustness, the solution of the standard problem does not guarantee a prescribed performance level in case of plant parameter uncertainty, where the parameters are allowed to stay anywhere in a given closed set.
(2) The solution aims at securing a prescribed performance level for all possible exogenous input signals, and it therefore entails a significant overdesign in cases where some of the input signals are measurable. These include the case where the disturbance is partially measurable and the case where the system is required to follow an *a priori* given, or a causally measured, reference signal.

The theory of H_∞ control and filtering has been recently extended to deal with structural uncertainties in a prescribed range.[5,6] The method of Reference 5 has a game theory meaning where nature can be considered as the adversary of the controller. Nature selects not only the exogenous signals to maximize the pay-off function, but also the parameters of the plant. The overdesign that was entailed in solving H_∞ control problems with measured signals has been elevated lately by Reference 7. A method has been introduced there which does not consider the measured signals as a part of nature's selection. A solution is proposed in Reference 7 for the various information patterns that are available about the measured signals.

The results of Reference 7 constitute the first attempt to deal with the tracking problem in cases where the statistics of the reference signal is unknown and where the controller possesses some measured information on this signal. The L_2 tracking problem of an *a priori* known reference signal has been solved long ago (e.g. Reference 10). The problem of tracking a reference signal with fixed preview has been investigated by Reference 11, where a knowledge of the statistics of the reference signal is required for the time interval that is not previewed. The problem of L_2 tracking with preview has also been treated in the frequency domain by Reference 12, where spectral factorization is used to derive the control signal.

None of the above works addresses the problem of parameter uncertainty. This issue is of special importance in the tracking problem since owing to the uncertainty the effect of a known reference signal cannot be completely cancelled. It is expected that the H_∞ approach of Reference 7 will exhibit some advantages over the traditional L_2 approach owing to the inherent robustness properties of the H_∞ design. The results of Reference 7 cannot guarantee, however, a required performance for the whole given set of the parameter uncertainty.

In the present paper we combine the methods of References 5 and 7 to obtain a game theory solution to the problem of achieving a prescribed level of tracking and disturbance attenuation in the presence of some measurable disturbance and reference signals and some uncertainty in the plant. The uncertain part of the plant state-space matrices is known to be norm-bounded and it may be time-varying. Three cases are considered. In the first, we treat the case where the tracking signal is perfectly known in advance. This case corresponds to situations where one wants to obtain a good tracking to, say, a step input, or else, in the stochastic case where the random reference signal has a known nonzero average component. The second case deals with a reference signal that is measured on-line, and the third one is the case where the reference signal is previewed in a fixed interval of time ahead and this future information on the signal is used to improve the tracking performance.

The paper is organized as follows. We begin in Section 2 by formulating the tracking problem and describing the plant uncertainty and the different information patterns that we treat. The performance index that we use is essentially the worst-case L_2-norm of the tracking error, for a specific reference signal, over all the initial states, the L_2 disturbances and measurement noise signals, and the admissible parameters. In Section 3 we augment the state-space description of the system to convert the parameter uncertainty into exogenous, energy-bounded, signals. An augmented game is then defined and solved. The saddle-point solution of the latter, if exists, guarantees the required H_∞ performance of the original tracking problem for all admissible parameters.

We demonstrate the use of the theory of Section 3 in an example in Section 4. In this example we solve the state feedback tracking problem for a second-order system using both the method of Reference 7, for the nominal plant, and the new method of the present paper. We compare the results in the case where the reference signal is not *a priori* known but is measured on line. We demonstrate in the example how preview improves the tracking performance for the worst possible value of the uncertain parameter of the system.

2. PROBLEM STATEMENT

Consider uncertain time-varying systems described by

$$(\Sigma) \qquad \dot{x} = (A + \Delta A)x + B_1 w + (B_2 + \Delta B)u + B_3 r; \quad x(0) = x_0 \tag{1a}$$

$$z = C_1 x + D_{12} u + D_{13} r \tag{1b}$$

$$y = (C_2 + \Delta C)x + v + (D_{22} + \Delta D)u \tag{1c}$$

where $x \in R^n$ is the state, x_0 is an unknown initial state, $w \in R^p$ is the disturbance input, $u \in R^m$ is the control input, $r \in R^r$ is a known or measurable reference signal, $y \in R^k$ is the measured output, $v \in R^k$ is the measurement noise, and $z \in R^q$ is the controlled output. The matrices A, B_1, B_2, B_3, C_1, C_2, D_{12}, D_{13}, and D_{22} are known real time-varying, piecewise continuous, bounded matrices of appropriate dimensions that describe the nominal system, and ΔA, ΔB, ΔC and ΔD are real-valued matrix functions representing time-varying parameter uncertainties. These uncertainties are assumed to be of the form

$$\begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F [E_1 \;\; E_2] \tag{2}$$

where $F \in R^{i \times j}$ is an unknown time-varying matrix with Lebesgue measurable elements satisfying

$$F^{\mathrm{T}}(t)F(t) \leqslant I, \quad \forall t \tag{3}$$

and H_1, H_2, E_1 and E_2 are known piecewise continuous matrix functions of appropriate dimensions that specify how the nominal system matrices A, B_2, C_2 and D_{22} are affected by the uncertain parameters of F.

The deterministic input signal $r(\cdot)$ provides a tracking trajectory $-D_{13}r(\cdot)$ for the system output $C_1 x + D_{12} u$, and the signal $z(\cdot)$ in (1b) represents the tracking error. Note that the input signal $r(\cdot)$ is also allowed to affect the system dynamics and that it may use an input channel different from that used by the control input $u(\cdot)$.

In this paper we are concerned with the following robust tracking problems: design a control law $u(\cdot)$ over the horizon $[0, T]$, using the available measurements $y(\cdot)$, and the known reference signal, $r(\cdot)$, to make $z(\cdot)$ uniformly small, in a certain sense, for any w, and v in $L_2[0, T]$ and $x_0 \in R^n$, and for all admissible uncertainties.

The admissible control law is assumed to be of the form

$$u = G_y y + G_r r \tag{4}$$

where G_y is a linear causal dynamic operator and G_r is a linear dynamic operator which can be either causal or non-causal, depending on whether the reference signal $r(\cdot)$ is, respectively, measured on line or known *a priori*.

Three tracking problems will be considered, depending on the information pattern for $r(\cdot)$. We define the history up to time t of the measurement y and the reference signal r by

$$Y_t = \{y(\tau), \quad 0 \leqslant \tau \leqslant t\} \quad \text{and} \quad R_t = \{r(\tau); \quad 0 \leqslant \tau \leqslant t\}$$

The tracking problems we shall investigate are:

(i) **Tracking of a non-causal r(·).** The signal r(·) is assumed to be known a priori for the whole interval [0, T]. The control law u(t), $\forall t \in [0, T]$ is based on Y_t and R_T.

(ii) **Tracking of a causal r(·).** The signal $r(\cdot)$ is measured on line but cannot be predicted. The control law $u(t)$, $\forall t \in [0, T]$ is based on Y_t and R_t.

(iii) **Fixed-review tracking.** It is assumed that at the current time the signal $r(\cdot)$ is previewed in a known fixed interval of time ahead. Given a positive scalar $\Delta < T$, the control law $u(t)$, $\forall t \in [0, T]$ is based on Y_t and $R_{t+\Delta}$.

The tracking performance that is used here is a standard quadratic pay-off function given by

$$J(r, u, v, w, x_0, F) = \|z\|_2^2 - \gamma^2[\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_{R^{-1}}^2] \tag{5}$$

where $\gamma > 0$ is a given scalar that indicates the level of tracking performance of the controlled system, and $R = R^{\mathrm{T}} > 0$ is a given weighting matrix for the initial state.

In the above, $\|x\|_A^2$ denotes $x^{\mathrm{T}} A x$, $L_2[0, T]$ stands for the space of square integrable real vector functions in $[0, T]$, and $\|\cdot\|_2$ denotes the $L_2[0, T]$ norm.

The robust tracking problems are to find a control law $u(\cdot) \in L_2[0, T]$ of the form (4), using the available information on $y(\cdot)$ and $r(\cdot)$, that minimizes

$$\sup_{v,w,x_0,F} J(r,u,v,w,x_0,F) \tag{6}$$

where $v, w \in L_2[0, T]$ and $x_0 \in R^n$.

The above tracking problems are indeed dynamic game problems, where given a specific information on the reference signal r, the controller plays against nature that can choose any initial state, any bounded energy disturbance input and measurement noise, and any uncertain matrix F satisfying (3). All the three games that are related to the tracking problems (i)–(iii) stem from the cost function (5). The only difference between these games is in the form by which the reference signal is available to the controller. The desired control law for each of the tracking problems (i)–(iii) corresponds indeed to the saddle-point minimizing strategy for each of the corresponding games.

Remark 2.1

Note that when the reference signal $r(\cdot)$ is identically zero, the above tracking problems reduce to the robust H_∞ control synthesis which has been analysed in References 5 and 8 for, respectively, continuous and discrete time systems.

Remark 2.2

We note that problems (i)–(iii) have been solved in Reference 7 for the case where there is no parameter uncertainty in (1). Also, it should be noted that in the absence of the reference signal $r(\cdot)$ and parameter uncertainty, the above problems reduce to the standard H_∞ control problem.

We end this section by introducing the following assumption which is standard in nonsingular H_∞ control problems.

Assumption A

$D_{12}(t)$ is of full column rank for all $t \in [0, T]$.

3. MAIN RESULTS

The game problem introduced in the previous section is very difficult to solve. We shall, therefore, solve an auxiliary game problem which does not involve uncertainty and is such that the worst-case value of its pay-off function is an upper bound for the worst-case value of the criterion in (6). The key idea behind this approach is to convert the parameter uncertainty into exogenous bounded-energy signals. Justification for this technique is provided in the sequel.

We introduce the following auxiliary system:

$$(\Sigma_a) \qquad \dot{x}_a = Ax_a + \begin{bmatrix} B_1 & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} w_a + B_2 u_a + B_3 r, \qquad x_a(0) = x_{a0} \tag{7a}$$

$$z_a = \begin{bmatrix} C_1 \\ \varepsilon E_1 \end{bmatrix} x_a + \begin{bmatrix} D_{12} \\ \varepsilon E_2 \end{bmatrix} u_a + \begin{bmatrix} D_{13} \\ 0 \end{bmatrix} r \tag{7b}$$

and

$$y_a = C_2 x_a + \begin{bmatrix} 0 & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} w_a + v_a + D_{22} u_a \tag{7c}$$

where $x_a \in R^n$ is the state, x_{a0} is an unknown initial state, $u_a \in R^m$ is the control signal, $w_a \in R^{p+j}$ is the disturbance input, $v_a \in R^k$ is the measurement noise signal, $y_a \in R^k$ is the measurement, $z_a \in R^{q+j}$ is the controlled output, r, A, B_1, B_2, B_3, C_1, C_2, D_{12}, D_{13}, D_{22}, H_1, H_2, E_1 and E_2 are as in (1)–(2) and $\varepsilon(t)$ is a piecewise continuous scaling function to be chosen that is nonzero for all $t \in [0, T]$. Associated with the system (Σ_a) we define the following performance index:

$$J_a(r, u_a, v_a, w_a, x_{a0}, \varepsilon) = \|z_a\|_2^2 - \gamma^2[\|w_a\|_2^2 + \|v_a\|_2^2 + \|x_{a0}\|_{R^{-1}}^2] \tag{8}$$

where γ and R are as in (5).

We have the following result:

Lemma 3.1

Consider the systems (Σ) and (Σ_a) together with the performance indices (5) and (8), respectively. Assume that u and u_a are generated by the same controller, namely, $u = G_y y + G_r r$ and $u_a = G_y y_a + G_r r$. Then, the following holds:

$$\sup_{v,w,x_0,F} [J(r, u, w, x_0, F)] \leq \sup_{v_a,w_a,x_{a0}} [J_a(r, u_a, v_a, w_a, x_{a0}, \varepsilon] \quad \text{for any admissible } \varepsilon.$$

Proof. For any given x_0, F, w, v, and r in (5) for the system (Σ), and any admissible ε, take

$$x_{a0} = x_0, \qquad w_a(t) = \begin{bmatrix} w(t) \\ \frac{\varepsilon}{\gamma} F[E_1 x_a(t) + E_2 u_a(t)] \end{bmatrix}, \qquad \text{and} \qquad v_a(t) = v(t)$$

Then, it is easy to verify that for all $t \in [0, T]$

$$x_a(t) = x(t), \qquad y_a(t) = y(t), \qquad u_a(t) = u(t) \qquad \text{and} \qquad z_a(t) = \begin{bmatrix} z(t) \\ \varepsilon[E_1 x(t) + E_2 u(t)] \end{bmatrix}$$

which implies that:

$$J_a(r, u_a, v_a, w_a, x_{a0}, \varepsilon) = \|z\|_2^2 - \gamma^2[\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_{R^{-1}}^2] + \varepsilon^2[\|E_1 x + E_2 u\|_2^2 - \|F(E_1 x + E_2 u)\|_2^2]$$

Now, considering (3) and (5) we obtain that $J_a(r, u_a, v_a, w_a, x_{a0}, \varepsilon) \geq J(r, u, v, w, x_0, F)$ and the result follows immediately. □

In view of Lemma 3.1, our approach for solving the robust tracking problems involves solving the game problem of (8) in lieu of the game of (5). More precisely, we will solve the following auxiliary game problem:

Find $\varepsilon \neq 0$, $u \in L_2[0, T]$, worst-case signals $v_a^*, w_a^* \in L_2[0, T]$, and a worst-case initial state $x_{a0}^* \in R^n$ satisfying the following saddle-point condition:

$$J_a(r, u_a^*, v_a, w_a, x_{a0}, \varepsilon) \leq J_a(r, u_a^*, v_a^* w_a^*, x_{a0}^*, \varepsilon) \leq J_a(r, u_a, v_a^*, w_a^*, x_{a0}^*, \varepsilon)$$

where the strategy $u_a^*(t)$, $t \in [0, T]$, is based on the available information at time t on the measurement $y_a(\cdot)$ and the reference signal $r(\cdot)$. Note that the system (Σ_a) is parameterized by ε, which is a scaling function to be searched in order that a saddle-point equilibrium in the game problem of (8) can be found.

The minimizing control law

$$u_a^* = G_y y_a + G_r r$$

with y_a replaced by y, will provide the control law of the related H_∞ tracking problem for the system (Σ).

Remark 3.1

We note that when the reference signal $r(\cdot)$ is identically zero, it has been shown in References 5 and 8 that the above controller guarantees the following H_∞ performance:

$$\|z\|_2 < \gamma[\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_{R^{-1}}^2]^{1/2}$$

for any w and v in $L_2[0,T]$ and $x_0 \in R^n$, and for all admissible uncertainties, whenever $\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_{R^{-1}}^2 \neq 0$.

A solution to the auxiliary game problem of (8) subject to (7) can be readily solved using the recent results of Reference 7. To this end, introduce the following Riccati differential equations (RDEs)

$$-\dot{X} = (A - B_2V_1\bar{D}_{12}^{\mathrm{T}}\bar{C}_1)^{\mathrm{T}}X + X(A - B_2V_1\bar{D}_{12}^{\mathrm{T}}\bar{C}_1) + X(\gamma^{-2}\bar{B}_1\bar{B}_1^{\mathrm{T}} - B_2V_1B_2^{\mathrm{T}})X_2 + \bar{C}_1^{\mathrm{T}}(I - \bar{D}_{12}V_1\bar{D}_{12}^{\mathrm{T}})\bar{C}_1; \qquad X(T) = 0 \quad (9)$$

and

$$\dot{Y} = (A - \bar{B}_1\bar{D}_{21}^{\mathrm{T}}V_2C_2)Y + Y(A - \bar{B}_1\bar{D}_{21}^{\mathrm{T}}V_2C_2)^{\mathrm{T}} + Y(\gamma^{-2}\bar{C}_1^{\mathrm{T}}\bar{C}_1 - C_2^{\mathrm{T}}V_2C_2)Y + \bar{B}_1(I - \bar{D}_{21}^{\mathrm{T}}V_2\bar{D}_{21})\bar{B}_1^{\mathrm{T}}; \quad Y(0) = R \quad (10)$$

where

$$\bar{B}_1 = \begin{bmatrix} B_1 & \frac{\gamma}{\varepsilon}H_1 \end{bmatrix}; \quad \bar{D}_{21} = \begin{bmatrix} 0 & \frac{\gamma}{\varepsilon}H_2 \end{bmatrix}; \quad \bar{C}_1 = \begin{bmatrix} C_1 \\ \varepsilon E_1 \end{bmatrix}; \quad \bar{D}_{12} = \begin{bmatrix} D_{12} \\ \varepsilon E_2 \end{bmatrix} \quad \text{(11a-d)}$$

and

$$V_1 = (\bar{D}_{12}^{\mathrm{T}}\bar{D}_{12})^{-1}; \quad V_2 = (I + \bar{D}_{21}\bar{D}_{21}^{\mathrm{T}})^{-1} \quad \text{(11e-f)}$$

We also consider the following conditions:

Condition 1. There exists an ε such that (9) has a solution $X(t)$ over $[0,T]$ satisfying $X(0) < \gamma^2R^{-1}$.

Condition 2. There exists an ε such that:

(a) The equation (9) has a solution $X(t)$ over $[0,T]$ satisfying $X(0) < \gamma^2R^{-1}$
(b) The equation (10) has a solution $Y(t)$ over $[0,T]$
(c) $I - \gamma^{-2}X(t)Y(t) > 0$ over $[0,T]$

Remark 3.2

Note that since $X(T) = 0$ and $Y(0) > 0$, it follows from well-known results on RDEs that if there exist solutions $X(t)$ and $Y(t)$ to (9) and (10) over $[0,T]$, respectively, then $X(t) = X^{\mathrm{T}}(t) \geqslant 0$ and $Y(t) = Y^{\mathrm{T}}(t) > 0$ over $[0,T]$.

We first deal with the case where perfect state measurements are available.

3.1. State feedback case

Our first theorem provides a solution to the auxiliary game problem of (8) where $v_a \equiv 0$ and the controller has access to the state x_a. In view of Theorem 3.1 in Reference 7 we have the following result.

Theorem 3.1

Consider the system (Σ_a) subject to the assumption of perfect state measurements, and where the time history R_{t+h}, $h \in [0,T]$ of the reference signal $r(\cdot)$ is available at time t. Then, the auxiliary game problem of (8) has a saddle-point solution if and only if Condition 1 is satisfied. Moreover, a saddle-point strategy is given by:

$$x^*_{a0} = [\gamma^2 R^{-1} - X(0)]^{-1}\theta(0) \tag{12}$$

$$w^*_a = \gamma^{-2} B_1^{\mathrm{T}}[X x^*_a + \theta] \tag{13}$$

$$u^*_a = -V_1[B_2^{\mathrm{T}} X + \bar{D}_{12}^{\mathrm{T}}\bar{C}_1)x^*_a + D_{12}^{\mathrm{T}} D_{13} r + B_2^{\mathrm{T}}\theta_c] \tag{14}$$

where x^* denotes the optimal trajectory of x_a with $u_a = u^*_a$, $w_a = w^*_a$ and $x_{a0} = x^*_{a0}$, $\theta(t)$, $\forall t \in [0, T]$ satisfies

$$\dot{\theta} = -A_\theta^{\mathrm{T}}\theta + B_r r, \quad \theta(T) = 0 \tag{15}$$

where

$$A_\theta = A - B_2 V_1 \bar{D}_{12}^{\mathrm{T}}\bar{C}_1 + (\gamma^{-2}\bar{B}_1\bar{B}_1^{\mathrm{T}} - B_2 V_1 B_2^{\mathrm{T}})X \tag{16a}$$

$$B_r = (XB_2 + \bar{C}_1^{\mathrm{T}}\bar{D}_{12})V_1 D_{12}^{\mathrm{T}} D_{13} - (XB_3 + C_1^{\mathrm{T}} D_{13}) \tag{16b}$$

and $\theta_c(t)$ is the 'causal' part of $\theta(\cdot)$ at time t, given by

$$\dot{\theta}_c(\tau) = -A_\theta^{\mathrm{T}}(\tau)\theta_c(\tau) + B_r(\tau)r(\tau), \qquad t < \tau \leqslant t_f, \qquad t_f \begin{cases} t+h & \text{if } t+h < T \\ T & \text{if } t+h \geqslant T \end{cases} \tag{17a}$$

and

$$\theta_c(t_f) = 0 \tag{17b}$$

Furthermore, the value of the game is

$$J_a(r, u^*_a, w^*_a, x^*_{a0}, \varepsilon) = \bar{J}(r, \varepsilon) + \| V_1^{1/2} B_2^{\mathrm{T}} \theta_1 \|_2^2 \tag{18}$$

where

$$\theta_1(t) = \theta(t) - \theta_c(t), \quad \forall t \in [0, T] \tag{19}$$

and

$$\bar{J}(r, \varepsilon) = \|D_{13} r\|_2^2 + \gamma^{-2}[\|\theta(0)\|^2_{P_0} + \|\bar{B}_1^{\mathrm{T}}\theta\|_2^2] - \|V_1^{1/2}(B_2^{\mathrm{T}}\theta + D_{12}^{\mathrm{T}} D_{13} r)\|_2^2 + 2\int_0^T \theta^{\mathrm{T}} B_3 r \, \mathrm{d}r \tag{20}$$

with

$$P_0 = [R^{-1} - \gamma^{-2} X(0)]^{-1} \tag{21}$$

Remark 3.3

Observe that in the case of a non-causal signal $r(\cdot)$, i.e. $h = T$, $\theta_c(t) = \theta(t)$ over $[0, T]$. Hence, it follows that the values of the game reduces to $\bar{J}(r, \varepsilon)$. On the other hand, when the signal $r(\cdot)$ is measured on line, i.e. $h = 0$, it follows from (17) that $\theta_c(t) = 0$ for all $t \in [0, T]$, and the value of the game is given by $\bar{J}(r, \varepsilon) + \|V_1^{1/2} B_2^{\mathrm{T}} \theta\|_2^2$.

In view of Theorem 3.1, we can easily obtain a solution to each of the three robust tracking problems (i)–(iii) when perfect state measurements are available. Indeed, the control law (14) with x_{a0} and w_a not playing their optimal strategy will provide a suitable control law for the tracking problems.

Corollary 3.1

Consider the system (Σ) subject to the assumption of perfect state measurements, and let $\gamma > 0$ be a given scalar. Then, if Condition 1 is satisfied we have the following results:

(a) A suitable control law for the problem of H_∞ tracking of a non-causal $r(\cdot)$ is given by

$$u = K_x x + K_r r + K_\theta \theta \tag{22}$$

where

$$K_x = -V_1(B_2^{\mathrm{T}} X + D_{12}^{\mathrm{T}} C_1 + \varepsilon^2 E_2^{\mathrm{T}} E_1); \quad K_r = -V_1 D_{12}^{\mathrm{T}} D_{13}, \quad \text{and} \quad K_\theta = -V_1 B_2^{\mathrm{T}} \quad \text{(23a–c)}$$

and where $\theta(\cdot)$ is as in (15). Furthermore, the controller guarantees the performance

$$\|z\|_2^2 < \gamma^2[\|w\|_2^2 + \|x_0\|_{R^{-1}}^2] + \bar{J}(r, \varepsilon)$$

(b) A suitable control law for the problem of H_∞ tracking of a causal $r(\cdot)$ is given by

$$u = K_x x + K_r r \tag{24}$$

with the guaranteed performance

$$\|z\|_2^2 < \gamma^2[\|w\|_2^2 + \|x_0\|_{R^{-1}}^2] + \bar{J}(r, \varepsilon) + \|V_1^{1/2} B_2^{\mathrm{T}} \theta\|_2^2$$

(c) A suitable control law for the problem of H_∞ fixed-preview tracking is given by

$$u = K_x x + K_r r + K_\theta \theta_c \tag{25}$$

with $\theta_c(\cdot)$ given by (17). Furthermore, this controller guarantees the performance

$$\|z\|_2^2 < \gamma^2[\|w\|_2^2 + \|x_0\|_{R^{-1}}^2] + \bar{J}(r, \varepsilon) + \|V_1^{1/2} B_2^{\mathrm{T}} \theta_1\|_2^2$$

where $\theta_1(\cdot)$ is as in (19).

Remark 3.4

In the case where $r(t)$ is identically zero over $[0, T]$, Corollary 3.1 reduces to the main result of Reference 9. In this situation, it has been shown there that the above controller guarantees the performance

$$\|z\|_2 < \gamma[\|w\|_2^2 + \|x_0\|_{R^{-1}}^2]^{1/2}$$

for any $w \in L_2[0, T]$ and $x_0 \in R^n$ and for all admissible uncertainties, whenever $\|w\|_2^2 + \|x_0\|_{R^{-1}}^2 \neq 0$.

3.2. Output feedback case

We shall first provide a solution to the auxiliary game problem of (8) via output measurements for a non-causal $r(\cdot)$. The result that follows is easily obtained from Theorem 3.3 in Reference 7.

Theorem 3.1

Consider the system (Σ_a) where the time history R_T of the reference signal $r(\cdot)$ is available in advance. Then, the auxiliary game problem of (8) has a saddle-point solution if and only if

Condition 2 is satisfied. Moreover, a saddle-point strategy is given by:

$$x_{a0}^* = [\gamma^2 R^{-1} - X(0)]^{-1}\theta(0) \tag{26}$$

$$v_a^* = 0 \tag{27}$$

$$w_a^* = \gamma^{-2}\bar{B}_1^{\mathrm{T}}(X\hat{x} + \theta) \tag{28}$$

and

$$u_a^* = -V_1[(B_2^{\mathrm{T}}X + \bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x} + D_{12}^{\mathrm{T}}D_{13}r + B_2^{\mathrm{T}}\theta] \tag{29}$$

where $\theta(t)$ is as in (15) and $\hat{x}(t)$ satisfy $\forall t \in [0, T]$:

$$\dot{\hat{x}} = A_e\hat{x} + B_2u_a^* + B_3r + L(y_a - C_2\hat{x} - D_{22}u_a^*) + B_e\theta, \quad \hat{x}(0) = 0 \tag{30}$$

with

$$A_e = A + \gamma^{-2}(\bar{B}_1 - L\bar{D}_{21})\bar{B}_1^{\mathrm{T}}X \tag{31a}$$

$$B_e = \gamma^{-2}(\bar{B}_1 - L\bar{D}_{21})\bar{B}_1^{\mathrm{T}} \tag{31b}$$

and

$$L = (I - \gamma^{-2}YX)^{-1}(YC_2^{\mathrm{T}} + \bar{B}_1\bar{D}_{21}^{\mathrm{T}})V_2 \tag{31c}$$

Furthermore, the value of the game is $\bar{J}(r, \varepsilon)$ of (20).

The next theorem, which follows from Theorem 3.1 in Reference 7 deals with the auxiliary game of (8) where the controller strategy $u_a(t)$ over $[0, T]$ is based on the information Y_t^a and R_{t+h}, with $0 \leq h < T$. The notation Y_t^a denotes the time history of $y_a(\cdot)$ up to time t, defined by $Y_t^a = \{y_a(\tau), 0 \leq \tau \leq t\}$.

Theorem 3.3

Consider the system (Σ_a) where the time history R_{t+h}, $h \in [0, T)$, of the reference signal $r(\cdot)$ is available at time t. Then, the auxiliary game problem of (8) has a saddle-point solution if and only if Condition 2 is satisfied. Moreover, a saddle-point strategy is given by (26)–(28) and

$$u_a^* = -V_1[(B_2^{\mathrm{T}}X + \bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x}_c + D_{12}^{\mathrm{T}}D_{13}r + B_2^{\mathrm{T}}\theta_c] \tag{32}$$

where $\theta_c(t)$ is given by (17) and $\hat{x}_c(t)$ is the 'causal part' of $\hat{x}(\cdot)$ at time t (with respect to $r(\cdot)$) given by

$$\dot{\hat{x}}_c = A_e\hat{x}_c + B_2u_a^* + B_3r + L(y - C_2\hat{x}_c - D_{22}u_a^*) + B_e\theta_c \tag{33}$$

and

$$\hat{x}_c(0) = [\gamma^2R^{-1} - X(0)]^{-1}\theta_c(0)$$

Furthermore, the value of the game is

$$J_a(r, u_a^*, v_a^*, w_a^*, x_{a0}^*, \varepsilon) = \bar{J}(r, \varepsilon) + \|V_1^{1/2}[B_2^{\mathrm{T}}\theta_1 + (B_2^{\mathrm{T}}X + \bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x}_1]\|_2^2$$

where $\theta_1(\tau)$ and $\bar{J}(r, \varepsilon)$ are given by (19) and (20), respectively, and

$$\hat{x}_1(t) = \hat{x}(t) - \hat{x}_c(t), \quad \forall t \in [0, T] \tag{34}$$

Remark 3.5

We note that similarly to Theorem 3.1, the conditions in Theorems 3.2 and 3.3 for the existence of a saddle-point equilibrium do not depend on the reference signal $r(\cdot)$. Indeed, $r(\cdot)$ affects only the saddle-point signal and the value of the game.

We also note that in the case when $r(\cdot)$ is causally measured, $h=0$, $\theta_c(t)=0$ and $\theta_1(t)=\theta(t)$ over $[0, T]$. Hence, in Theorem 3.3 we have $\hat{x}_c(0)=0$.

The results of Theorems 3.2 and 3.3 can be easily specialized to provide output feedback controllers for each of the H_∞ tracking problems (i)–(iii). The control law u_a^* of (29) and (32) with $y_a(\cdot)$ replaced by $y(\cdot)$ will provide a suitable control law for the tracking problems.

In view of Theorem 3.2 we can easily derive the following result.

Corollary 3.2 (Robust tracking of a non-causal $r(\cdot)$)

Consider the system (Σ) where the reference signal $r(\cdot)$ is known in advance for the whole interval $[0, T]$ and let $\gamma>0$ be a given scalar. Then, if Condition 2 is satisfied define the control law

$$u=-V_1[(B_2^{\mathrm{T}}X+\bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x}+D_{12}^{\mathrm{T}}D_{13}r+B_2^{\mathrm{T}}\theta] \tag{35}$$

where $\theta(\cdot)$ satisfies (15) and $\hat{x}(\cdot)$ is given by

$$\dot{\hat{x}}=A_e\hat{x}+B_2u+B_3r+L(y-C_2\hat{x}-D_{22}u)+B_e\theta, \quad \hat{x}(0)=0$$

This control law will guarantee the performance

$$\|z\|_2^2<\gamma^2[\|w\|_2^2+\|v\|_2^2+\|x_0\|_{R^{-1}}^2]+\bar{J}(r, \varepsilon)$$

where $\bar{J}(r, \varepsilon)$ is as in (20).

A solution to the robust tracking with a causal $r(\cdot)$ and the robust fixed-preview tracking are obtained directly from Theorem 3.3 and are provided in the next two corollaries.

Corollary 3.3. (Robust tracking of a causal $r(\cdot)$)

Consider the system (Σ) where the reference signal $r(\cdot)$ is measured on line, and let $\gamma>0$ be a given scalar. Then, if Condition 2 is satisfied, the control law

$$\dot{\hat{x}}_c=A_e\hat{x}_c+B_2u+B_3r+L(y-C_2\hat{x}_c-D_{22}u); \quad \hat{x}_c(0)=0 \tag{36}$$

$$u=-V_1[(B_2^{\mathrm{T}}X+\bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x}_c+D_{12}^{\mathrm{T}}D_{13}r] \tag{37}$$

will guarantee the performance

$$\|z\|_2^2<\gamma^2[\|w\|_2^2+\|v\|_2^2+\|x_0\|_{R^{-1}}^2]+\bar{J}(r, \varepsilon)+\|V_1^{1/2}[B_2^{\mathrm{T}}\theta+(B_2^{\mathrm{T}}X+\bar{D}_{12}^{\mathrm{T}}\bar{C}_1)\hat{x}_1\|_2^2$$

with $\theta(\cdot)$, $\theta_1(\cdot)$ and $\bar{J}(r, \varepsilon)$ satisfying (15), (19) and (20), respectively.

It is easy to see that the above controller can be rewritten in the following observer-based structure:

$$\dot{\hat{x}}_c=(A+\Delta A^*)\hat{x}_c+B_1w^*+B_2u+B_3r+L[y-(C_2+\Delta C^*)\hat{x}_c-D_{22}u], \quad \hat{x}_c(0)=0 \tag{38}$$

$$u=K_x\hat{x}_c+K_rr \tag{39}$$

where K_x and K_r are as in (23) and

$$\Delta A^* = \frac{1}{\varepsilon^2} H_1 H_1^T X; \qquad \Delta C^* = \frac{1}{\varepsilon^2} H_2 H_1^T X; \qquad w^* = \gamma^{-2} B_1^T X \hat{x}_c \tag{40a-c}$$

Remark 3.6

In the tracking controller (38)–(39), $K_x \hat{x}_c$ can be viewed as the estimate of the signal $K_x x$ of the state feedback control law (24) in the presence of the 'worst-case' input disturbance w^* and 'worst-case' parameter uncertainties. The observer (38) is a modified Luenberger observer where ΔA^* and ΔC^* reflect the effect of the 'worst-case' parameter uncertainties on the estimation of $K_x x$.

Remark 3.7

The result of Corollary 3.3 can be used to solve the important case where some of the disturbances are measurable. Replacing $B_1 w$ in (1a) by $B_1 w + B_m w_m$, where w_m describes the measurable disturbances, and it is assumed that nature can arbitrarily choose w. The augmented reference signal is then $r_a = [r^T,\ w_m^T]^T$, and B_3 and D_{13} are replaced by $[B_3\ B_m]$ and $[D_{13}\ 0]$, respectively. If Condition 2 is satisfied, the control law is then given by (36) and (37) with $B_3 r$ replaced, in (36), by $B_3 r + B_m w_m$.

Corollary 3.4 (Robust fixed-preview tracking)

Consider the system (Σ) where the reference signal $r(\cdot)$ is previewed Δ second ahead and let $\gamma > 0$ be a given scalar. Then, if Condition 2 is satisfied the control law

$$\dot{\hat{x}}_c = A_e \hat{x}_c + B_2 u + B_3 r + L(y - C_2 \hat{x}_c - D_{22} u) + B_e \theta_c \qquad \hat{x}_c(0) = [\gamma^2 R^{-1} - X(0)]^{-1} \theta_c(0) \tag{41}$$

$$u = -V_1 [(B_2^T X + \bar{D}_{12}^T \bar{C}_1) \hat{x}_c + D_{12}^T D_{13} r + B_2^T \theta_c] \tag{42}$$

where $\theta_c(\cdot)$ is given by (17) with $h = \Delta$, will guarantee the performance

$$\|z\|_2^2 < \gamma^2 [\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_{R^{-1}}^2] + \bar{J}(r, \varepsilon) + \|V_1^{1/2} [B_2^T \theta_1 + (B_2^T X + \bar{D}_{12}^T \bar{C}_1) \hat{x}_1\|_2^2$$

with $\theta_1(\cdot)$, $\bar{J}(r, \varepsilon)$ and $\hat{x}_1(\cdot)$ satisfying (19), (20) and (34), respectively.

4. EXAMPLE

Consider the system of (1) where:

$$A = \begin{bmatrix} 0 & -1+\delta \\ 1 & -0.5 \end{bmatrix}, \quad B_1 = 0.225 \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$D_{12} = 10^{-2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad D_{13} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

We assume that δ is not known and that it may lie anywhere in the interval $[-0.7\quad 0.7]$. We further assume that the two states are available for measurement, so that we can apply state feedback, and we consider the stationary case, where $T \to \infty$. Using the notation of Section 2 we have $H_1 = [1 \quad 0]^T$, $E_1 = [0 \quad 0{\cdot}7]$, $E_2 = 0$ and $H_2 = 0$.

We first solve the tracking problem by applying the method of Reference 7 for the nominal plant ($\delta = 0$), using $\gamma = 0{\cdot}42$ and ignoring the plant uncertainty. We obtain the following result:

$$X_1 = \begin{bmatrix} 3{\cdot}558 & -0{\cdot}967 \\ -0{\cdot}967 & 0{\cdot}365 \end{bmatrix}, \qquad K_{x1} = [483{\cdot}52 - 182{\cdot}5]$$

where X_1 is the solution of the appropriate algebraic Riccati equation and K_{x1} is the corresponding state feedback gain. The resulting closed-loop transfer function from r to the first component of z of (1b) is

$$G_1 = \frac{2s + 19{\cdot}2}{s^2 + 9{\cdot}625s + q_1}$$

where for the nominal case $q_1 = 25{\cdot}176$. For $\delta = 0{\cdot}7$ and $\delta = -0{\cdot}7$ the corresponding values of q_1 become $7{\cdot}553$ and $42{\cdot}8$, respectively. Note that in this case the state-feedback gain K_{x1} is determined for $\delta = 0$, and the resulting values of q_1 are obtained for the same K_{x1} by varying A. The magnitude frequency responses of $G_1 - 1$ for the three values of δ are depicted in Figure 1.

The same problem has been solved using the theory of Section 3 for the same value of γ. The steady-state solution X of (9) and the corresponding gain matrices K_x and K_r of (24) are:

$$X = \begin{bmatrix} 746{\cdot}448 & -192{\cdot}235 \\ -192{\cdot}235 & 49{\cdot}675 \end{bmatrix}, \qquad K_x = [96117 - 24837], \quad \text{and} \quad K_r = 0$$

where the latter is obtained for zero preview. The transference from r to the first component of z of the resulting closed-loop is described by

$$G_2 = \frac{2s + 2484{\cdot}7}{s^2 + 12\,42s + q_2}$$

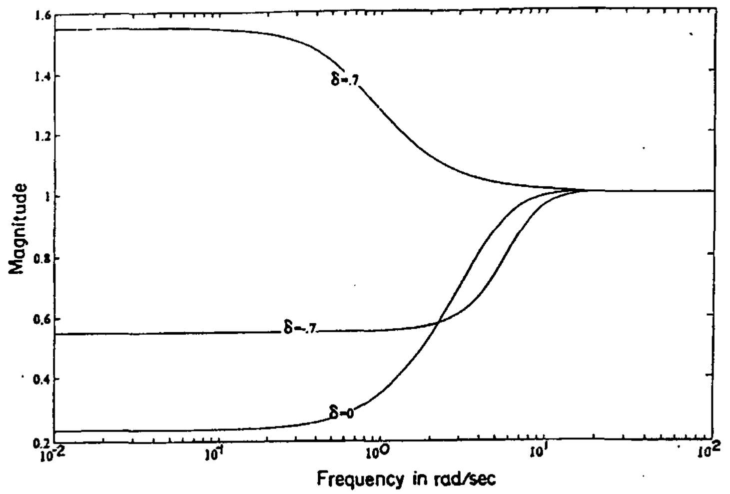

Figure 1. Magnitude frequency plots of the transmission from r to $[1\ 0]C_1x - r$ for three values of δ. Using the method of [7] for $\gamma = 0.42$.

where for the nominal plant $q_2 = 4806{\cdot}9$, and for $\delta = 0{\cdot}7$ and $\delta = -0{\cdot}7$ the values of q_2 are $1442{\cdot}1$ and $8171{\cdot}7$, respectively. The corresponding magnitude frequency responses of $G_2 - 1$ are depicted in Figure 2. Comparing the results of Figures 1 and 2 it becomes clear that the variation and the absolute values of $G_i - 1$, $i = 1, 2$ are much larger at the low-frequency range in the non-robust design of Figure 1. Tracking of, say, $r = \sin(0{\cdot}1t)$, using on line measurement of r, will clearly yield better results when the method of the present paper is used.

We demonstrate next the use of preview on the worst case of $\delta = 0{\cdot}7$. We bring in Figure 3 the

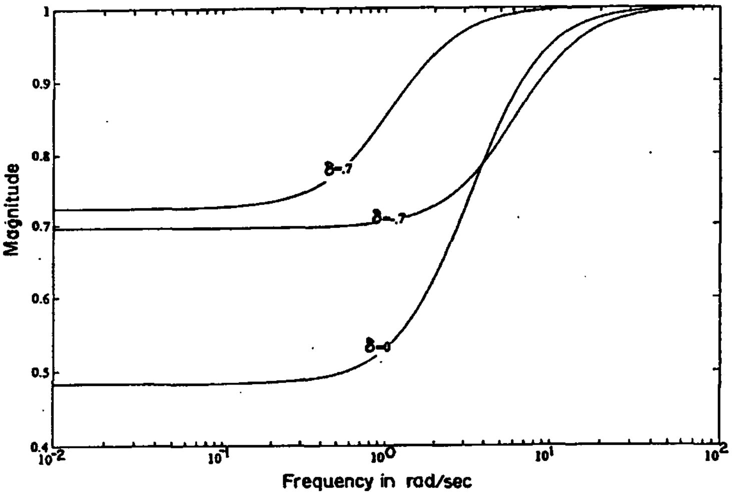

Figure 2. Magnitude frequency plots of the transmission from r to $[1\ 0]C_1x - r$ for three values of δ. Using the method of Section 3 with-out preview.

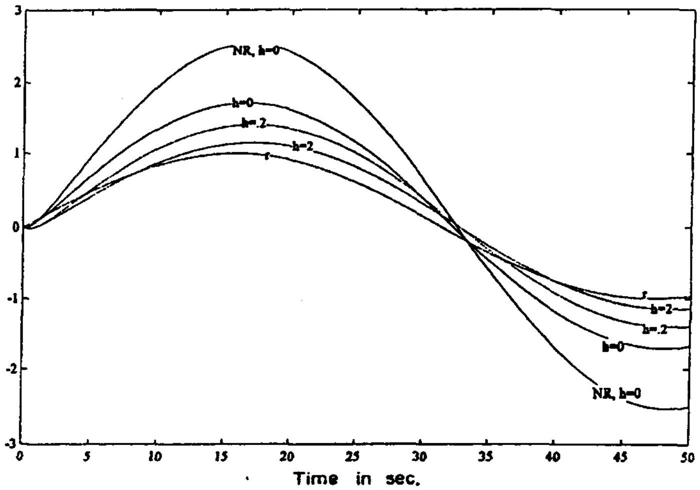

Figure 3. Simulation results for tracking $r = \sin(0.1t)$. The results of the method of Section 3 are derived for preview length of $h = 0$, 0.2, and 2 seconds. These results are compared with the nonrobust design of [7] ($h = 0$).

simulation results that are obtained by the theory of Section 3, for $r = \sin(0{\cdot}1t)$, using preview of $h = 0{\cdot}2$ and $h = 2$ seconds. We also bring in the figure the corresponding results for $h = 0$, using the new method and the result that is obtained by using the above K_{x1} for the non-robust (NR) case.

The example clearly shows the benefits of applying preview, when available. It should be noted here, however, that the preview is usually aimed at improving the tracking performance in the worst case (in our example $\delta = 0{\cdot}7$). Its effect may not be so impressive for other values of the plant parameters.

5. CONCLUSIONS

Problems of robust H_∞ tracking are solved for time-varying linear systems. Three tracking problems are investigated depending on whether the reference signal is perfectly known in advance, measured on line, or previewed in a fixed intervals of time ahead. Dynamic games are defined that convert the plant uncertainty to energy bounded pseudo-disturbance signals. It is shown that the saddle-point equilibrium solutions to these games, if exist, guarantee the prescribed level of tracking performance for the worst possible disturbance and noise signals, in spite of the uncertainty in the plant. Conditions are obtained for the existence of equilibria in these game and the resulting saddle-point tracking strategies are derived.

The resulting H_∞ trackers are in general time-varying. They do not assume an *a priori* knowledge of the reference signal structure, nor do they require an *a priori* information on the statistics of the unmeasurable exogenous signals. The obtained results supplement the two existing methods for worst-case tracking. The first method allows the reference to be a part of nature's strategy in its attempt to maximize the pay-off function. This approach does not fully utilize the information that is obtained by measuring the reference signal. The other approach is to assume some model whose response to a random input yields the reference signal. The first approach usually leads to an overdesign, whereas the model that is assumed by the second approach is inaccurate and in many cases is hardly available. The results of the present paper can be used to reduce the overdesign that is entailed in the first approach and the difficulties that are encountered, in practice, using the second approach. These results fully utilize the information that is gathered by measuring the reference signal and they provide the best possible tracking under the prescribed plant uncertainty.

The method that is developed in the paper can also deal with cases where some *a priori* information is available on the reference signal. This information should not be necessarily accurate and it may involve some parameter uncertainty. The uncertain model that is known to produce the signal can be incorporated into the state-space description of the system, together with its parameter uncertainty, and the method of the paper can be applied to the augmented system.

The theory of the paper can also be used in cases of measurable disturbances. The measurable part of the disturbance is then off-limits for nature which can only use the unmeasurable part of the disturbance in its game against the controller. This can be achieved by considering the measurable part of the disturbance as a reference signal that is also an input to the system. Unlike the tracking problem this signal does not appear as a part of the controlled output. Since we allow for arbitrary reference and measurable disturbance signals, there is hardly any advantage in using the measured signals in the state feedback case (with zero preview) when $D_{12}^{\mathrm{T}} D_{13} = 0$. In the output feedback case, both measured signals appear in the observer equation of (36), independently of the geometry of D_{12} and D_{13}.

In the present paper we have treated time-varying systems in a finite-time framework. A question may arise what happens when T tends to infinity. The answer to this question is entailed in the fact that the feedback loop of our solution is identical to the one obtained in the robust regulator of Reference 5. It is shown in Reference 5 that under mild assumptions this loop is robust in the limiting infinite-horizon case.

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