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H_∞ ESTIMATION FOR DISCRETE-TIME LINEAR UNCERTAIN SYSTEMS

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SUMMARY

This paper is concerned with the problem of H_∞ estimation for linear discrete-time systems with time-varying norm-bounded parameter uncertainty in both the state and output matrices. We design an estimator such that the estimation error dynamics is quadratically stable and the induced operator norm of the mapping from noise to estimation error is kept within a prescribed bound for all admissible uncertainties. A Riccati equation approach is proposed to solve the estimation problem and it is shown that the solution is related to two algebraic Riccati equations.

KEY WORDS H_∞ estimation Uncertain discrete-time systems Robust estimation Algebraic Riccati equations

1. INTRODUCTION

The problem of optimal filtering has been well studied over the past decades with much attention being focused on systems subject to input and measurement noises, which are assumed to be 'white' processes with known spectral density; see, for instance, the celebrated Kalman filtering approach.[1] In many situations, however, the statistics of the noise sources are not fully known. In order to cope with this problem, considerable interest has been devoted to the problem of estimation for systems with noise of partially unknown statistics, e.g. noise with bounded energy or bounded amplitude; see, for instance, References 3, 6, 8, 10, 12, 13 and 15–19. More specifically, H_∞ estimation has been developed to deal with energy-bounded noises, i.e. where only upper bounds on the spectral density of the input and the measurement noises are known.

Most of the work in H_∞ estimation has been focused on continuous-time systems. Initially, a frequency-domain approach was proposed to solve this problem, e.g. Reference 12. Following the dramatic development of the H_∞ control theory (see, for example, References 5 and 7 and the references therein), many results on the H_∞ estimation have been derived based on the Riccati equation approach; see, for example, References 3 and 15–19. In addition, the interpolation theory has recently been used by Reference 8 in H_∞ estimation for both continuous-time and discrete-time systems. The discrete-time H_∞ estimation has also been discussed in References 6 and 13 using a polynomial equation approach whereas a Riccati equation approach has been used in Reference 19. Note that all of the above work was accomplished for systems where the only uncertainty in the model is in the form of a bounded energy noise and thus it cannot be applied directly to systems with parameter uncertainty.

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Very recently, there has been some attempt to deal with the H_∞ estimation of uncertain systems. The basic characteristics of this problem is that the system considered contains not only uncertainty in the form of a bounded energy noise but also parameter uncertainty. This problem is referred to as ***robust H_∞ estimation*** and has been solved via both the interpolation theory and the Riccati equation approaches in the continuous-time context in Reference 10. The focal point of Reference 10 is to convert the parameter uncertainty to an extra parameter (constant) scaled noise with bounded energy.

In this paper we solve the problem of robust H_∞ estimation for discrete-time systems. The linear discrete-time systems under consideration are subject to time-varying norm-bounded parameter uncertainty and input and measurement noises with bounded energy. The parameter uncertainty appears in both the state and output matrices. We are concerned with the following robust H_∞ estimation problem: designing an estimator such that the estimation error dynamics is quadratically stable and the induced operator norm of the mapping from the noise to the estimation error is kept within a prescribed bound for all admissible parameter uncertainties. The paper can be regarded as the discrete-time counterpart of Reference 10. Although the interpolation approach can also be used as in Reference 10, only the Riccati equation approach is presented for simplicity. Similar to the continuous time case,[10] two algebraic Riccati equations (AREs) are required to solve the problem, which raises no duality to the state feedback robust H_∞ control problem[9] where only one Riccati equation suffices. Compared with the H_∞ estimator for systems without parameter uncertainty, the results in this paper indicate that the estimator structure should take into account the parameter uncertainty. Finally, we point out that when there is no parameter uncertainty in the system, the robust H_∞ estimator proposed in this paper will recover well-known H_∞ estimation results.

2. PROBLEM AND PRELIMINARIES

Consider the class of discrete-time uncertain systems described by a state-space model of the the form

$$(\Sigma_1)\colon\ x(k+1) = [A + \Delta A(k)]x(k) + Bw(k) \tag{1a}$$

$$y(k) = [C + \Delta C(k)]x(k) + Dw(k) \tag{1b}$$

$$z(k) = Lx(k) \tag{1c}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the noise which belongs to $l_2[0, \infty)$, $y(k) \in \mathbb{R}^r$ is the measured output, $z(t) \in \mathbb{R}^p$ is a linear combination of the state variables to be estimated, A, B, C, D and L are known real constant matrices that describe the nominal system and $\Delta A(k)$ and $\Delta C(k)$ represent the time-varying parameter uncertainties which have the following structure:

$$\begin{bmatrix} \Delta A(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(k)E \tag{2}$$

with $F(\cdot)\colon \mathbf{Z} \to \mathbb{R}^{i\times j}$ being an unknown matrix satisfying

$$F^{\mathrm{T}}(k)F(k) \leqslant I, \qquad \forall k = 0, 1, \ldots \tag{3}$$

and H_1, H_2 and E being known real constant matrices with appropriate dimensions.

In this paper, we are concerned with designing an estimator for $z(k)$ of the form

$$(\Sigma_e)\colon\ x_e(k+1) = A_e x_e(k) + K_e y(k) \tag{4a}$$

$$z_e(k) = L_e x_e(k) \tag{4b}$$

where $x_e(k) \in \mathbb{R}^l$ is the estimator state, the matrices A_e, K_e and L_e and the dimension l are to be chosen. The estimation error is defined by

$$e(k) \triangleq z(k) - z_e(k) = Lx(k) - L_e x_e(k) \tag{5}$$

Before proposing the problem of robust H_∞ estimation, we introduce the following definition of stability for uncertain systems.

Definition 2.1.[2]

Consider the uncertain system

$$x(k+1) = [A + \Delta A(k)]x(k)$$

This system is said to be *quadratically stable* if there exists a matrix $P = P^T > 0$ and a scalar $\alpha > 0$ such that this system admits a Lyapunov function $V(x) = x^T Px$ satisfying

$$\begin{aligned} V[x(k+1)] - V[x(k)] &= x^T(k)\{[A + \Delta A(k)]^T P[A + \Delta A(k)] - P\}x(k) \\ &< -\alpha x^T(k)x(k) \end{aligned}$$

for all admissible uncertainty $\Delta A(k)$, $x(k) \in \mathbb{R}^n$ and $k = 0, 1, 2, \ldots$. □

In this paper, we consider the following robust H_∞ estimation problem: given a prescribed level of noise attenuation $\gamma > 0$, find an estimator of the form (4) for the system (1) such that the following conditions are satisfied:

(a) *The augmented system of* (Σ_1) *and the estimator* (Σ_e) *is quadratically stable*;
(b) *With zero initial conditions for* $x(k)$ *and* $x_e(k)$, *the induced operator norm of the mapping* $\mathscr{G}$ *from the noise*, w, *to the estimation error*, e, *satisfies the following condition*:

$$\| \mathscr{G} \| < \gamma \tag{6}$$

for all admissible $F(k)$ *satisfying (3).*

Note that the condition (a) of the above definition is needed to guarantee the uniform asymptotic stability of the error dynamics in the presence of time-varying parameter uncertainty; see, for example, References 2 and 9.

In connection with the robust H_∞ estimation problem for the system (1), we introduce the following parameterized discrete-time system

$$(\Sigma_2): \; x(k+1) = Ax(k) + \begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \tilde{w}(k) \tag{7a}$$

$$y(k) = Cx(k) + \begin{bmatrix} D & \frac{\gamma}{\epsilon} H_2 \end{bmatrix} \tilde{w}(k) \tag{7b}$$

where $x(k) \in \mathbb{R}^n$ is the state, $\tilde{w}(k) \in \mathbb{R}^{m+i}$ is the noise which belongs to $l_2[0, \infty)$, $y(k) \in \mathbb{R}^r$ is the measured output, A, B, C, D, H_1 and H_2 are the same as in the system (1), $\varepsilon > 0$ is a parameter to be chosen and $\gamma > 0$ is the level of noise attenuation we wish to achieve for the estimator. Associated with the system (7), let the following linear combination of the state variables of (Σ_2) be estimated:

$$\tilde{z}(k) = \begin{bmatrix} \tilde{z}_1(k) \\ \tilde{z}_2(k) \end{bmatrix} = \begin{bmatrix} L \\ \varepsilon E \end{bmatrix} x(k) \tag{8}$$

Moreover, we define an estimate of $\tilde{z}(k)$ obtained from the estimator (4) as given by

$$\tilde{z}_e(k) = \begin{bmatrix} z_e(k) \\ 0 \end{bmatrix} = \begin{bmatrix} L_e \\ 0 \end{bmatrix} x_e(k) \tag{9}$$

Hence, the corresponding estimation error reads

$$\tilde{e}(k) = \tilde{z}(k) - \tilde{z}_e(k) = \begin{bmatrix} \tilde{e}_1(k) \\ \tilde{e}_2(k) \end{bmatrix} = \begin{bmatrix} L & -L_e \\ \varepsilon E & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix} \in \mathbb{R}^{(p+j)} \tag{10}$$

Also, we denote by $\tilde{T}(z)$ the transfer function from the noise, $\tilde{w}$, to the estimation error, $\tilde{e}$.

The following theorem establishes the relationship between the robust H_∞ estimation problem associated with (1) and the H_∞ estimation problem associated with (7) and estimation error (10).

Theorem 2.1.

Given a prescribed level of noise attenuation $\gamma > 0$, if there exist some constant $\varepsilon > 0$ and an estimator in the form of (4) such that

$$\| \tilde{T}(z) \|_\infty < \gamma \tag{11}$$

then the robust H_∞ estimation problem for the uncertain system (1) is solvable with the same estimator (4).

Proof See Appendix A.

In the above, $\| H(z) \|_\infty \triangleq \sup_{0 \leqslant \omega \leqslant 2\pi} \sigma_{\max}[H(e^{j\omega})]$, where $\sigma_{\max}(\cdot)$ stands for the maximum singular value.

By Theorem 2.1, the robust H_∞ estimation problem for system (Σ_1) can be solved by finding a constant $\varepsilon > 0$ such that the H_∞ estimation problem for the system (7) with estimation error (10) is solvable via the estimator (Σ_e). Although there is no parameter uncertainty in the system (7), the zero entry in (9) makes this estimation problem a non-standard one. As it can be seen from (10), the estimator has no influence on $\tilde{e}_2$, which renders the choice of the matrices in (Σ_e) less trivial.

In the remainder of this section we shall list some results on H_∞ estimation for linear time-invariant discrete-time systems. These results can be established by dualizing the H_∞ control results for discrete-time systems in References 11, 14 and 20.

Consider the system

$$(\Sigma_3):\ x(k+1) = Ax(k) + Bw(k) \tag{12a}$$

$$y(k) = Cx(k) + Dw(k) \tag{12b}$$

$$z(k) = Lx(k) \tag{12c}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the noise which belongs to $l_2[0, \infty)$, $y(k) \in \mathbb{R}^r$ is the measured output, $z(k) \in \mathbb{R}^p$ is a linear combination of state variables to be estimated, A, B, C, D and L are constant matrices with appropriate dimensions. We also make the following assumptions:

Assumptions A

(A.1) (C, A) is detectable;
(A.2) $R = DD^{\mathrm{T}} > 0$;

(A3) $\operatorname{rank}\begin{bmatrix} A - e^{j\omega} I & B \\ C & D \end{bmatrix} = n + m, \qquad \forall \omega \in [0, 2\pi].$

With the estimator (4), the estimation error associated with (Σ_3) is given by

$$e(k) = z(k) - z_e(k) = Lx(k) - L_e x_e(k) \tag{13}$$

Also, let $G(z)$ be the transfer function from the noise, w, in (12) to the estimation error, e. Then, the following result holds.

Theorem 2.2.

Consider the system (Σ_3) satisfying Assumptions A and let $\gamma > 0$ be a prescribed level of noise attenuation. Then there exists an estimator (Σ_e) for $z(k)$ such that

$$\| G(z) \|_\infty < \gamma$$

if and only if there exists a solution $P = P^{\mathrm{T}} \geqslant 0$ to the ARE

$$P = APA^{\mathrm{T}} - (APC_l^{\mathrm{T}} + BD_l^{\mathrm{T}})(C_l P C_l^{\mathrm{T}} + R_l)^{-1}(APC_l^{\mathrm{T}} + BD_l^{\mathrm{T}})^{\mathrm{T}} + BB^{\mathrm{T}} \tag{14}$$

such that

(a) $U = I - \gamma^{-2} LPL^{\mathrm{T}} > 0$;

(b) The matrix

$$A - (APC_l + BD_l^{\mathrm{T}})(C_l P C_l^{\mathrm{T}} + R_l)^{-1} C_l$$

is asymptotically stable, where

$$C_l = \begin{bmatrix} C \\ \gamma^{-1} L \end{bmatrix}$$

$$D_l = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

$$R_l = \begin{bmatrix} DD^{\mathrm{T}} & 0 \\ 0 & -I \end{bmatrix}$$

Moreover, if the above conditions are satisfied, a suitable estimator is given by

$$x_e(k+1) = (A - K_e C)x_e(k) + K_e y(k) \tag{15a}$$

$$z_e(k) = Lx_e(k) \tag{15b}$$

where

$$K_e = (BD^{\mathrm{T}} + AVC^{\mathrm{T}})(CVC^{\mathrm{T}} + DD^{\mathrm{T}})^{-1} \tag{15c}$$

$$V = P + \gamma^{-2} PL^{\mathrm{T}} U^{-1} LP \tag{15d}$$

Remark 2.1.

Note that Assumption (A.2) together with condition (a) will guarantee the non-singularity of the matrix $C_l P C_l^{\mathrm{T}} + R_l$. Moreover, it is easy to see that when $\gamma \to \infty$ the above result recovers that of the Kalman filter for system (Σ_3). □

Finally, we introduce the following assumptions for the system (1) which will be used to guarantee the existence of a desired estimator and the stability of the estimation error dynamics:

Assumptions B

(B.1) The nominal system matrix A is stable and invertible;
(B.2) (C, A) is detectable;
(B.3) $DD^{\mathrm{T}} > 0$.

Note that Assumption (B.3) is similar to the standard assumption in Kalman filtering for the nominal system (A, B, C, D) which amounts to that all components of the measured output are noisy. It should be pointed out that cases with noise-free measurements are rarely encountered in practice.

3. RICCATI EQUATION APPROACH TO ROBUST H_∞ ESTIMATION

This section is devoted to solving the robust H_∞ estimation problem via a Riccati equation approach. This will be accomplished by using Theorem 2.1 which relates the robust H_∞ estimation problem to an H_∞ estimation problem and the latter is solved via two algebraic Riccati equations.

Consider the system (Σ_2) and denote by $G_1(z)$ and $G_2(z)$ the transfer functions from $\tilde{w}(k)$ to $y(k)$ and to $\tilde{z}(k)$ respectively, i.e.

$$G_1(z) = C(zI - A)^{-1}\begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} + \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \tag{16}$$

$$G_2(z) = \begin{bmatrix} G_{21}(z) \\ G_{22}(z) \end{bmatrix} = \begin{bmatrix} L(zI - A)^{-1}\begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \\ \varepsilon E(zI - A)^{-1}\begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \end{bmatrix} \tag{17}$$

Observe that $G_1(z)$ and $G_2(z)$ are parameterized by ε. Also, consider that the estimator (Σ_e) is used to estimate $\tilde{z}_1(k)$ and we denote by $G_e(z)$ the transfer function between $y(k)$ and $z_e(k)$ of (Σ_e), i.e.

$$z_e(z) = G_e(z)y(z) \tag{18}$$

With the above notations, (9) and (10) imply that the transfer function from $\tilde{w}(k)$ to $\tilde{e}(k)$ is given by

$$\tilde{T}(z) = \begin{bmatrix} \tilde{T}_1(z) \\ \tilde{T}_2(z) \end{bmatrix} = \begin{bmatrix} G_{21}(z) - G_e(z)G_1(z) \\ G_{22}(z) \end{bmatrix} \tag{19}$$

By taking into account that

$$\tilde{T}^{\mathrm{T}}(1/z)\tilde{T}(z) = \tilde{T}_1^{\mathrm{T}}(1/z)\tilde{T}_1(z) + \tilde{T}_2^{\mathrm{T}}(1/z)\tilde{T}_2(z) \tag{20}$$

it is now clear that to find an estimator (Σ_e) such that for some $\varepsilon > 0$,

$$\| \tilde{T}(z) \|_\infty < \gamma \tag{21}$$

it is necessary that

$$\| G_{22}(z) \|_\infty < \gamma \tag{22}$$

for the same $\varepsilon > 0$.

It should be noted that condition (22) will allow us to decompose $[I - \gamma^{-2}G_{22}^{\mathrm{T}}(1/z)G_{22}(z)]^{-1}$ in the following co-spectral factorization form

$$[I - \gamma^{-2}G_{22}^{\mathrm{T}}(1/z)G_{22}(z)]^{-1} = V(z)V^{\mathrm{T}}(1/z) \tag{23}$$

where $V(z)$ is a $(m+i) \times (m+i)$ invertible rational matrix with both $V(z)$ and $V^{-1}(z)$ being stable.[7]

Hence, it follows that (21) holds if and only if

(i) $\| G_{22}(z) \|_\infty < \gamma$ and (24a)

(ii) $\| [G_{21}(z) - G_e(z) G_1(z)] V(z) \|_\infty < \gamma$ (24b)

Remark 3.1.

Note that finding an estimator (Σ_e) to satisfy (24b) is equivalent to designing an estimator (Σ_e) for the linear combination, $\tilde{z}_1(k)$, of the state variables of (Σ_2) such that $\| \tilde{z}_1 - z_e \|_2 < \gamma \| \hat{w} \|_2$, where $\hat{w}$ is a new noise belonging to $l_2[0, \infty)$ that generates $\tilde{w}$ in (Σ_2) via $V(z)$, i.e.

$$\tilde{w}(z) = V(z) \hat{w}(z) \tag{25}$$

In the above, $\|\cdot\|_2$ stands for the usual $l_2[0, \infty)$ norm.

This implies that the H_∞ estimation problem for (7) can be solved using existing results on H_∞ estimation, provided that a state-space realization for $V(z)$ is available. □

The following lemma provides a suitable state-space realization for the co-spectral factor $V(z)$.

Lemma 3.1.

Consider the transfer function matrix $G_{22}(z)$ satisfying Assumption (B.1) and $\| G_{22}(z) \|_\infty < \gamma$. Then, there exists a co-spectral factor $V(z)$ of (23) with state-space realization

$$V(z) = \tilde{C}(zI - \tilde{A})^{-1} \tilde{B} + \tilde{D} \tag{26}$$

where

$$\tilde{D} = (I - \gamma^{-2} \bar{B}^{\mathrm{T}} P \bar{B})^{-1/2} \tag{27}$$

$$\bar{B} = \begin{bmatrix} B & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \tag{28}$$

$$\tilde{B} = \gamma^{-1} \bar{B} \tilde{D} \tag{29}$$

$$\tilde{C} = \tilde{D} \tilde{B}^{\mathrm{T}} P A \tag{30}$$

$$\tilde{A} = A + \tilde{B} \tilde{B}^{\mathrm{T}} P A \tag{31}$$

and $P = P^{\mathrm{T}} \geqslant 0$ is a solution to the following ARE:

$$A^{\mathrm{T}} P A - P + \gamma^{-2} A^{\mathrm{T}} P \bar{B} (I - \gamma^{-2} \bar{B}^{\mathrm{T}} P \bar{B})^{-1} \bar{B}^{\mathrm{T}} P A + \varepsilon^2 E^{\mathrm{T}} E = 0 \tag{32}$$

which guarantees that

$$I - \gamma^{-2} \bar{B}^{\mathrm{T}} P \bar{B} > 0 \tag{33a}$$

and that

$$A + \gamma^{-2} \bar{B} (I - \gamma^{-2} \bar{B}^{\mathrm{T}} P \bar{B})^{-1} \bar{B}^{\mathrm{T}} P A \tag{33b}$$

is asymptotically stable.

Proof. See Appendix B.

Remark 3.2.

It should be noted that the existence of the matrix P in Lemma 3.1 is guaranteed by the given assumptions. In fact, by the results in Reference 4, the existence of such a matrix P is equivalent to $\| G_{22}(z) \|_\infty < \gamma$ and A being stable. □

By considering (25) and assuming $\| G_{22}(z) \|_\infty < \gamma$, Lemma 3.1 leads to the following state-space representation for the system (7) and $\tilde{z}_1(k)$:

$$(\Sigma_4)\colon\ x_a(k+1) = A_a x_a(k) + B_a \hat{w}(k \tag{34a}$$

$$y(k) = C_a x_a(k) + D_a \hat{w}(k) \tag{34b}$$

$$\tilde{z}_1(k) = L_a x_a(k) \tag{34c}$$

where

$$x_a(k) = [x^T(k) \quad \eta^T(k)]^T \tag{34d}$$

$$A_a = \begin{bmatrix} A & \bar{B}\tilde{C} \\ 0 & \tilde{A} \end{bmatrix} \tag{34e}$$

$$B_a = \begin{bmatrix} \bar{B}\tilde{D} \\ \tilde{B} \end{bmatrix} \tag{34f}$$

$$C_a = \begin{bmatrix} C & \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \tilde{C} \end{bmatrix} \tag{34g}$$

$$D_a = \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \tilde{D} \tag{34h}$$

$$L_a = [L \quad 0] \tag{34i}$$

and with $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ given as in Lemma 3.1.

It can be observed from (7), (24b) and the definition of $V(z)$ that there are n pairs of stable zero-pole cancellation in (34). Therefore, the system (34) can be reduced to nth order by using a linear transformation on $x_a(k)$. Indeed, by introducing the following transformation:

$$U = \begin{bmatrix} I & \gamma I \\ 0 & I \end{bmatrix}$$

and considering zero initial conditions, the system (34) reduces to:

$$(\Sigma_5)\colon\ x(k+1) = \hat{A}x(k) + \hat{B}\hat{w}(k) \tag{35a}$$

$$y(k) = \hat{C}x(k) + \hat{D}\hat{w}(k) \tag{35b}$$

$$\tilde{z}_1(k) = Lx(k) \tag{35c}$$

where

$$\hat{A} = \tilde{A} = A + \gamma^{-2}\bar{B}(I - \gamma^{-2}\bar{B}^T P \bar{B})^{-1}\bar{B}^T P A \tag{35d}$$

$$\hat{B} = \gamma\tilde{B} = \bar{B}(I - \gamma^{-2}\bar{B}^T P \bar{B})^{-1/2} \tag{35e}$$

$$\hat{C} = C + \gamma^{-2}\begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix}(I - \gamma^{-2}\bar{B}^T P \bar{B})^{-1}\bar{B}^T P A \tag{35f}$$

$$\hat{D} = \begin{bmatrix} D & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix}(I - \gamma^{-2}\bar{B}^T P \bar{B})^{-1/2} \tag{35g}$$

Now, in view of Remark 3.1, the design of an estimator (Σ_e) for the system (7) to guarantee that (24b) holds is reduced to finding an estimator (Σ_e) for the system (Σ_5) and estimation error

$$\tilde{e}_1(k) = Lx(k) - L_e x_e(k) \tag{35h}$$

such that $\| \tilde{e}_1 \|_2 < \gamma \| \hat{w} \|_2$. Note that this problem is a standard H_∞ estimation problem and therefore (Σ_e) can now be determined by using Theorem 2.2. A complete solution is presented in the next theorem. In particular, the estimator (Σ_e) is of nth order.

Theorem 3.1.

Consider the uncertain system (1) satisfying Assumptions B. Given a prescribed level of noise attenuation $\gamma > 0$, the associated robust H_∞ estimation problem is solvable if for some $\varepsilon > 0$ the following conditions are satisfied:

(a) There exists a solution $P = P^{\mathrm{T}} \geqslant 0$ to the ARE:

$$A^{\mathrm{T}}PA - P + \gamma^{-2}A^{\mathrm{T}}P\bar{B}(I - \gamma^{-2}\bar{B}^{\mathrm{T}}P\bar{B})^{-1}\bar{B}^{\mathrm{T}}PA + \varepsilon^2 E^{\mathrm{T}}E = 0 \tag{36}$$

such that

$$I - \gamma^{-2}\bar{B}^{\mathrm{T}}P\bar{B} > 0 \tag{37}$$

and the matrix $A + \gamma^{-2}\bar{B}(I - \gamma^{-2}\bar{B}^{\mathrm{T}}P\bar{B})^{-1}\bar{B}^{\mathrm{T}}PA$ is stable;

(b) There exists a solution $Q = Q^{\mathrm{T}} \geqslant 0$ to the ARE:

$$Q = \hat{A}Q\hat{A}^{\mathrm{T}} - (\hat{A}Q\hat{C}_l^{\mathrm{T}} + \hat{B}\hat{D}_l^{\mathrm{T}})(\hat{C}_lQ\hat{C}_l^{\mathrm{T}} + \hat{R}_l)^{-1}(\hat{A}Q\hat{C}_l^{\mathrm{T}} + \hat{B}\hat{D}_l^{\mathrm{T}})^{\mathrm{T}} + \hat{B}\hat{B}^{\mathrm{T}} \tag{38}$$

such that

(i) $\hat{U} = I - \gamma^{-2}LQL^{\mathrm{T}} > 0$;

(ii) The matrix

$$\hat{A} - (\hat{A}Q\hat{C}_l^{\mathrm{T}} + \hat{B}\hat{D}_l^{\mathrm{T}})(\hat{C}_lQ\hat{C}_l^{\mathrm{T}} + \hat{R}_l)^{-1}\hat{C}_l$$

is asymptotically stable, where

$$\hat{C}_l = \begin{bmatrix} \hat{C} \\ \gamma^{-1}L \end{bmatrix}, \qquad \hat{D}_l = \begin{bmatrix} \hat{D} \\ 0 \end{bmatrix}, \qquad \hat{R}_l = \begin{bmatrix} \hat{D}\hat{D}^{\mathrm{T}} & 0 \\ 0 & -I \end{bmatrix}$$

In the above, $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$ are the same as in (35). Moreover, if the above conditions are satisfied, a suitable estimator is given by:

$$x_e(k+1) = \hat{A}x_e(k) + K_e[y(k) - \hat{C}x_e(k)] \tag{39a}$$

$$z_e(k) = Lx_e(k) \tag{39b}$$

where

$$K_e = (\hat{B}\hat{D}^{\mathrm{T}} + \hat{A}\hat{V}\hat{C}^{\mathrm{T}})(\hat{C}\hat{V}\hat{C}^{\mathrm{T}} + \hat{D}\hat{D}^{\mathrm{T}})^{-1} \tag{39c}$$

$$\hat{V} = Q + \gamma^{-2}QL^{\mathrm{T}}\hat{U}^{-1}LQ \tag{39d}$$

Remark 3.3.

Note that the estimator (39) can be rewritten as

$$x_e(k+1) = (A + \Delta A_{\mathrm{worst}})x_e(k) + K_e[y(k) - (C + \Delta C_{\mathrm{worst}})x_e(k)] \tag{40}$$

where

$$\Delta A_{\text{worst}} = \gamma^{-2}\bar{B}(I - \gamma^{-2}\bar{B}^{\mathrm{T}}P\bar{B})^{-1}\bar{B}^{\mathrm{T}}PA \tag{41}$$

$$\Delta C_{\text{worst}} = \gamma^{-2}\begin{bmatrix} D & \frac{\gamma}{\varepsilon}H_2 \end{bmatrix}(I - \gamma^{-2}\bar{B}^{\mathrm{T}}P\bar{B})^{-1}\bar{B}^{\mathrm{T}}PA \tag{42}$$

Similar to the continuous-time case,[10] ΔA_{worst} and ΔC_{worst} can be interpreted as the worst case uncertainty in the state and output matrices, respectively. Also, it can be seen from (39c) that the estimator gain matrix, K_e, depends on the uncertainty. When there is no parameter uncertainty in system (1), then $P = 0$ and thus the condition (a) in Theorem 3.1 will be superfluous. In this case, the estimator (39) will reduce to that in Section 2. □

Remark 3.4.

The result in Theorem 3.1 can be easily extended to the case when the uncertainty in (2) and (3) is of a block-diagonal form, i.e.

$$F(k) = \operatorname{diag}\{F_1(k), F_2(k), \ldots, F_\alpha(k)\}$$

with $F_l^{\mathrm{T}}(k)F_l(k) \leqslant I, l = 1, 2, \ldots, \alpha$. However, in this situation the corresponding scaled H_∞ estimation problem will involve α scaling parameters. This can be obtained by applying Theorem 3.1 on the scaled (2) given by the following:

$$\begin{bmatrix} \Delta A(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} H_1(\varepsilon) \\ H_2(\varepsilon) \end{bmatrix} F(k)E(\varepsilon) \tag{43}$$

where

$$H_l(\varepsilon) = H_l \operatorname{diag}\{\varepsilon_1^{-1}I_{i_1\times i_1}, \varepsilon_2^{-1}I_{i_2\times i_2}, \ldots, \varepsilon_\alpha^{-1}I_{i_\alpha\times i_\alpha}\}, l = 1, 2$$
$$E(\varepsilon) = \operatorname{diag}\{\varepsilon_1 I_{j_1\times j_1}, \varepsilon_2 I_{j_2\times j_2}, \ldots, \varepsilon_\alpha I_{j_\alpha\times j_\alpha}\}E$$
$$\varepsilon_l > 0, l = 1, 2, \ldots, \alpha$$

The details are omitted. □

Remark 3.5.

It should be noted that the choices of the structure matrices H_1, H_2 and E in (2) are not unique. Therefore, the following interesting question arises: does the choice of these matrices affect the solvability of the robust H_∞ estimation problem? The answer to this problem is not clear in the general case. However, it is easy to verify that rescaling and the so-called unitary transformation of these matrices do not affect the solvability. By rescaling, we mean to rewrite H_1, H_2 and E, respectively, as $H_1/\lambda, H_2/\lambda$ and λE for some $\lambda > 0$; and by unitary transformation, we mean to rewrite $[H_1^{\mathrm{T}}, H_2^{\mathrm{T}}]^{\mathrm{T}}, E$ and $F(k)$, respectively, as $[H_1^{\mathrm{T}}, H_2^{\mathrm{T}}]^{\mathrm{T}}U_1$, U_2E and $U_1^{-1}F(k)U_2^{-1}$ for some unitary matrices U_1 and U_2. Indeed, the scaling parameter λ can be absorbed by ε; and the introduction of U_1 and U_2 does not change (3) and the AREs (36) and (38). □

CONCLUSION

In this paper we have solved the discrete-time H_∞ estimation problem for systems subject to time-varying norm-bounded parameter uncertainty in both the state and output matrices. It is shown that the robust H_∞ estimation problem can be converted to solving a discrete-time co-spectral factorization and an H_∞ estimation problem for a discrete-time system without parameter uncertainty. A Riccati equation approach has been proposed and a solution to the robust H_∞ estimation problem has been given in terms of two algebraic Riccati equations. Since the state feedback robust H_∞ control can be solved in terms of only one algebraic Riccati equation, no duality result has arisen between this problem and the corresponding robust H_∞ estimation problem.

APPENDIX A. PROOF OF THEOREM 2.1.

The augmented system associated with (4), (7) and (10) is given by

$$\xi(k+1) = A_c\xi(k) + \begin{bmatrix} B_c & \frac{\gamma}{\varepsilon}H_c \end{bmatrix}\tilde{w}(k) \tag{A.1a}$$

$$\tilde{e}(k) = \begin{bmatrix} C_c \\ \varepsilon E_c \end{bmatrix}\xi(k) \tag{A.1b}$$

where

$$\xi(k) = [x^T(k) \quad x_e^T(k)]^T \tag{A.1c}$$

$$A_c = \begin{bmatrix} A & 0 \\ K_eC & A_e \end{bmatrix} \tag{A.1d}$$

$$B_c = \begin{bmatrix} B \\ K_eD \end{bmatrix} \tag{A.1e}$$

$$H_c = \begin{bmatrix} H_1 \\ K_eH_2 \end{bmatrix} \tag{A.1f}$$

$$C_c = [L \quad -L_e] \tag{A.1g}$$

$$E_c = [E \quad 0] \tag{A.1h}$$

Suppose condition (11) is satisfied for some $\varepsilon > 0$. Then, by Lemma 2.1 of Reference 11 there exists a matrix $X = X^T > 0$ such that

$$A_c^TXA_c + \left\{X^{-1} - \gamma^{-2}\begin{bmatrix} B_c & \frac{\gamma}{\varepsilon}H_c \end{bmatrix}\begin{bmatrix} B_c & \frac{\gamma}{\varepsilon}H_c \end{bmatrix}^T\right\}^{-1} + \begin{bmatrix} C_c \\ \varepsilon E_c \end{bmatrix}^T\begin{bmatrix} C_c \\ \varepsilon E_c \end{bmatrix} < 0 \tag{A.2}$$

Now, using the same argument as in the proof of Theorem 3.2 of Reference 9, (A.2) implies that there exists a symmetric matrix $P > 0$ such that

$$[A_c + H_cF(k)E_c]^TP[A_c + H_cF(k)E_c] - P + \gamma^{-2}PB_c(I + \gamma^{-2}B_c^TPB_c)^{-1}B_c^TP + C_c^TC_c < 0 \tag{A.3}$$

for all $F(k) \in \mathbb{R}^{i\times j}$ satisfying (3).

On the other hand, the augmented system associated with (1), (4) and (5) is of the form

$$\xi(k+1) = [A_c + H_cF(k)E_c]\,\xi(k) + B_cw(k) \tag{A.4a}$$

$$e(k) = C_c\xi(k) \tag{A.4b}$$

Finally, by Lemma 2.1 of Reference 9, it follows from (A.3) that (A.4) is quadratically stable and condition (6) holds.

APPENDIX B. PROOF OF LEMMA 3.1

Initially note that since A is stable and $\| G_{22}(z) \|_\infty < \gamma$, it follows[4] that the required solution $P = P^T \geqslant 0$ to the ARE (32) exists.

Considering that

$$G_{22}(z) = \varepsilon E(zI - A)^{-1}\bar{B}$$

we have that a state-space realization for $[I-\gamma^{-2}G_{22}^{\mathrm{T}}(1/z)G_{22}(z)]^{-1}$ is

$$[I-\gamma^{-2}G_{22}^{\mathrm{T}}(1/z)G_{22}(z)]^{-1}=\left[\begin{array}{cc|c} A^{-\mathrm{T}} & \varepsilon^2A^{-\mathrm{T}}E^{\mathrm{T}}E & 0 \\ -\gamma^{-2}\bar{B}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}} & A-\gamma^{-2}\varepsilon^2\bar{B}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}E^{\mathrm{T}}E & \gamma^{-1}\bar{B} \\ \hline -\gamma^{-1}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}} & -\varepsilon\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}E^{\mathrm{T}}E & I \end{array}\right]$$

where the matrix notation for a state-space realization of a transfer function is used, i.e.

$$C(zI-A)^{-1}B+D\triangleq\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$$

Now, let us introduce the linear transformation matrix

$$U=\begin{bmatrix} -P & I \\ I & 0\end{bmatrix}$$

Using (27)–(32), it is easy to show that

$$U^{-1}\begin{bmatrix} A^{-\mathrm{T}} & \varepsilon^2A^{-\mathrm{T}}E^{\mathrm{T}}E \\ \gamma^{-2}\bar{B}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}} & A-\gamma^{-2}\varepsilon^2\bar{B}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}E^{\mathrm{T}}E\end{bmatrix}U=\begin{bmatrix}\tilde{A} & -\tilde{B}\tilde{B}\tilde{A}^{-\mathrm{T}} \\ 0 & \tilde{A}^{-\mathrm{T}}\end{bmatrix}$$

$$U^{-1}\begin{bmatrix}0 \\ \gamma^{-1}\bar{B}\end{bmatrix}=\begin{bmatrix}\gamma^{-1}\bar{B} \\ \gamma^{-1}P\bar{B}\end{bmatrix}$$

$$=\begin{bmatrix}\tilde{B}(\tilde{D}^{\mathrm{T}}-\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}}) \\ \tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}}\end{bmatrix}$$

$$[-\gamma^{-1}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}} \quad -\varepsilon^2\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}E^{\mathrm{T}}E]\,U=[\gamma^{-1}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}P-\varepsilon^2\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}E^{\mathrm{T}}E \quad -\gamma^{-1}\bar{B}^{\mathrm{T}}A^{-\mathrm{T}}]$$
$$=[\tilde{C} \quad -\tilde{D}\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}}]$$

and

$$\tilde{D}(\tilde{D}^{\mathrm{T}}-\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}})=I$$

Therefore

$$[I-\gamma^{-2}G_{22}^{\mathrm{T}}(1/z)G_{22}(z)]^{-1}=\left[\begin{array}{cc|c}\tilde{A} & -\tilde{B}\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}} & \tilde{B}(\tilde{D}^{\mathrm{T}}-\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}}) \\ 0 & \tilde{A}^{-\mathrm{T}} & \tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}} \\ \hline \tilde{C} & -\tilde{D}\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}} & \tilde{D}(\tilde{D}^{\mathrm{T}}-\tilde{B}^{\mathrm{T}}\tilde{A}^{-\mathrm{T}}\tilde{C}^{\mathrm{T}})\end{array}\right]$$

which is a state-space realization of $V(z)V^{\mathrm{T}}(1/z)$.

Finally, note that both $V(z)$ and $V^{-1}(z)$ are stable because both A and $\tilde{A}$ are stable and $\tilde{D}$ is non-singular.

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