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Quantized feedback control for linear uncertain systems

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SUMMARY

This paper studies robust control problems under the setting of quantized feedback. We consider both the static and dynamic logarithmic quantizers. In the static quantization case, the quantizer has an infinite number of levels, and the design problem is to find the minimal quantization density required to achieve a given control objective. In the dynamic quantization case, the problem is to minimize the number of quantization levels to achieve a given control objective. We present a number of results for different controller-quantizer configurations. These results are developed using the so-called *sector bound approach* for quantized feedback control, which was initiated by the authors previously for systems without uncertainties. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Control using quantized feedback can be traced back to the work of Kalman [1] in 1956, which studied the limit cycle behavior of a system with a finite-alphabet quantizer in the control loop. Widrow [2] in 1961 also studied quantization errors for sampled-data systems using statistical analysis methods. Early work can also be found in Tou [3], Larson [4], Curry [5], and Fischer [6]. Many of these researchers studied linear quadratic control under quantized feedback. More recent works include [7, 8], which aim at understanding and mitigation of quantization effects.

Recently, there is a surge of interest in quantized feedback control, with the aim to understand the required quantization density or information rate for control purposes. Noticeable works include [9–13].

Two most pertinent references to this paper are the work by Elia and Mitter [13] and a follow-up work by Fu and Xie [14]. In [13], the problem of quadratic stabilization of discrete-time single-input–single-output (SISO) linear time-invariant systems using quantized feedback is studied. The quantizer is assumed to be static and time invariant (i.e. memoryless and with fixed quantization levels). It is proved in [13] that for a quadratically stabilizable system, the quantizer needs to be *logarithmic* (i.e. the quantization levels are linear in logarithmic scale). Furthermore,

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the coarsest quantization density is given explicitly in terms of the system's unstable poles. The work of Elia and Mitter [13] is also generalized to some extent to guaranteed performance control [15], stabilization of two-input systems [16], and multi-input systems [17].

In Fu and Xie [14], the work of Elia and Mitter [13] is generalized to general multi-input–multi-output (MIMO) systems and to control problems requiring performances. This is done using the so-called *sector bound method*, which is based on using a simple sector bound to model the quantization error. For a SISO system with quantized state feedback (which is the most fundamental problem), the sector bound method gives an identical result as in [13]. But the main advantage of the sector bound method is that it is easy to understand and easy to generalize to more complicated quantized feedback control scenarios such as those mentioned above.

In this paper, we study the problem of robust stabilization for linear uncertain systems via logarithmic quantized feedback. Our work is based on the sector bound method in [14]. We study both static (memoryless) logarithmic quantizers and dynamic ones. For the former, we give conditions under which there exists a quadratic stabilizing controller for a given quantization density. For the latter, we introduce a simple dynamic scaling method in combination with a static logarithmic quantizer. This allows us to achieve robust stabilization using a finite-level quantizer. Three cases of quantized feedback control are considered: state feedback; output feedback with quantization occurring at the control input; and output feedback with quantization occurring at the measured output. The results on robust stabilization are then generalized to robust performance control for two types of performance measures: H_∞ and linear quadratic costs. Several examples are also given to illustrate our results and to demonstrate how the required quantization density or quantization bit rate increases as the level of system uncertainties increases.

The rest of this paper is organized as follows: Section 2 studies robust stabilization using static quantized feedback. Section 3 uses a dynamic quantizer for the same purpose. Section 4 generalizes the robust stabilization results to robust performance control. Section 5 gives illustrative examples. Section 6 concludes the paper.

2. ROBUST STABILIZATION VIA MEMORYLESS QUANTIZED FEEDBACK

The uncertain system we consider is in the following form:

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k)+(B+\Delta B)u(k) \\ y(k) &= (C+\Delta C)x(k)+(D+\Delta D)u(k) \end{aligned} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}$ is the (single) control input, $y(k)\in\mathbb{R}^r$ is the measured output, the matrices A, B, C, D represent a 'nominal' system, the Δ terms represent the uncertainties in the system, and they satisfy

$$\begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(k)[E_1 \;\; E_2], \quad \|F(k)\|\leqslant 1 \tag{2}$$

where $F(k)\in\mathbb{R}^{n_1\times n_2}$ represents norm-bounded uncertainty and the matrices H_1, H_2, E_1, and E_2 characterize the structure of uncertainty. This type of uncertainty is very common in the robust control literature; see, e.g. [18].

A memoryless quantizer is a static nonlinear mapping from input α to output β, i.e.

$$\beta = Q(\alpha) \tag{3}$$

In this paper, we assume that both α and β are scalar variables for simplicity, i.e. $\alpha\in\mathbb{R}$ and $\beta\in\mathscr{V}$, which are a finite or countable subset of $\mathbb{R}$. Following the works of [13, 14] for quantized feedback control for stabilization, we consider using an infinite-level logarithmic quantizer, which can be written as

$$Q(\alpha) = \begin{cases} \rho^i & \text{if } \dfrac{1}{1+\delta}\rho^i < \alpha \leqslant \dfrac{1}{1-\delta}\rho^i \\ & \quad i = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{if } \alpha = 0 \\ -Q(-\alpha) & \text{if } \alpha < 0 \end{cases} \tag{4}$$

where $0<\rho<1$ represents the *quantization density* of $Q(\cdot)$, and δ is related to ρ by

$$\delta = \frac{1-\rho}{1+\rho} \tag{5}$$

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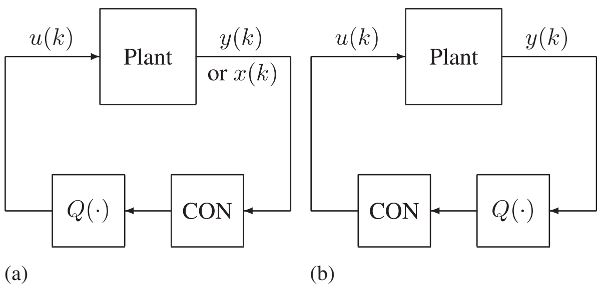

Figure 1. Quantized feedback control: (a) Cases 1 and 2 and (b) Case 3.

The associated quantized set $\mathscr{V}$ is given by

$$\mathscr{V}=\{\pm\rho^i, i=0,\pm 1,\pm 2,\ldots\}\cup\{0\} \tag{6}$$

We study three cases, as depicted in Figure 1:

Case 1: The full state is available for feedback design, i.e. $y(k)=x(k)$ in (1). We will call this case the *quantized state feedback*.

Case 2: The control input is quantized. In this case, the construction of a pre-quantized control signal is done at the output end where the measured output perfectly available. The control signal is then quantized and transmitted to the input side. We will call this case the *output feedback with quantized input*.

Case 3: The measured output is quantized. In this case, the construction of the control signal is done at the input end using quantized output signal. No more quantization happens to the control input signal. We will call this case the *output feedback with quantized output*.

In all the cases above, we use a single quantizer. Obviously, it is possible to have more complicated scenarios. For example, quantization may happen to both measured output and control input [19]; a MIMO system may require multiple quantizers, one for each feedback channel [14].

For Case 1, we use a static quantized state feedback controller, i.e.

$$u(k)=Q(Kx(k)) \tag{7}$$

Dynamic output feedback is allowed for Cases 2 and 3. For Case 2, we use

$$\begin{aligned} x_c(k+1) &= A_c x_c(k)+B_c\bar{y}(k)+B_1 u(k)\\ v(k) &= C_c x_c(k)+D_c\bar{y}(k)+D_1 u(k)\\ u(k) &= Q(v(k)) \end{aligned} \tag{8}$$

where

$$\bar{y}(k)=y(k)-Du(k)=(C+\Delta C)x(k)+\Delta Du(k) \tag{9}$$

$x_c(k)$ is the state of the controller with its dimension with $x_c(0)=0$ and matrices A_c, B_c, C_c, D_c, B_1, and D_1 to be designed. Note that using $\bar{y}(k)$ instead of $y(k)$ does not alter the available feedback information.

For Case 3, we use

$$\begin{aligned} v(k) &= Q(y(k))\\ x_c(k+1) &= A_c x_c(k)+B_c v(k)\\ u(k) &= C_c x_c(k)+D_c v(k) \end{aligned} \tag{10}$$

with $x_c(0)=0$. Note that the B_1 and D_1 terms are not needed because the mapping from v to u is linear.

Denote by $\tilde{x}$ the state of the closed-loop system for any of the three cases above, it can be verified that

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\Delta)\tilde{x}(k)+\tilde{B}(\Delta)\beta(k)\\ \alpha(k) &= \tilde{C}(\Delta)\tilde{x}(k)+\tilde{D}(\Delta)\beta(k)\\ \beta(k) &= Q(\alpha(k)) \end{aligned} \tag{11}$$

where $\tilde{A}(\Delta)$, $\tilde{B}(\Delta)$, $\tilde{C}(\Delta)$, and $\tilde{D}(\Delta)$ depend on $F(k)$ in (2).

Definition 2.1

The uncertain quantized system (11) is said to be quadratically stable if there exists a *constant* Lyapunov matrix $\tilde{P}=\tilde{P}'>0$ such that the corresponding Lyapunov function $V(\tilde{x})=\tilde{x}'\tilde{P}\tilde{x}$ satisfies:

$$\begin{aligned} &V(\tilde{x}(k+1))-V(\tilde{x}(k))<0\\ &\forall\tilde{x}(k)\neq 0,\quad \|F(k)\|\leqslant 1 \end{aligned} \tag{12}$$

along the trajectory of (11).

We now study the three cases separately.

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2.1. *Case 1: Quantized state feedback*

The use of static state feedback implies that $\tilde{x}(k)=x(k)$. In this case, we denote $V(x)=x'Px$. The corresponding closed-loop system (11) becomes

$$x(k+1)=(A+\Delta A)x(k)+(B+\Delta B)Q(Kx(k)) \quad (13)$$

with (2) simplified to

$$[\Delta A \;\; \Delta B]=HF(k)[E_1 \;\; E_2], \quad \|F(k)\|\leqslant 1 \quad (14)$$

The admissible quantization density ρ depends on $V(x)$ (or P) and K. This raises the key question: What is the coarsest density among all possible P and K? In [13], this problem is studied for systems without uncertainties, and the answer is given for a specially chosen K:

$$K=K_{GD}=-\frac{B^{\mathrm{T}}PA}{B^{\mathrm{T}}PB} \quad (15)$$

More specifically, for $K=K_{GD}$, the coarsest density is

$$\rho_{\inf}=\frac{\prod_i |\lambda_i^u|-1}{\prod_i |\lambda_i^u|+1} \quad (16)$$

where λ_i^u are the unstable eigenvalues of A. It is shown in [14] that the result on ρ (or δ) remains the same even when K is allowed to be a free variable.

When the system is subject to uncertainties, the approach in [13] seems to be difficult to generalize. It turns out that the coarsest quantization density is in general difficult to characterize. We therefore aim to search for an upper bound of it, which guarantees quadratic stabilizability. That is, we consider the following problem: Given a (logarithmic) quantization density $\rho>0$, determine (possibly sufficient) conditions under which there exists a quadratically stabilizing quantized state feedback controller with a given quantization density. Once an algorithm is found for this problem, the required quantization density can be easily searched by repeatedly applying the algorithm.

To solve the above problem, we resort to the sector bound method used in [14]. This method uses the following simple observation: For a given quantization density $\rho>0$, the quantization error is bounded by

$$Q(v(k))-v(k)=\Delta(k)v(k), \quad |\Delta(k)|\leqslant\delta \quad (17)$$

for all k, where δ is related to ρ by (5). When there are no uncertainties, it is shown in [14] that the quantized state feedback controller (13) and (15) is quadratically stabilizing if and only if the (unquantized) state feedback controller (15) is quadratically stabilizing in the presence of the sector bound uncertainty (17). That is, the quantized feedback stabilization problem is equivalent to the well-known quadratic stabilization problem with a sector-bounded uncertainty. This is a key observation that allows [14] to generalize the work of [13] to stabilization problem for MIMO systems and performance control problems.

Now let us return to the uncertain system (1). Given a quantization density $\rho>0$, we apply the sector bound (17). For this purpose, we define the following auxiliary system:

$$\begin{aligned} x(k+1) &= Ax(k)+Bv(k)+[B \;\; \tau^{-1}H]w(k) \\ \zeta(k) &= \begin{bmatrix} \delta v(k) \\ \tau(E_1x(k)+E_2v(k)+[E_2 \;\; 0]w(k)) \end{bmatrix} \end{aligned} \quad (18)$$

where $w(k)$ is an exogenous input, $v(k)$ is the control input, $\zeta(k)$ is the controlled output, and $\tau>0$ is a scaling parameter, which can be searched numerically.

Theorem 2.1

The system (1) is quadratically stabilizable for a given a quantization density $\rho>0$ if there exists a scaling parameter $\tau>0$ and a state feedback controller

$$v(k)=Kx(k) \quad (19)$$

for the auxiliary system (18) such that the H_∞ norm of the transfer function from w to ζ is less than 1. Further, the control gain K and Lyapunov matrix P for the auxiliary system will also work for the uncertain system (1).

Proof

Combining (13) and (17) gives

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k) \\ &\quad +(B+\Delta B)(1+\Delta(k))v(k) \end{aligned} \quad (20)$$

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Define

$$\begin{aligned}\bar{\Delta}(k) &= \Delta(k)/\delta\\ \xi(k) &= \Delta(k)v(k) = \bar{\Delta}(k)\delta v(k)\\ \eta(k) &= F(k)(E_1x(k)+E_2v(k)+E_2\xi(k))\\ w(k) &= [\xi^{\mathrm{T}}(k)\ \ \tau\eta^{\mathrm{T}}(k)]^{\mathrm{T}}\end{aligned} \tag{21}$$

for any $\tau>0$. Using (14) and (20) becomes

$$x(k+1)=Ax(k)+Bv(k)+B\xi(k)+H\eta(k)$$

which is the same as (18) using $w(k)$ defined in (21). From (18) and (21), $w(k)$ and $\zeta(k)$ are related by

$$w(k)=\mathrm{diag}\{\bar{\Delta}(k), F(k)\}\zeta(k) \tag{22}$$

which implies

$$w^{\mathrm{T}}(k)w(k)\leqslant\zeta^{\mathrm{T}}(k)\zeta(k) \tag{23}$$

It is well known [18] that the system (18) subject to (23) is quadratically stabilizable if the H_∞ norm for the closed-loop transfer function of (18)–(19) from w to ζ is less than 1 for some $\tau>0$. Hence, our result is proved. □

If the system (1) does not involve any uncertainty, Theorem 2.1 reduces to the following result (see [14]).

Corollary 2.1
Suppose the system (1) does not involve any uncertainty (i.e. $H=0, E_1=0, E_2=0$). Then, it is quadratically stabilizable for a given quantization density $\rho>0$ *if and only if* there exists a state feedback controller (19) for the following auxiliary system:

$$\begin{aligned}x(k+1) &= Ax(k)+Bv(k)+Bw(k)\\ \zeta(k) &= \delta v(k)\end{aligned} \tag{24}$$

such that the H_∞ norm of the transfer function from w to ζ is less than 1. Furthermore, the control gain K and Lyapunov matrix P for the auxiliary system (24) will also work for the system (1).

2.2. *Case 2: Output feedback with quantized input*

We first consider the special case when no uncertainties exist in the system. This case has been studied in [14], and the result is that the output feedback with quantized input is equivalent to quantized state feedback for quadratic stabilization, provided the system is detectable. That is, if the state feedback can quadratically stabilize the system for a given quantization density, so can the output feedback. The corresponding quantized output feedback controller is an observer-based one, taking the following form:

$$\begin{aligned}x_c(k+1) &= Ax_c(k)+L(\bar{y}(k)-Cx_c(k))+Bu(k)\\ v(k) &= Kx_c(k)\\ u(k) &= Q(v(k))\end{aligned} \tag{25}$$

with $x_c(0)=0$, where L is the observer gain and K is the state feedback gain. Note that $L=B_c,\ B_1=B$, and $D_1=0$ if we compare (25) with (8). Returning to the uncertain system (1), motivated by the above, we restrict the quantized feedback controller (8) to be

$$\begin{aligned}x_c(k+1) &= A_cx_c(k)+B_c\bar{y}(k)+Bu(k)\\ v(k) &= C_cx_c(k)+D_c\bar{y}(k)\\ u(k) &= Q(v(k))\end{aligned} \tag{26}$$

with $x_c(0)=0$, i.e. we set $B_1=B$ and $D_1=0$.

Next, we define an auxiliary system

$$\begin{aligned}x(k+1) &= Ax(k)+Bv(k)+[B\ \tau^{-1}H_1]w(k)\\ \bar{y}(k) &= Cx(k)+[0\ \tau^{-1}H_2]w(k)\\ \zeta(k) &= \begin{bmatrix}\delta v(k)\\ \tau(E_1x(k)+E_2v(k)+[E_2\ 0]w(k))\end{bmatrix}\end{aligned} \tag{27}$$

where δ is computed from a given quantization density $\rho>0$, and $\tau>0$ is a scaling parameter. We also define an auxiliary controller

$$\begin{aligned}x_c(k+1) &= A_cx_c(k)+B_c\bar{y}(k)+Bv(k)+[B\ 0]w(k)\\ v(k) &= C_cx_c(k)+D_c\bar{y}(k)\end{aligned} \tag{28}$$

with $x_c(0)=0$.

Theorem 2.2
Consider the uncertain system (1) and a given quantization density $\rho>0$. Suppose there exists an auxiliary output feedback controller (28) for the auxiliary

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system (27) such that the H_∞ norm of the transfer function from w to ζ is less than 1. Then, the system (1) is quadratically stabilizable via the quantized feedback controller (26) with the same controller parameters A_c, B_c, C_c, and D_c.

Proof
The proof is similar to that of Theorem 2.1. Using (1) and (8), it is straightforward to verify that the closed-loop system (11) is given by

$$
\begin{aligned}
x(k+1) &= (A+\Delta A)x(k)+(B+\Delta B)u(k)\\
x_c(k+1) &= B_c(C+\Delta C)x(k)+A_c x_c(k)\\
&\quad +B_c\Delta D u(k)+Bu(k)\\
v(k) &= D_c(C+\Delta C)x(k)+C_c x_c(k)+D_c\Delta D u(k)\\
u(k) &= Q(v(k))
\end{aligned}
$$

Using (2), (17) and the definition (21), the above can be rewritten as

$$
\begin{aligned}
x(k+1) &= Ax(k)+Bv(k)+B\xi(k)+H_1\eta(k)\\
x_c(k+1) &= A_c x_c(k)+B_c\bar{y}_k+Bv(k)+B\xi(k)\\
\bar{y}(k) &= Cx(k)+H_2\eta(k)\\
v(k) &= C_c x_c(k)+D_c\bar{y}(k)
\end{aligned}
$$

which is the same as (27)–(28) with $w(k)$ and $\zeta(k)$ related as in (22)–(23). As for the proof of Theorem 2.1, the system (27)–(28) subject to (23) is quadratically stabilizable if the H_∞ norm for the closed-loop transfer function of (27)–(28) from w to ζ is less than 1 for some $\tau>0$. □

When no uncertainty is involved in the system, Theorem 2.2 reduces to the following [14].

Corollary 2.2
Suppose the system (1) does not involve any uncertainty (i.e. $H_1=0, H_2=0, E_1=0, E_2=0$) and the quantization density $\rho>0$ is given. Then, the following three problems are equivalent:

(a) The system (1) is quadratically stabilizable via output feedback with quantized input.

(b) The following auxiliary system:

$$
\begin{aligned}
x(k+1) &= Ax(k)+Bv(k)+Bw(k)\\
\bar{y}(k) &= Cx(k)\\
\zeta(k) &= \delta v(k)
\end{aligned}
\tag{29}
$$

can be controlled by the following auxiliary controller:

$$
\begin{aligned}
x_c(k+1) &= A_c x_c(k)+B_c\bar{y}(k)\\
&\quad +Bv(k)+Bw(k)\\
v(k) &= C_c x_c(k)+D_c\bar{y}(k)
\end{aligned}
\tag{30}
$$

with $x_c(0)=0$, such that the H_∞ norm of the closed-loop transfer function from w to ζ is less than or equal to 1.

(c) The system (1) can be quadratically stabilized via quantized state feedback with the given density ρ.

2.3. Output feedback with quantized output

In this case, the corresponding auxiliary system is given by

$$
\begin{aligned}
x(k+1) &= Ax(k)+Bu(k)+[0\ \tau^{-1}H_1]w(k)\\
v(k) &= Cx(k)+Du(k)+[1\ \tau^{-1}H_2]w(k)\\
\zeta(k) &= \begin{bmatrix}\delta(Cx(k)+Du(k)+[0\ \tau^{-1}H_2]w(k))\\ \tau(E_1x(k)+E_2u(k))\end{bmatrix}
\end{aligned}
\tag{31}
$$

We have the following result:

Theorem 2.3
Consider the uncertain system (1) and a given quantization density $\rho>0$. Suppose there exists an output feedback controller (10), without quantization (i.e. $v(k)=y(k)$), for the auxiliary system (31) such that the H_∞ norm of the transfer function from w to ζ is less than 1. Then, the system (1) is quadratically stabilizable via the same controller with quantized output and quantization density ρ.

Proof
The proof is similar to those of Theorems 2.1–2.2. The details are omitted. □

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When the system uncertainties disappear, again we have the following special result [14]:

Corollary 2.3
Suppose there is no uncertainty in (1). Then the following are equivalent:

(a) The system (1) is quadratically stabilizable via output feedback with quantized output and quantization density $\rho > 0$.
(b) There exists an unquantized output feedback controller for the following auxiliary system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ v(k) &= Cx(k) + Du(k) + w(k) \\ \zeta(k) &= \delta(Cx(k) + Du(k)) \end{aligned} \tag{32}$$

such that the H_∞ norm of the transfer function from w to ζ is less than 1.

3. ROBUST STABILIZATION VIA DYNAMIC QUANTIZATION

We now turn our attention to dynamic quantization. For simplicity, we only consider the case of output feedback control with quantized output (i.e. Case 3), but the ideas here can be easily generalized to other quantization settings. The control problem is the same as described in the previous section, except that we now want to use a dynamic quantizer. The goal is to achieve robust stabilization by using a quantizer with only a finite number of quantization levels.

The approach we take here follows from [20]. More precisely, we first design a robust quantized feedback controller using a static logarithmic quantizer, then use a dynamic *scaling parameter* to scale the input signal to the quantizer. At the same time, we truncate the scaled input when it is either too small or too large. That is, we replace the infinite-level logarithmic quantizer with a finite-level one. By adjusting the scaling parameter appropriately, robust stabilization can still be achieved.

We define an $2N$-level logarithmic quantization with quantization density $\rho > \rho_{\text{inf}}$ as

$$\mathscr{V} = \{\pm\rho^i, i = 0, 1, 2, \ldots, N-1\} \tag{33}$$

The associated quantizer $Q(\cdot)$ becomes:

$$Q(y) = \begin{cases} \rho^i & \text{if } \dfrac{1}{1+\delta}\rho^i < y \leqslant \dfrac{1}{1-\delta}\rho^i \\ & \quad 0 < i < N-1 \\ \rho^{N-1} & \text{if } 0 \leqslant y \leqslant \dfrac{1}{1-\delta}\rho^{N-1} \\ 1 & \text{if } y > \dfrac{1}{1+\delta} \\ -Q(-y) & \text{if } y < 0 \end{cases} \tag{34}$$

The idea of dynamic scaling is to scale $y(k)$ so that it is within the quantization range as much as possible. That is, we use

$$v(k) = g_k^{-1} Q(g_k y(k)) \tag{35}$$

where g_k is the scaling parameter at time k defined by

$$g_{k+1} = \begin{cases} g_k \gamma_1 & \text{if } |Q(g_k y_k)| = 1 \\ g_k/\gamma_2 & \text{if } |Q(g_k y_k)| = \rho^{N-1} \\ g_k & \text{otherwise} \end{cases} \tag{36}$$

with any initial $g_0 > 0$. The constants $\gamma_1, \gamma_2 \in (0, 1)$ are design parameters to be discussed later.

The following result comes from [20]:

Lemma 1
Suppose the system (1) without uncertainties can be quadratically stabilized by the quantized feedback controller (10) with a density-ρ (static) logarithmic quantizer, $0 < \rho < 1$. Then there exist $\gamma_1, \gamma_2 \in (0, 1)$ and some finite $N > 0$ such that when the quantizer in (10) is replaced with the dynamically scaled $2N$-level quantizer (35)–(36), the quantized feedback controller (10) asymptotically stabilizes the system (1) for any $g_0 > 0$.

Remark 3.1
The role of γ_1 and γ_2 is to keep the scaled input $g_k y(k)$ to be within the quantization range as much as possible. Roughly speaking, γ_1 makes the g_{k+1} smaller than g_k, thus plays a zoom-out role; Similarly, γ_2 plays a zoom-in role. In particular, γ_1 is chosen to be small such that $\gamma_1 A$ to be stable; γ_2 is chosen to close to 1 such that $\tilde{A}/\gamma_2$ is also stable. In the above, A is the open-loop

state matrix and $\tilde{A}$ is the closed-loop state matrix. These two parameters together with the initial g_0 can be optimized to give a minimal N. See details in [20].

Combining Lemma 1 and Theorem 2.3, we have the following main result:

Theorem 3.1
Consider the uncertain system (1). Suppose its auxiliary system (31) can be quadratically stabilized via a quantized feedback controller (10) with a density-ρ (static) logarithmic quantizer, $0<\rho<1$. Then, (1) can be robustly asymptotically stabilized using the same controller (10) and a dynamically scaled finite-level quantizer (35)–(36) for some finite N and constants $\gamma_1, \gamma_2 \in (0,\ 1)$ and any $g_0>0$.

Proof
By Lemma 1, quadratic stabilization of the auxiliary system (31) via (10) with a density-ρ (static) logarithmic quantizer implies that (31) can be asymptotically stabilized by the same controller when the quantizer is replaced by (35)–(36) for some finite N and constants $\gamma_1, \gamma_2 \in (0,\ 1)$. By Theorem 2.3 (or by following its proof to be more precise), we conclude that the original uncertain system (1) is asymptotically stabilized by the same dynamically scaled quantized feedback controller. □

Remark 3.2
A typical behavior of the system is as follows: If the initial state is very large, the feedback signal tends to be saturated, forcing g_k to decrease fast. This would result in a period of overshoot. Once g_k is sufficiently small, saturation will stop and the state decays exponentially. When the state is sufficiently small, g_k will increase gradually, causing the quantizer to bounce back and forth between the dead zone and logarithmic region. During this phase, the state also decays exponentially, but at a lower rate.

Remark 3.3
One may think that the number of quantization levels needs to be high in order to achieve the above behavior. In reality, a moderate number of quantization levels is typically sufficient. This will be seen in examples in Section 5.

4. ROBUST PERFORMANCE CONTROL UNDER QUANTIZED FEEDBACK

In this section, we show that the sector bound approach used in this paper can be applied to study robust performance control problems as well. Two performance control problems are to be treated: linear quadratic control and H_∞ control.

4.1. Robust linear quadratic control

In this subsection, we consider the robust linear quadratic regulation with quantized input. For simplicity, we consider the state feedback control. But our treatment can be easily generalized to output feedback control.

Consider the system (1)–(2) with the following performance cost function:

$$J(x(0)) = \sum_{k=0}^{\infty} x^{\mathrm{T}}(k)Qx(k)+ru^2(k) \qquad Q=Q^{\mathrm{T}}\geqslant 0, \quad r>0 \tag{37}$$

Our objective is to design a quantized state feedback control $u(k)=Q(Kx(k))$ such that an upper bound of the above cost function for all admissible uncertainties satisfying (2) is minimized.

As in Section II, a logarithmic quantizer is adopted.

In view of (17), the closed-loop system of (1) under the quantized state feedback can be described by

$$x(k+1)=A_\Delta(k)x(k) \tag{38}$$

where

$$A_\Delta(k)=(A+\Delta A)+(B+\Delta B)(1+\Delta(k))K$$

Furthermore, the cost function (37) can be given by

$$J(x(0))=\sum_{k=0}^{\infty} x^{\mathrm{T}}(k)(Q+r(1+\Delta(k))^2K^{\mathrm{T}}K)x(k) \tag{39}$$

Let $V(x)=x^{\mathrm{T}}Px, P=P^{\mathrm{T}}>0$ be the associated Lyapunov function candidate for considering the quadratic stability of (38). Denote

$$\Delta V(x(k))=V(x(k+1))-V(x(k)) \tag{40}$$

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Then, using (38) the performance cost function is given by

$$J(x(0)) = x^{\mathrm{T}}(0)Px(0) + \sum_{k=0}^{\infty} x^{\mathrm{T}}(k)\Omega(\Delta A, \Delta B, \Delta(k))x(k) \quad (41)$$

where

$$\Omega(\Delta A, \Delta B, \Delta(k)) = A_{\Delta}^{\mathrm{T}}(k)PA_{\Delta}(k) - P + Q + r(1+\Delta(k))^2 K^{\mathrm{T}}K$$

In the absence of system uncertainty and quantization, $\Omega(\Delta A, \Delta B, \Delta(k)) = 0, \forall k$ and the optimal cost can be obtained by minimizing $\mathrm{tr}(P)$. With the uncertainty and quantization, we formulate our performance control problem called *quantized guaranteed cost control* (*QGCC*) as follows: Given a performance bound γ and quantization density $\rho>0$, find P and K, if exist, such that

$$\mathrm{tr}(P) < \gamma \quad (42)$$

subject to

$$\Omega(\Delta A, \Delta B, \Delta(k)) < 0 \quad (43)$$

with ΔA and ΔB satisfying (2) and $|\Delta(k)| \leqslant \delta$, where δ is related to ρ by (5).

The following theorem provides a solution to the QGCC problem.

Theorem 4.1
Consider the uncertain system (1)–(2) and the cost function (37). Given a quantization density $\rho>0$ and a performance bound $\gamma>0$, the QGCC problem has a solution if the following linear matrix inequalities (LMIs):

$$\mathrm{tr}(\tilde{P}) < \gamma, \quad \begin{bmatrix} -\tilde{P} & I \\ I & -S \end{bmatrix} \leqslant 0 \quad (44)$$

$$\begin{bmatrix} -S & * & * & * & * & * \\ AS+BW & -S+\tau HH^{\mathrm{T}}+\delta^2 BB^{\mathrm{T}} & * & * & * & * \\ W & \delta^2 B^{\mathrm{T}} & -r^{-1}+\delta^2 & * & * & * \\ E_1S+E_2W & \delta^2 E_2 B^{\mathrm{T}} & \delta^2 E_2 & -\tau I+\delta^2 E_2 E_2^{\mathrm{T}} & * & * \\ W & 0 & 0 & 0 & -I & 0 \\ Q^{1/2}S & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (45)$$

have a solution $(\tilde{P}, S, W, \tau)$, where $*$ denotes a transposed term. In this case, P and K are related to S and W as follows:

$$P = S^{-1}, \quad K = WP \quad (46)$$

Proof
By Schur complement, (45) is equivalent to

$$\begin{bmatrix} -S+SQS & * & * & * \\ AS+BW & -S+\tau HH^{\mathrm{T}} & * & * \\ W & 0 & -r^{-1} & * \\ E_1S+E_2W & 0 & 0 & -\tau I \end{bmatrix} + \delta^2 M_1^{\mathrm{T}} M_1 + N_1^{\mathrm{T}} N_1 < 0$$

where

$$M_1 = [0 \;\; B^{\mathrm{T}} \;\; 1 \;\; E_2^{\mathrm{T}}], \quad N_1 = [W \;\; 0 \;\; 0 \;\; 0]$$

Since $\delta^2 M_1^{\mathrm{T}} M_1 + N_1^{\mathrm{T}} N_1 \geqslant M_1^{\mathrm{T}} \Delta(k) N_1 + N_1^{\mathrm{T}} \Delta(k) M_1$, the above implies that

$$\begin{bmatrix} -S+SQS & * & * & * \\ AS+B(1+\Delta(k))W & -S+\tau HH^{\mathrm{T}} & * & * \\ (1+\Delta(k))W & 0 & -r^{-1} & * \\ E_1S+E_2(1+\Delta(k))W & 0 & 0 & -\tau I \end{bmatrix} < 0$$

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Again, by Schur complement, the above leads to

$$\begin{bmatrix} -S+SQS+r(1+\Delta(k))^2 W^{\mathrm{T}}W & * \\ AS+B(1+\Delta(k))W & -S \end{bmatrix} + \tau\bar{H}\bar{H}^{\mathrm{T}}+\tau^{-1}\bar{E}^{\mathrm{T}}\bar{E}<0 \tag{47}$$

where

$$\bar{H}=\begin{bmatrix} 0 \\ H \end{bmatrix}, \quad \bar{E}=[E_1 S+E_2(1+\Delta(k))W \ 0]$$

Since

$$\tau\bar{H}\bar{H}^{\mathrm{T}}+\tau^{-1}\bar{E}^{\mathrm{T}}\bar{E}\geqslant\bar{H}F(k)\bar{E}+\bar{E}^{\mathrm{T}}F(k)^{\mathrm{T}}\bar{H}$$

for any $\tau>0$ and $\|F(k)\|\leqslant 1$, (47) implies that

$$\begin{bmatrix} -S+SQS+r(1+\Delta(k))^2 W^{\mathrm{T}}W & * \\ (A+\Delta A)S+(B+\Delta B)(1+\Delta(k))W & -S \end{bmatrix}<0$$

for all admissible uncertainties satisfying (2). By multiplying the above from the left and right, respectively, by $\mathrm{diag}\{S^{-1},S^{-1}\}$ and taking into account (46), we have

$$\begin{bmatrix} -P+Q+r(1+\Delta(k))^2 K^{\mathrm{T}}K & * \\ PA_{\Delta} & -P \end{bmatrix}<0$$

By Schur complement, (43) follows. Finally, using Schur complement, (42) is equivalent to $\mathrm{tr}(\tilde{P})<\gamma$, $-\tilde{P}+P\leqslant 0$, which is equivalent to (44) by removing the variable $\tilde{P}$. □

Remark 4.1

We note in Theorem 4.1 that the scaling parameter τ appears linearly in the LMIs (44)–(45). Therefore, τ is automatically optimized when the LMIs are solved.

4.2. Robust H_∞ control with quantized output

We shall further extend the studies in Section 2 to the quantized H_∞ control. For simplicity, we consider the H_∞ control with quantized output.

Here, we consider the system

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k)+(B+\Delta B)u(k) \\ &\quad +B_1 w(k) \\ y(k) &= (C+\Delta C)x(k)+(D+\Delta D)u(k) \\ &\quad +D_1 w(k) \\ z(k) &= L_1 x(k)+L_2 u(k)+L_3 w(k) \end{aligned} \tag{48}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}$ is the control input, $w(k)\in\mathbb{R}^m$ is the exogenous input, which is energy bounded, $y(k)\in\mathbb{R}^r$ is the measured output, and $z(k)\in\mathbb{R}^p$ is the controlled output defined by the matrices L_1, L_2, and L_3. The uncertainties $(\Delta A,\Delta B,\Delta C,\Delta D)$ are given by (2).

The robust H_∞ control with quantized input is stated as: Given a scalar $\gamma>0$, find a quantized output feedback control such that the closed-loop system is asymptotically stable and under zero initial condition,

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k)<\gamma^2\sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k) \tag{49}$$

for all admissible uncertainties (2) and any non-zero disturbance input.

By adopting a logarithmic quantizer with density $\rho>0$, we introduce an auxiliary system

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ &\quad +[0\ \tau^{-1}H\ \gamma^{-1}B_1]\bar{w}(k) \\ v(k) &= Cx(k)+Du(k) \\ &\quad +[1\ \tau^{-1}H_2\ \gamma^{-1}D_1]\bar{w}(k) \\ \bar{z}(k) &= \begin{bmatrix} L_1 \\ \delta C \\ \tau E_1 \end{bmatrix} x(k)+\begin{bmatrix} L_2 \\ \delta D \\ \tau E_2 \end{bmatrix} u(k) \\ &\quad +\begin{bmatrix} 0 & 0 & \gamma^{-1}L_3 \\ 0 & \delta\tau^{-1}H_2 & 0 \\ \tau E_2 & 0 & 0 \end{bmatrix}\bar{w}(k) \end{aligned} \tag{50}$$

where $\bar{w}(k)$ and $\bar{z}(k)$ are the associated with exogenous input and controlled output, respectively, δ is given by (5) and $\tau>0$ is a scaling parameter.

We have the following result.

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Theorem 4.2
Consider the system (48) with uncertainties satisfying (2). Given a prescribed H_∞ performance $\gamma>0$ and a logarithmic quantizer with density $\rho>0$, the robust H_∞ control with quantized output problem is solvable by a quantized output feedback controller if there exists a scaling parameter $\tau>0$ such that the controller asymptotically stabilizes the auxiliary system (50) with $\|G_{\bar{z}\bar{w}}(z)\|_\infty<1$, where $G_{\bar{z}\bar{w}}(z)$ is the closed-loop transfer function from $\bar{w}$ to $\bar{z}$.

Proof
It can be easily obtained by the following arguments in [21] and the proofs of Theorems 2.1–2.2. □

5. ILLUSTRATIVE EXAMPLES

In this section, we give some examples to demonstrate the results we have obtained so far.

Example 1
The first example is to show the effects of three quantization schemes as studied in Section 2. The system to be considered is given by (1) with

$$A=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix},\quad C=[-3\ \ 1],\quad D=0$$
$$H_1=\varepsilon\begin{bmatrix}1\\ 1\end{bmatrix},\quad H_2=\varepsilon,\quad E_1=[1\ \ 0],\quad E_2=1 \tag{51}$$

In the above, the parameter $\varepsilon>0$ controls the size of uncertainties.

When $\varepsilon=0$, the uncertainties vanish and the transfer function of the system becomes $G(z)=C(zI-A)^{-1}B=(z-3)/z(z-2)$. This example is analyzed in [14]. When quantized state feedback is used, the coarsest quantization density is computed to be $\rho=\frac{1}{3}$. The same quantization density is reached when output feedback with quantized control input is used. For output feedback with quantized output, the coarsest quantization density turns out to be $\rho=0.8182$. That is, the latter scheme requires a much denser quantizer.

When $\varepsilon>0$, Theorems 2.1–2.3 are applied and the coarsest quantization densities are searched by solving the H_∞ control problems associated with the auxiliary systems for various τ and ρ. The parameter τ is numerically optimized to minimize the associated ρ. The results for ρ are plotted in Figure 2.

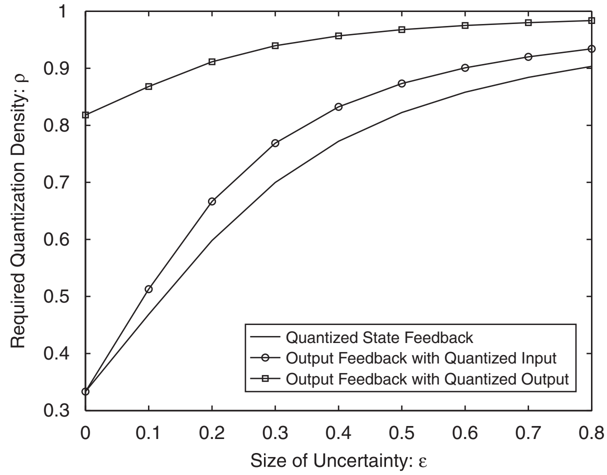

Figure 2. Required quantization density versus size of uncertainty.

It is clear that increasing ε will increase the required quantization density. Also, the output feedback with quantized output requires a denser quantizer compared with output feedback with quantized input. Finally, although the output feedback with quantized input requires the same quantization density as the quantized state feedback when there is no uncertainty, the former requires a denser quantizer when ε increases. This is because the existence of uncertainty makes it difficult to recover the state information from the output measurement.

The next two examples are modified from those used in [20] by adding some uncertain parameters in the system model. The purpose of these examples is to show how robust stabilization works under finite-level quantization.

Example 2
We consider a first-order system

$$x(k+1)=(1+\theta_k)ax(k)+u(k) \tag{52}$$
$$y(k)=x(k) \tag{53}$$

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where $a>1$ and $|\theta_k| \leqslant \theta$ is an uncertain parameter. The corresponding uncertainty model (2) has $H_1=1$, $E_1=\theta$, $H_2=0$, $E_2=0$. In order to stabilize the system using a logarithmic quantizer (31) with density ρ, the controller $H(z)=h$ becomes a constant because of full state feedback. The closed-loop system is given by

$$\begin{aligned} x(k+1) &= ((1+\theta_k)a + h(1+\Delta_k))x(k) \\ |\Delta_k| &\leqslant \delta \end{aligned} \tag{54}$$

In [20], we showed, for the case without θ_k, that the optimal stabilizing h which leads to the minimum number of quantization levels is given by $h=-a$ and the corresponding N is given by $N \geqslant N_0$, where

$$N_0 = 1 + \frac{\log(\gamma_2 a^{-1} - \delta)}{\log(1-\delta) - \log(1+\delta)}, \quad \delta < a^{-1} \tag{55}$$

In the presence of the uncertainty parameter θ_k, h needs to ensure robust stability of (54). Since this is a scalar system, it is sufficient to ensure the stability of (54) for $\theta_k = \theta$. The modified minimum N is given by

$$N_0(\theta) = 1 + \frac{\log(\gamma_2 (a(1+\theta))^{-1} - \delta)}{\log(1-\delta) - \log(1+\delta)}, \quad \delta < a^{-1} \tag{56}$$

The result is shown in Figure 3, which contains two curves for the required bit rate

$$N_b = \lceil \log_2(2N) \rceil, \quad N \geqslant \lceil N_0(\theta) \rceil \tag{57}$$

where $\lceil \cdot \rceil$ is the integer round-up function. One curve is for $\theta=0$ and another for $\theta=1$. These curves are compared with the minimum bit rate $\lceil \log_2(a) \rceil$ given in [12]. The value for $\gamma_2 = 0.9$ is used. We see that very few bits of quantization are required for robust stabilization.

Example 3
This example aims at demonstrating the use of a finite-level quantizer for robust stabilization and the robustness of the dynamic scaling method under process noises. Consider the system (1) with

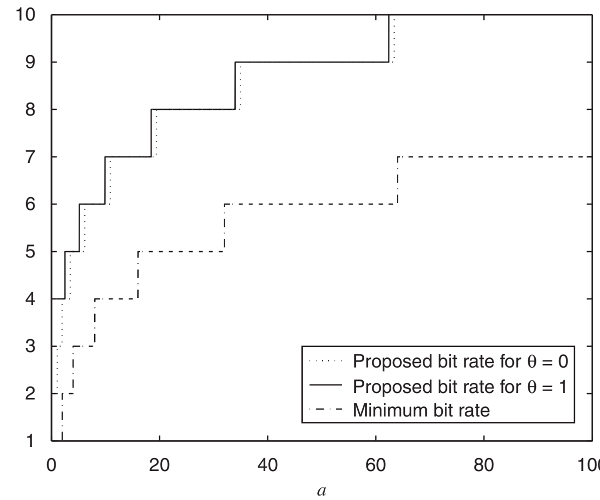

Figure 3. Bit rate comparison for a first order system.

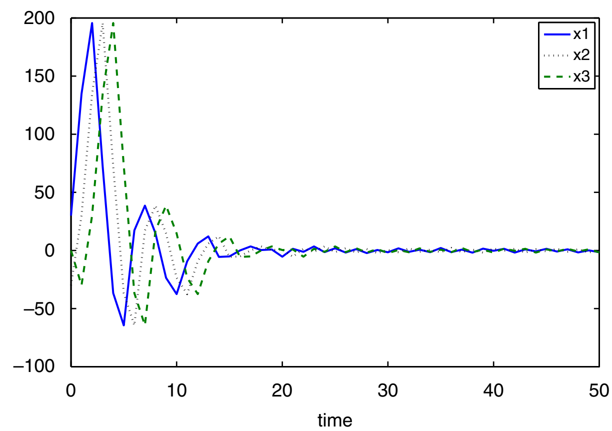

Figure 4. State response of the closed-loop system with $N=8$.

$$A = \begin{bmatrix} 2.7 & -2.41 & 0.507 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [1 \ \ -0.5 \ \ 0.04], \quad D = 0$$

$$H_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad H_2 = 0, \quad E_1 = [\theta \ \ 0 \ \ 0], \quad E_2 = 0$$

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That is, only the (1,1)th element of A is perturbed to become $2.7+\theta_k$ with $|\theta_k|\leqslant\theta$.

The nominal system is unstable with two unstable open-loop poles at $1.2\pm i0.5$ but without unstable zero and the relative degree is 1. It follows from [14] that

$$\delta_{\sup}=|1.2\pm i0.5|^{-2}=0.5917, \quad \rho_{\inf}=0.2565$$

for the nominal system.

We choose the logarithmic quantizer with $\rho=0.6653$ and controller with

$$A_c=\begin{bmatrix} -255.6834 & 46.7502 & 217.854 \\ 616.3274 & -111.8387 & -523.9270 \\ -431.7862 & 79.0425 & 368.0348 \end{bmatrix}$$

$$B_c=\begin{bmatrix} 5.8122 \\ -14.0003 \\ 9.8161 \end{bmatrix}$$

$$C_c=[81.6699 \quad -15.0325 \quad -69.6715]$$

$$D_c=-1.8594$$

We also choose $\gamma_1=0.2, \gamma_2=0.8$, $\mu_0=1$, and $N=8$ (which corresponds to $N_b=4$ bits). Note that the minimal bit rate required for stabilizing the nominal system is 1 bit [12].

The state response of the closed-loop nominal system ($\theta_k\equiv 0$) with the initial state $x_0=[30-300]^{\mathrm{T}}$, $g_0=0.1$ is shown in Figure 4. The scaling gain g_k is shown in Figure 5.

When $\theta_k\equiv 0.3$, the state response of the closed-loop system with the same controller and the initial condition becomes that given in Figure 6. It is clear that the quantized feedback controller is still able to stabilize the system.

If we have a good estimate $\tilde{x}_0$ of the initial state x_0, we may set the initial scaling gain g_0 to improve the transient performance. For example, we may set $g_0=1/|Cx_0|$ for a given x_0. For the example, if $x_0=[30-300]^{\mathrm{T}}$ is given, the state responses of the closed-loop system with $\theta_k\equiv 0.3$ and $N=8$ are shown in Figure 7. Clearly, the overshoot is much smaller than that of Figure 6 where x_0 is not assumed known.

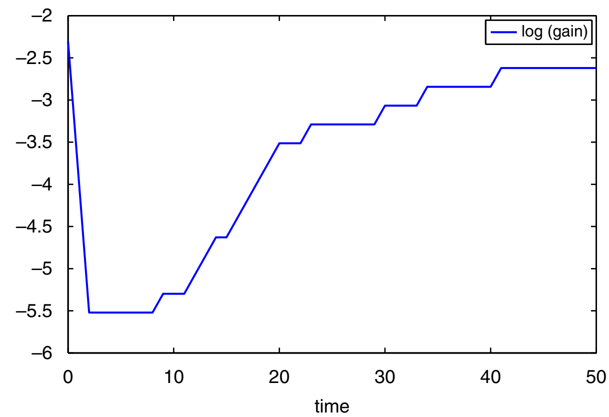

Figure 5. The scaling factor g_k for the noise-free case with $N=8$.

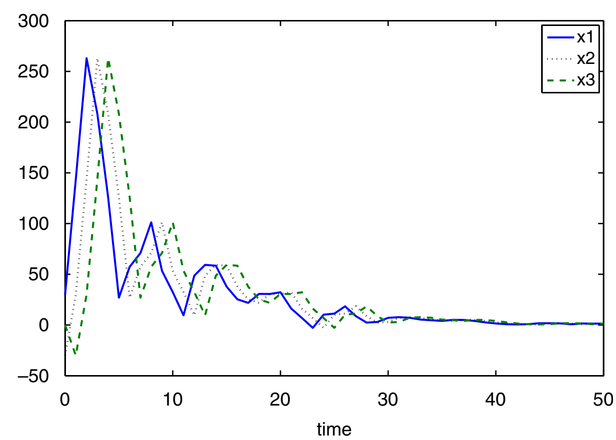

Figure 6. State response of the closed-loop system with $\theta_k\equiv 0.3$ and $N=8$.

Next, we study the robustness of the closed-loop system. This is done by adding some process noise to the system, i.e. we modify (1) to be

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k)+(B+\Delta B)u(k)+\eta(k) \\ y(k) &= (C+\Delta C)x(k)+(D+\Delta D)u(k) \end{aligned} \tag{58}$$

where $\|\eta(k)\|\leqslant\bar{\eta}$ for some constant $\bar{\eta}>0$. In the simulation below, we take η_k to be a saturated Gaussian white noise with zero mean, covariance matrix $Q_\eta=3I$ and $\bar{\eta}=100$. The same quantized controller is used

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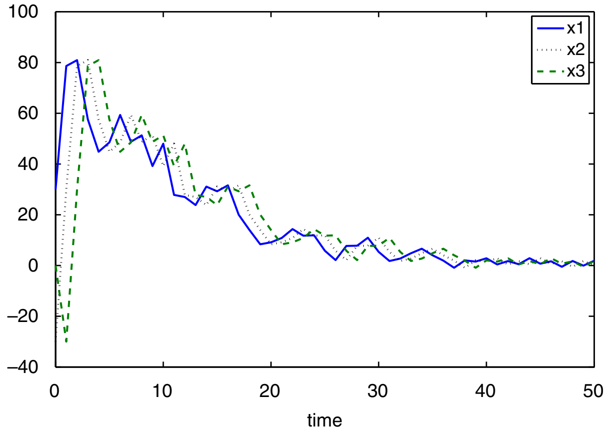

Figure 7. State response of the closed-loop system with $N=8$ and with known x_0.

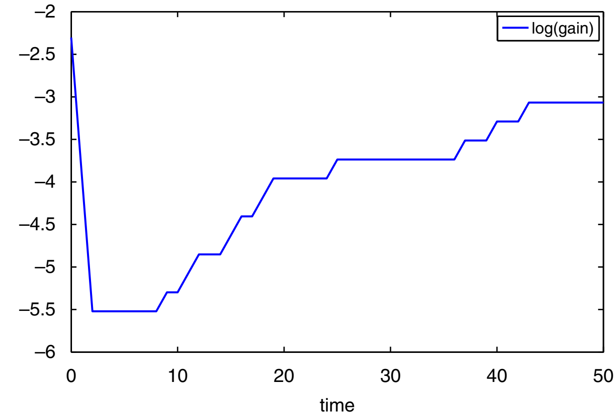

Figure 9. The scaling factor g_k for the noisy case with $Q_\eta = 3I$.

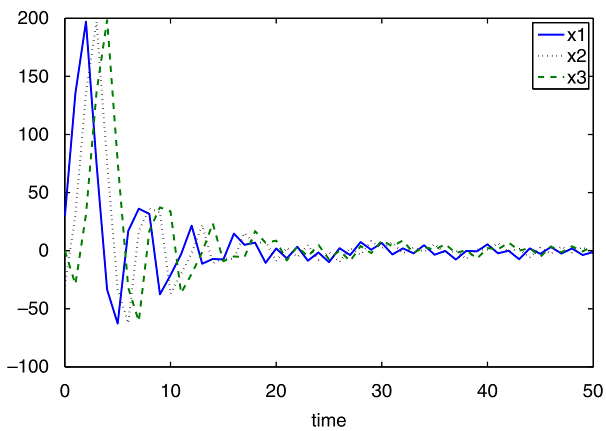

Figure 8. Closed-loop response under input noise with $Q_w = 3I$.

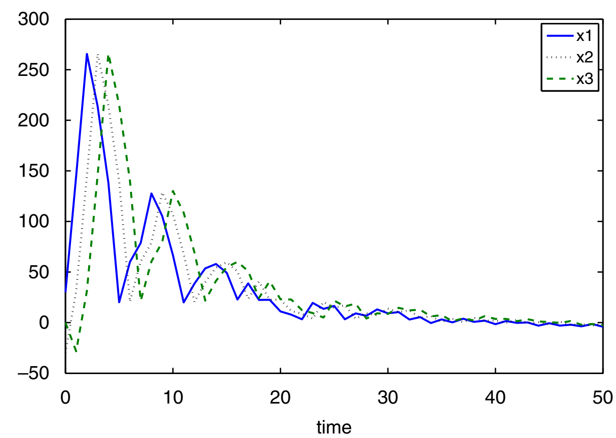

Figure 10. Closed-loop response under input noise with $Q_\eta = 3I$ and $\theta_k \equiv 0.3$.

as before. The state response of the closed-loop nominal system with $x_0 = [30 \quad -30 \quad 0]^{\mathrm{T}}$ is shown in Figure 8 with the corresponding scaling gain g_k in Figure 9. It can be observed that the final state converges to a bounded region.

In the presence of the uncertainty of $\theta_k \equiv 0.3$, the state response of the closed-loop system with the same noise input and the initial condition is shown in Figure 10.

6. CONCLUSIONS

We have studied a number of robust stabilization problems and robust performance problems associated with logarithmic quantized feedback. We have considered three quantization cases, namely quantized state feedback, output feedback with quantized input, and output feedback with quantized output. In each of these cases, we have shown the connection between quadratic stabilizability for a given quantization density and H_∞ control for a corresponding auxiliary system. This allows us to use the standard H_∞ design tools to deal with quantized feedback control for uncertain systems.

In the output feedback control case, we have noted an interesting phenomenon that the quadratic stabilizability depends on where the quantization occurs. In particular, a coarser quantization density can be achieved in general when quantization occurs at the control input rather than at the measured output. Intuitively, this is because the measured information is better preserved in the former case.

Although the sector bound method gives only sufficient conditions for quantized feedback stabilization, we make two points: (1) The results become non-conservative when uncertainties are not present, as shown in Corollaries 2.1–2.3 and (2) The technical difficulties for quantized feedback stabilization of the uncertain systems as in Theorems 2.1–2.3 are essentially the same as quadratic stabilization of systems with two blocks of uncertainties (one from $F(k)$ and one from $\Delta(k)$). This problem has been studied for a long time in the robust control literature, and there is no non-conservative solution to it.

For finite-level quantized feedback control, the proposed method in this paper involves a simple dynamic scaling parameter that plays a similar role to the well-known 'zoom-in/zoom-out' method in [9–11]. Although this use of a logarithmic quantizer is not optimal in minimizing the bit rate, it is a relatively simple method, and the required bit rate is typically moderate, as seen in our examples.

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