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Consensusability of linear multi-agent systems with time delay

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SUMMARY

This paper studies the consensusability of a continuous-time linear time-invariant multi-agent system (MAS) with time delay in an undirected network with N nodes. We show that the MAS can achieve consensus if and only if $N-1$ time-delay subsystems associated with the eigenvalues of the Laplacian matrix of the network are simultaneously asymptotically stable. By employing a linear matrix inequality (LMI) method, we present a controller design method for a MAS to reach consensus. We also obtain a bound on the maximum time delay for consensusability for a MAS with first-order integrator dynamics by using frequency-domain analysis. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the last decade or so, the consensus control problem has been widely studied because of its relevance in multi-agent coordinated control, distributed computation, biological group behaviors such as swarms and flocks [1], and so on. An important problem of consensus control is to design an appropriate consensus protocol by using locally exchanged information such that all the agents in a network agree upon certain quantities of common interest.

The seminal work [2] solved the consensus problem and average-consensus problem of first-order integrator networks with and without time delay by using algebraic graph theory and frequency-domain analysis. In [3], consensus protocols were designed for both the first-order integral multi-agent systems (MASs) and discrete-time MASs. In [4] Ma *et al.* considered the conditions for consensusability of linear MASs without delay and showed that the consensusability of MASs depends on the dynamic structure of each agent and the communication topology among agents. Reference [5] studied the consensus conditions of first-order integrator systems under both directed and undirected communication network topologies. Reference [6] studied the consensusability problems of discrete-time MASs under the effect of network topology and communication data rate. Consensus using quantized information has also been considered in [7]. Reference [8] has given consensus convergence rate analysis of MASs by using stochastic approximation approach.

All of the existing works on the consensus problem focus on special first-order integrator systems or discrete-time systems [9–11] or systems without delay [12, 13]. However, time delay is common in the process of information exchange between agents in practice. As we all know, the frequency-domain tool is effective for stability analysis of linear time-invariant systems both with and without time delay. But this method is only applicable to some special systems such as scalar systems because the corresponding characteristic equation becomes a transcendental function that

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is difficult to deal with for high-order systems. The linear matrix inequality method has been used for stability analysis of general linear systems with time delay [14, 15]. References [16] and [17] have studied the consensus problems of double integrator systems and kth-order consensus in MASs with time delays by using frequency-domain method, and they both obtained an implicit formulation of the critical time delay. However, they both consider the case of integrator system for every order dynamics.

In this paper, we consider the consensus control problem for general (high-order) continuous-time linear MASs with time delay in an undirected network with N nodes. The time delay that we consider arises from communication delays between agents, meaning that our consensus protocol contains a delay in the relative state information. The generality of state and input matrix (arising from high-order dynamics of each agent) and the existence of time delay bring difficulties to the study of consensusability. By introducing some linear transformation, we prove that the consensusability of N agents is equivalent to that $N-1$ time-delay subsystems associated with the eigenvalues of Laplacian matrix of the network are simultaneously asymptotically stable. So the first contribution of this paper is that we find the relationship between the consensusability problem and the stability problem.

There is a common gain matrix K in all of the $N-1$ time-delay subsystems. Hence, the key to consensusability becomes the existence of the common K for the simultaneous stability. The second contribution of this paper is that we turned the state time-delay stability problem into the state feedback control problem with input time delay in designing the common gain matrix K. We obtain a sufficient condition for the stability of the $N-1$ time-delay subsystems by using linear matrix inequality (LMI) method and Lyapunov stability theory. The third contribution of this paper is that we reduce the number of the inequalities by considering of the linearity and symmetry of these inequalities. When the corresponding LMI holds, the gain matrix K can be constructed for the consensus protocol to guarantee the consensusability of the MASs.

For a MAS with first-order dynamics for each agent, we employ a more direct frequency-domain analysis method to derive an explicit condition for consensusability. We find that a bound of time delay proves that if the eigenvalues of the Laplacian matrix of the network satisfy certain simple relationship, then the MAS can always achieve consensus regardless of the time delay.

We will use the following notations in this paper:

- $R^{m\times n}$ denotes the family of $m \times n$ dimensional real matrices.
- R denotes the real number field.
- I_m denotes the $m \times m$ dimensional identity matrix.
- $\mathbf{1}_m$ denotes the m dimensional column vector with all components 1.
- $\mathbf{0}_m$ denotes the m dimensional column vector with all components 0.
- $\otimes$ denotes the Kronecker product.
- For a vector or matrix X, X^T denotes its transpose, and $\|X\|$ denotes its Euclidean norm.
- For a square matrix X, X^{-1} denotes its inverse (if exists), and $\det(X)$ denotes its determinant.
- $X < 0$ means that the matrix X is negative definite, and $X^T = X$.
- $\lambda_i(X)$ denotes the ith eigenvalue of a matrix X.
- $v_{i,k\sim l}$ denotes a column vector composed of the kth to lth components of column vector v_i.
- $[v_1, v_2, \ldots, v_n]$ denotes a matrix composed of vector v_i, $i = 1, 2, \ldots, n$ of the same dimension.
- $*$ represents the elements below the main diagonal of a symmetric matrix.

2. PROBLEM FORMULATION

2.1. Algebraic graph theory

Let a simple graph (no self-loops or multiple edges) $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ denote the undirected communication topology between multi-agents with the set of vertices $\mathcal{V} = \{1, 2, \ldots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The ith vertex represents the ith agent, and the edge (i, j) denotes the communication channel between agent i and agent j. The set $\mathcal{E} \subset \{(i, j) : i, j \in \mathcal{V}\}$ is the edge set. The set of neighbors of the ith agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. $\mathcal{A} = [a_{ij}] \in R^{N\times N}$ is called

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the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements and $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$ and $i \neq j$; $a_{ij} = 0$ otherwise. The degree of the ith vertex is denoted by $d_i = \sum_{j \in \mathcal{N}_i} a_{ij} = \sum_{j=1}^{N} a_{ij}$ and the degree matrix $\mathcal{D} = \text{diag}\{d_1, d_2, \ldots, d_N\}$. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Obviously, for an undirected graph, $\mathcal{L}$ is a symmetric, positive semi-definite matrix, and all its eigenvalues are nonnegative. Note that $\mathcal{L}\mathbf{1}_N = \mathbf{0}_N$. For an undirected connected graph, the eigenvalues of $\mathcal{L}$ can be arranged as follows:

$$0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leqslant \ldots \leqslant \lambda_N(\mathcal{L}).$$

2.2. *Consensus and consensusability*

In this paper, we will consider a network consists of N agents, and the dynamics of the ith agent is given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $A \in R^{n \times n}$ and $B \in R^{n \times p}$ are general constant matrices. $x_i \in R^n$ and $u_i \in R^p$ are the state and the control input of the ith agent, respectively.

Definition 1 (Consensus)
The agents in the network achieve consensus if

$$\lim_{t \to \infty} \|x_j(t) - x_i(t)\| = 0, \quad \forall i, j \in \{1, 2, \ldots, N\}$$

for any initial value $x_i(0)$.

The consensus protocol of the ith agent is of the following form:

$$u_i(t) = K \sum_{j=1}^{N} a_{ij} \left(x_j(t - \tau) - x_i(t - \tau) \right), \quad i = 1, 2, \ldots, N, \tag{2}$$

where τ denotes the communication delay time in the network and $K \in R^{p \times n}$ is a constant gain matrix to be designed.

Remark 1
Although in practice, the communication time delay of distinct agents may be different, we can adopt the maximum consensus algorithm of Reference [18] to obtain the maximum delay time, and we can use this delay time in our consensus protocol (2).

Define

$$\mathcal{U} \triangleq \left\{ u(t) : [0, \infty) \to R^{pN} \mid u(t) = \left[u_1^T(t), u_2^T(t), \ldots, u_N^T(t) \right]^T, \ u_i(t) \text{ is defined by (2)} \right\}. \tag{3}$$

Definition 2 (Consensusable)
If there exists a $u(t) \in \mathcal{U}$ such that system (1) reaches consensus, then we say that system (1) is consensusable w.r.t. $\mathcal{U}$ (so simply, consensusable).

Remark 2
It is obvious that if A is stable, then system (1) is consensusable by taking $u(t) = 0$, because all the systems of (1) are asymptotically stable. So for the sake of making this problem meaningful, without loss of generality, we assume that A is unstable (including the case where A has eigenvalues with zero real part).

In this paper, we focus on the consensusability condition for system (1) under the admissible consensus protocol (3). Note from [4] that, when $\tau = 0$, the corresponding delay-free system can reach consensus under the following assumptions, which we will adopt for the delay case as well.

Assumption 1
The network topology $\mathcal{G}$ is an undirected connected graph.

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Assumption 2
(A, B) is stabilizable.

We list some lemmas that will be used in the proof of our main results.

Lemma 1 (Schur complement)
For a given symmetric matrix S with the form $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$, $S_{11} \in R^{r \times r}$, $S_{12} \in R^{r \times (n-r)}$, $S_{22} \in R^{(n-r)\times(n-r)}$, then $S < 0$ if and only if

(i) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$; or equivalently,
(ii) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 2
For any constant matrix $M \in R^{m \times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $w : [0, \gamma] \to R^m$, such that the integrations in the following are well defined, then

$$\gamma \int_{t-\gamma}^{t} w^T(s) M w(s)\, ds \geqslant \left(\int_{t-\gamma}^{t} w(s)\, ds \right)^T M \left(\int_{t-\gamma}^{t} w(s)\, ds \right).$$

3. TIME-DOMAIN METHOD FOR CONSENSUS

In this section, we will prove that system (1) is consensusable if and only if $N-1$ time-delay subsystems associated with the eigenvalues of the Laplacian matrix of the network are simultaneously asymptotically stable.

Let $\delta_i(t) \triangleq x_1(t) - x_i(t),\ \ i = 2, 3, \ldots, N$. Then,

$$\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0,\ \ \forall i, j \in \{1, 2, \ldots, N\}$$

is equivalent to

$$\lim_{t\to\infty} \|\delta_i(t)\| = 0,\ \ i = 2, 3, \ldots, N.$$

Moreover, we have the dynamics of $\delta_i(t)$ as follows:

$$\dot{\delta}_i(t) = A\delta_i(t) + BK \left[\sum_{j=2}^{N} (a_{ij} - a_{1j}) \delta_j(t-\tau) - d_i \delta_i(t-\tau) \right].$$

Define $\delta(t) \triangleq [\delta_2^T(t), \delta_3^T(t), \ldots, \delta_N^T(t)]^T$. Then the dynamics of the whole network has the form of

$$\dot{\delta}(t) = (I_{N-1} \otimes A)\delta(t) - \left[\left(L_{22} + \mathbf{1}_{N-1}\alpha^T \right) \otimes BK \right] \delta(t-\tau) \tag{4}$$

where

$$L_{22} = \begin{bmatrix} d_2 & -a_{23} & \cdots & -a_{2N} \\ -a_{32} & d_3 & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & d_N \end{bmatrix}, \quad \alpha = \begin{bmatrix} a_{12} \\ a_{13} \\ \vdots \\ a_{1N} \end{bmatrix}.$$

It is obvious that system (1) that is consensusable is equivalent to $\lim_{t\to\infty} \|\delta(t)\| = 0$. So next, we will focus on the asymptotically stable condition for system (4).

Note that

$$\mathcal{L} = \begin{bmatrix} d_1 & -\alpha^T \\ -\alpha & L_{22} \end{bmatrix}$$

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is the Laplacian matrix of the network and it is a symmetric matrix. Taking the transformation matrix $S = \begin{bmatrix} 1 & \mathbf{0}_{N-1}^T \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}$, then we have

$$S^{-1}\mathcal{L}S = \begin{bmatrix} 0 & -\alpha^T \\ \mathbf{0}_{N-1} & L_{22} + \mathbf{1}_{N-1}\alpha^T \end{bmatrix}. \tag{5}$$

From (5), we can see that the eigenvalues of $L_{22} + \mathbf{1}_{N-1}\alpha^T$ are the nonzero eigenvalues of $\mathcal{L}$, that is, $\lambda_2(\mathcal{L}), \lambda_3(\mathcal{L}), \cdots, \lambda_N(\mathcal{L})$. Based on this analysis, we can obtain the following result on the relationship between the eigenvectors of $\mathcal{L}$ and $L_{22} + \mathbf{1}_{N-1}\alpha^T$.

Lemma 3
Let v_i denote the right eigenvector corresponding to the eigenvalue $\lambda_i(\mathcal{L})$, $i = 2, 3, \ldots, N$. Then $v_{i,2\sim N}^T$ is the left eigenvector of $L_{22} + \mathbf{1}_{N-1}\alpha^T$ corresponding to the same eigenvalue $\lambda_i(\mathcal{L})$, $i = 2, 3, \ldots, N$.

Proof
In fact, we have

$$\mathcal{L}v_i = \begin{bmatrix} d_1 v_{i,1} - \alpha^T v_{i,2\sim N} \\ -\alpha v_{i,1} + L_{22} v_{i,2\sim N} \end{bmatrix} = \begin{bmatrix} \lambda_i(\mathcal{L}) v_{i,1} \\ \lambda_i(\mathcal{L}) v_{i,2\sim N} \end{bmatrix},$$

so

$$-\alpha v_{i,1} = \lambda_i(\mathcal{L}) v_{i,2\sim N} - L_{22} v_{i,2\sim N}. \tag{6}$$

It is obvious that $\mathbf{1}_N^T v_i = 0$, $i = 2, 3, \ldots, N$; thus, we have

$$-\alpha v_{i,1} = \alpha \mathbf{1}_{N-1}^T v_{i,2\sim N}. \tag{7}$$

From (6) and (7), we have

$$v_{i,2\sim N}^T \left(L_{22} + \mathbf{1}_{N-1}\alpha^T\right) = \lambda_i(\mathcal{L}) v_{i,2\sim N}^T.$$

That is to say, the component of the right eigenvector of $\mathcal{L}$ is the left eigenvector of $L_{22} + \mathbf{1}_{N-1}\alpha^T$ corresponding to the same eigenvalue $\lambda_i(\mathcal{L})$, $i = 2, 3, \ldots, N$.

Because $\mathcal{L}$ is a symmetric matrix, it can be diagonalized by using its eigenvector matrix; in other words, there exists a matrix $U = [u_1, u_2, \ldots, u_N]$ composed of the mutually orthogonal eigenvectors of $\mathcal{L}$ such that

$$U^{-1}\mathcal{L}U = \mathrm{diag}\{0, \lambda_2(\mathcal{L}), \cdots, \lambda_N(\mathcal{L})\}.$$

□

We take $V = [u_{2,2\sim N}, u_{3,2\sim N}, \ldots, u_{N,2\sim N}]^T$; then $U = \begin{bmatrix} 1 & \beta^T \\ \mathbf{1}_{N-1} & V^T \end{bmatrix}$ with $\beta = [u_{2,1}, u_{3,1}, \ldots, u_{N,1}]^T$; thus, V is an invertible matrix (because if V is singular, then there exist linearly dependent columns in V^T; without loss of generality, we can assume that $u_{2,2\sim N} = k u_{3,2\sim N}$, $k \in R$; then $u_{2,1} = -\mathbf{1}_{N-1}^T u_{2,2\sim N} = -k\mathbf{1}_{N-1}^T u_{3,2\sim N} = k u_{3,1}$; thus, we can obtain that $u_2 = k u_3$, but this contradicts with the invertibility of U), and we have

$$V\left(L_{22} + \mathbf{1}_{N-1}\alpha^T\right)V^{-1} = \mathrm{diag}\{\lambda_2(\mathcal{L}), \lambda_3(\mathcal{L}), \ldots, \lambda_N(\mathcal{L})\}.$$

Let $\tilde{\delta}(t) \triangleq (V \otimes I_n)\delta(t)$; then we can obtain its dynamics as follows:

$$\dot{\tilde{\delta}}(t) = (I_{N-1} \otimes A)\tilde{\delta}(t) - [\mathrm{diag}\{\lambda_2(\mathcal{L}), \lambda_3(\mathcal{L}), \cdots, \lambda_N(\mathcal{L})\} \otimes BK]\,\tilde{\delta}(t-\tau).$$

Take $\tilde{\delta}(t) = \left[\tilde{\delta}_2^T(t), \tilde{\delta}_3^T(t), \cdots, \tilde{\delta}_N^T(t)\right]^T$; then we can obtain the following theorem:

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Theorem 1
Systems (1) and (2) achieve consensus if and only if the following $N-1$ time-delay subsystems are simultaneously asymptotically stable:

$$\dot{\tilde{\delta}}_i(t) = A\tilde{\delta}_i(t) - \lambda_i(\mathcal{L})BK\tilde{\delta}_i(t-\tau), \quad i = 2, 3, \ldots, N. \tag{8}$$

In order to find a gain matrix K such that every time-delay subsystem of (8) is asymptotically stable, we first consider the stability of the following time-delay system:

$$\dot{\eta}(t) = A\eta(t) - aBF\eta(t-\tau), \quad a \in [\lambda_2(\mathcal{L}), \lambda_N(\mathcal{L})], \tag{9}$$

It is obvious that system (9) is equivalent to

$$\dot{\eta}(t) = A\eta(t) - aBu(t-\tau), \quad a \in [\lambda_2(\mathcal{L}), \lambda_N(\mathcal{L})], \tag{10}$$

with a state feedback control law of form

$$u(t) = F\eta(t). \tag{11}$$

And for this problem of (10) and (11), we have the following result.

Lemma 4
For the given constant $\bar{\tau} > 0$, if there exist symmetric positive definite matrices W, X , Z, and Y of appropriate dimension such that the following linear matrix inequalities hold,

$$\begin{bmatrix} XA^T + AX - XWX & -aBY + XWX & XA^TW \\ * & -XWX & -a\bar{\tau}Y^TB^TW \\ * & * & -W \end{bmatrix} < 0, \tag{12}$$

and

$$XA^TZX + XZAX - aXZBY - aY^TB^TZX < 0, \tag{13}$$

then systems (10) and (11) are asymptotically stable for any $\tau \in [0, \bar{\tau}]$, and the gain matrix can be taken as $F = YX^{-1}$.

Proof
The stability of systems (10) and (11) is equivalent to that of the closed-loop system (9). Set $X = S^{-1}$. Then we can define a Lyapunov functional for (9) as follows:

$$V(\eta(t)) \triangleq \eta^T(t)S\eta(t) + \bar{\tau}\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\eta}^T(s)W\dot{\eta}(s)\,ds\,d\theta.$$

Thus, we can obtain the derivative of $V(\eta(t))$ about t as follows:

$$\begin{aligned}\dot{V}(\eta(t)) = {} & \eta^T(t)\left(A^TS + SA\right)\eta(t) - a\eta^T(t-\tau)F^TB^TS\eta(t) - a\eta^T(t)SBF\eta(t-\tau) \\ & + \bar{\tau}\tau\dot{\eta}^T(t)W\dot{\eta}(t) - \bar{\tau}\int_{t-\tau}^{t}\dot{\eta}^T(s)W\dot{\eta}(s)\,ds.\end{aligned}$$

Then from Lemma 2, we have

$$-\bar{\tau}\int_{t-\tau}^{t}\dot{\eta}^T(s)W\dot{\eta}(s)\,ds \leqslant -\frac{\bar{\tau}}{\tau}\left(\int_{t-\tau}^{t}\dot{\eta}^T(s)\,ds\right)W\left(\int_{t-\tau}^{t}\dot{\eta}(s)\,ds\right).$$

Because $W > 0$ and $\tau \leqslant \bar{\tau}$, then we have

$$\bar{\tau}\tau\dot{\eta}^T(t)W\dot{\eta}(t) \leqslant \bar{\tau}^2\dot{\eta}^T(t)W\dot{\eta}(t).$$

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Thus,

$$\begin{aligned}\dot{V}(\eta(t)) &\leqslant \eta^T(t)\left(A^TS + SA + \bar{\tau}^2A^TWA\right)\eta(t) - a\eta^T(t)\left(SBF + \bar{\tau}^2A^TWBF\right)\eta(t-\tau)\\ &\quad - a\eta^T(t-\tau)\left(F^TB^TS + \bar{\tau}^2F^TB^TWA\right)\eta(t)\\ &\quad - \left(\eta^T(t) - \eta^T(t-\tau)\right)W\left(\eta(t) - \eta(t-\tau)\right)\\ &= \xi^T(t)\Phi\xi(t),\end{aligned}$$

where $\xi(t) = [\eta^T(t), \eta^T(t-\tau)]^T$ and

$$\Phi = \begin{bmatrix} A^TS + SA + \bar{\tau}^2A^TWA - W & -aSBF - a\bar{\tau}^2A^TWBF + W \\ * & a^2\bar{\tau}^2F^TB^TWBF - W \end{bmatrix}.$$

Using the (ii) of Schur complement, we see that $\Phi < 0$ is equivalent to

$$\begin{bmatrix} A^TS + SA - W & -aSBF + W & A^T \\ * & -W & -aF^TB^T \\ * & * & -\frac{1}{\bar{\tau}^2}W^{-1} \end{bmatrix} < 0.$$

Note that the last matrix is nonlinear. By premultiplying and postmultiplying diag$\{S^{-1}, S^{-1}, \bar{\tau}W\}$, we can obtain the following equivalent linear matrix inequality:

$$\begin{bmatrix} S^{-1}A^T + AS^{-1} - S^{-1}WS^{-1} & -aBFS^{-1} + S^{-1}WS^{-1} & \bar{\tau}S^{-1}A^TW \\ * & -S^{-1}WS^{-1} & -a\bar{\tau}S^{-1}F^TB^TW \\ * & * & -W \end{bmatrix} < 0.$$

Because $X = S^{-1}$ and we define $Y = FX$, so we have

$$\begin{bmatrix} XA^T + AX - XWX & -aBY + XWX & \bar{\tau}XA^TW \\ * & -XWX & -a\bar{\tau}Y^TB^TW \\ * & * & -W \end{bmatrix} < 0.$$

That is to say, $\dot{V}(\eta(t)) < 0$ if the linear matrix inequality (12) holds; then from Lyapunov stability theorem, we know that systems (9)–(11) are asymptotically stable. In addition, if the linear matrix inequality (13) holds, then we have $A^TZ + ZA - aZBYX^{-1} - aX^{-1}Y^TB^TZ < 0$, so the matrix $A - aBYX^{-1}$ is a stable matrix. That is to say, when $\tau = 0$, the corresponding delay-free system is asymptotically stable.

In summary, if the linear matrix inequalities (12) and (13) hold, then system (9) (or systems (10) and (11)) is asymptotically stable for any $\tau \in [0, \bar{\tau}]$ by taking the gain matrix to be $F = YX^{-1}$. □

Remark 3

Note the linearity and symmetry of inequalities (12) and (13). So if they hold for boundary values $\lambda_2(\mathcal{L})$ and $\lambda_N(\mathcal{L})$, then they hold in $[\lambda_2(\mathcal{L}), \lambda_N(\mathcal{L})]$. Thus, we can obtain the following result.

From Theorem 1, Lemma 4, and Remark 3, we can obtain the following result:

Theorem 2

For a given constant $\bar{\tau} > 0$, if there exist symmetric positive definite matrices W, X, Z, and Y of appropriate dimension such that the following linear matrix inequalities hold,

$$\begin{bmatrix} XA^T + AX - XWX & -\lambda_2(\mathcal{L})BY + XWX & XA^TW \\ * & -XWX & -\lambda_2(\mathcal{L})\bar{\tau}Y^TB^TW \\ * & * & -W \end{bmatrix} < 0, \tag{14}$$

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$$\begin{bmatrix} XA^T + AX - XWX & -\lambda_N(\mathcal{L})BY + XWX & XA^T W \\ * & -XWX & -\lambda_N(\mathcal{L})\bar{\tau}Y^T B^T W \\ * & * & -W \end{bmatrix} < 0, \tag{15}$$

$$XA^T ZX + XZAX - \lambda_2(\mathcal{L})XZBY - \lambda_2(\mathcal{L})Y^T B^T ZX < 0, \tag{16}$$

and

$$XA^T ZX + XZAX - \lambda_N(\mathcal{L})XZBY - \lambda_N(\mathcal{L})Y^T B^T ZX < 0, \tag{17}$$

then system (1) is consensusable for $\tau \in [0, \bar{\tau}]$ with the common gain matrix taken as $K = YX^{-1}$.

Remark 4
Our results in Theorem 2 has a close relationship with the second smallest eigenvalue of Laplacian matrix, that is, $\lambda_2(\mathcal{L})$, and the largest eigenvalue of Laplacian matrix, that is, $\lambda_N(\mathcal{L})$. From Remark 3, we know that, as long as we know that the linear matrix inequalities (14)–(17) hold for the lower bound of $\lambda_2(\mathcal{L})$ and the upper bound of $\lambda_N(\mathcal{L})$, then system (1) is consensusable. We have estimates for them as follows.

Lemma 5 ([19]*)*
Let $\mathcal{G}$ be a simple graph on N vertices. Then

$$\lambda_N(\mathcal{L}) \leqslant \max\{d_u + d_v | (u, v) \in \mathcal{E}\},$$

where d_u is the degree of vertex u. And if $N \geqslant 3$, then

$$\lambda_2(\mathcal{L}) \geqslant d^*,$$

with equality if $\mathcal{G}$ is a complete bipartite graph, where d^* denotes the second largest degree of $\mathcal{G}$.

4. FREQUENCY-DOMAIN CRITERIA FOR CONSENSUS

Because we know that the system is asymptotically stable if and only if all the eigenvalues of its characteristic equation have the negative real part, so from Theorem 1, we have the following result.

Lemma 6
System (1) is consensusable if and only if there exists a matrix K such that all the roots s of the following characteristic equations

$$\det[sI - A + \lambda_i(\mathcal{L})BKe^{-\tau s}] = 0, \quad i = 2, 3, \ldots, N \tag{18}$$

have the negative real parts.

Remark 5
Apparently, the analysis of the distribution of the roots of (18) is not easy, but for some special systems (such as first-order integrator systems), it is possible to have some more explicit results.

For first-order integrator systems,

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N, \tag{19}$$

the consensus protocol of the ith agent is given by

$$u_i(t) = k \sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t-\tau)), \quad i = 1, 2, \ldots, N \tag{20}$$

where $k \in R$ is the common gain and $\tau \geqslant 0$ is the delay time. We have the following results.

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Lemma 7
System (19) is consensusable if and only if all the roots s of the following equations

$$s + \lambda_i(\mathcal{L})ke^{-\tau s} = 0, \quad i = 2, 3, \ldots, N \tag{21}$$

have the negative real parts.

The proof of this lemma directly follows from Theorem 1 and is thus omitted.

Theorem 3
System (19) is consensusable if and only if $k > 0$ and $\tau < \tau_{\max}$, where

$$\tau_{\max} = \frac{\pi}{2k\lambda_N(\mathcal{L})}.$$

Proof
We should note that k should stabilize the delay-free system of (19) and (20); that is to say, when $\tau = 0$, the following equations

$$s + \lambda_i(\mathcal{L})k = 0, \quad i = 2, 3, \ldots, N$$

have the negative real parts; thus, there must be $k > 0$.

By the continuous dependence of the roots on parameter τ, we only need to consider that there is no imaginary axis root and zero root of (19) when $\tau < \tau_{\max}$, where $\tau_{\max}$ is a parameter to be determined.

Note that the complex roots of (19) are distributed symmetrically on the complex plane with respect to the real axis; thus, we only need to consider the case of $s = \iota w_i$ ($\iota^2 = -1$), $w_i \geqslant 0$, $i = 2, 3, \ldots, N$ for (19). That is to say, we only need to guarantee that the following equations

$$\iota w_i + \lambda_i(\mathcal{L})ke^{-\iota w_i \tau} = 0, \quad i = 2, 3, \cdots, N \tag{22}$$

do not have solutions $w_i \geqslant 0, i = 2, 3, \cdots, N$.

If there is a $i^* \in \{2, \cdots, N\}$ such that

$$\iota w_{i^*} + \lambda_{i^*}(\mathcal{L})ke^{-\iota w_{i^*}\tau} = 0, \tag{23}$$

where $w_{i^*} \geqslant 0$, then by separating the real part and imaginary part of (23), we obtain that

$$\lambda_{i^*}(\mathcal{L})k\cos(w_{i^*}\tau) = 0, \tag{24}$$

$$w_{i^*} - \lambda_{i^*}(\mathcal{L})k\sin(w_{i^*}\tau) = 0. \tag{25}$$

From (24), we have $\cos(w_{i^*}\tau) = 0$, so $w_{i^*} \neq 0$, and there must be $w_{i^*}\tau = l\pi + \frac{\pi}{2}, l = 0, 1, \ldots$. From (25), we have $\frac{w_{i^*}}{\sin(w_{i^*}\tau)} = \lambda_{i^*}(\mathcal{L})k > 0$, so $\sin(w_{i^*}\tau) > 0$; thus, $w_{i^*}\tau = 2l\pi + \frac{\pi}{2}, l = 0, 1, \ldots$. That is to say, $k\lambda_{i^*}(\mathcal{L}) = w_{i^*} = \frac{2l\pi + \frac{\pi}{2}}{\tau}, l = 0, 1, \ldots$. So we have $\tau = \frac{2l\pi + \frac{\pi}{2}}{k\lambda_{i^*}(\mathcal{L})}$; thus, we can see that (22) has no solution of $s = \iota w_i$, $w_i \geqslant 0, i = 2, 3, \cdots, N$ if and only if $\tau < \frac{\pi}{2k\lambda_N(\mathcal{L})}$. □

Remark 6
From Theorem 3, we know that by choosing a relatively small (or large) gain $k > 0$, we can guarantee that system (19) is consensusable when the time delay is large (or small).

5. SIMULATION

We consider a network of three agents (Figure 1) with first-order integrator dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, 3. \tag{26}$$

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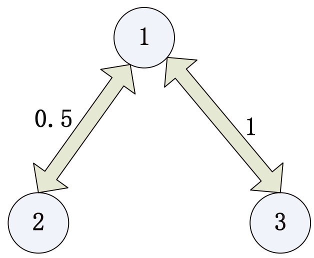

Figure 1. Network topology 1.

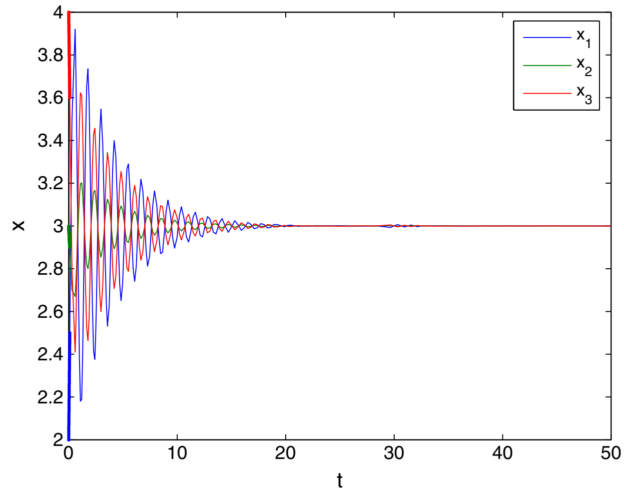

Figure 2. $\tau < \tau_{\max}$.

The consensus protocol of the ith agent is given by

$$u_i(t) = 2\sum_{j=1}^{3} a_{ij}(x_j(t-\tau) - x_i(t-\tau)). \tag{27}$$

From Theorem 3, we know that this network can reach consensus if and only if $\tau < \tau_{\max} = \frac{\pi}{2(3+\sqrt{3})} \approx 0.33$. Figure 2 displays the simulation result of $\tau = 0.3 < \tau_{\max}$, and the system reached consensus. Figure 3 is the simulation result of $\tau = 0.335 > \tau_{\max}$, and in this case, the states are divergent. So the simulation results are consistent with our conclusion in Theorem 3.

Also, we can consider a network of seven agents with first-order integrator dynamics like (26), and the corresponding adjacency matrix is given by

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 & 4 \\ 2 & 0 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The consensus protocol is the same as (27) in form with $k = 0.5$. Similarly, we know that this network can reach consensus if and only if $\tau < \tau_{\max} \approx 0.1732$. Figures 4 and 5 display the simulation results of $\tau = 0.17 < \tau_{\max}$ and $\tau = 0.18 > \tau_{\max}$, respectively. Apparently, the simulation results are also consistent with our conclusion in Theorem 3.

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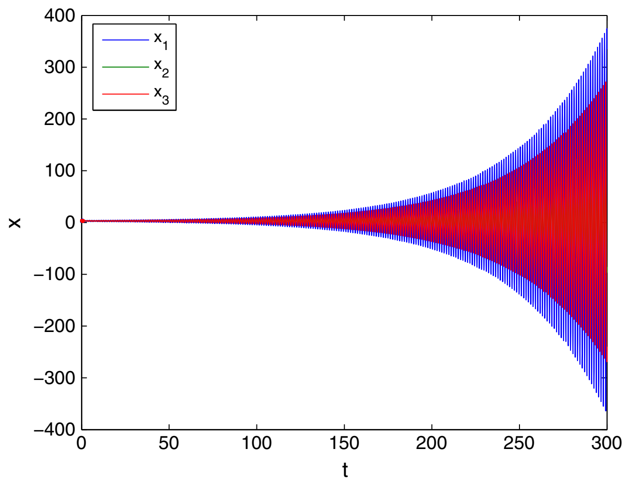

Figure 3. $\tau > \tau_{\max}$.

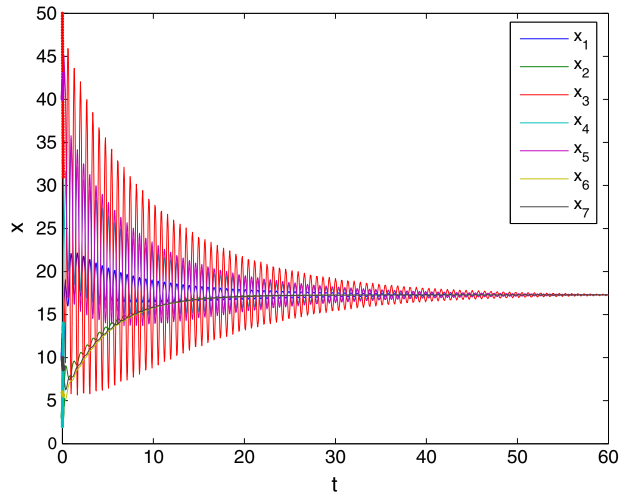

Figure 4. $\tau < \tau_{\max}$.

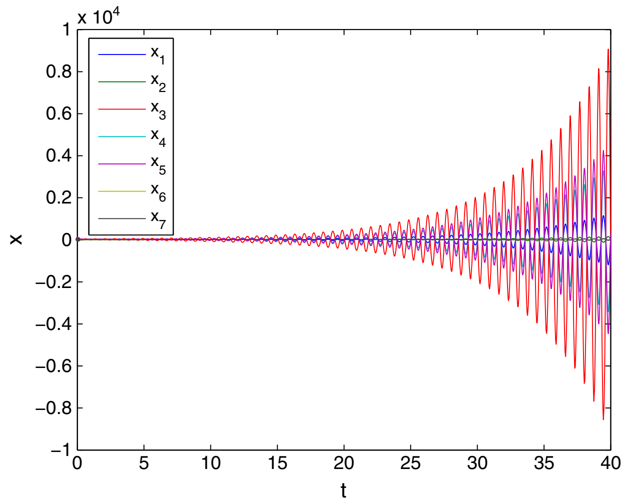

Figure 5. $\tau > \tau_{\max}$.

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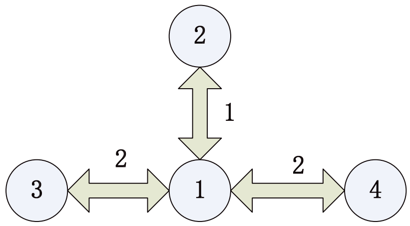

Figure 6. Network topology 2.

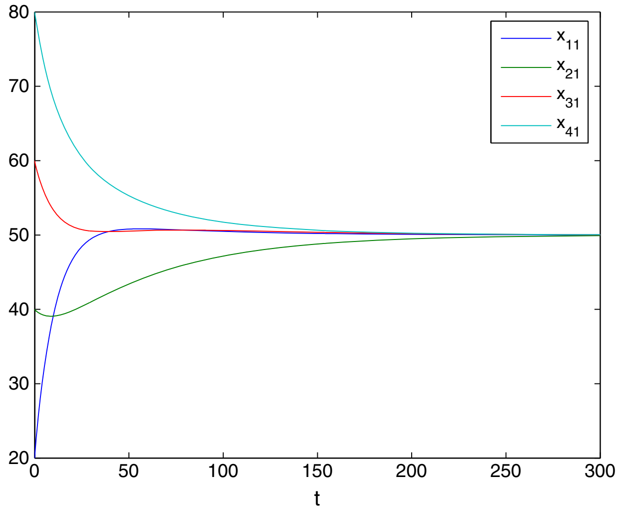

Figure 7. The first component of system (28).

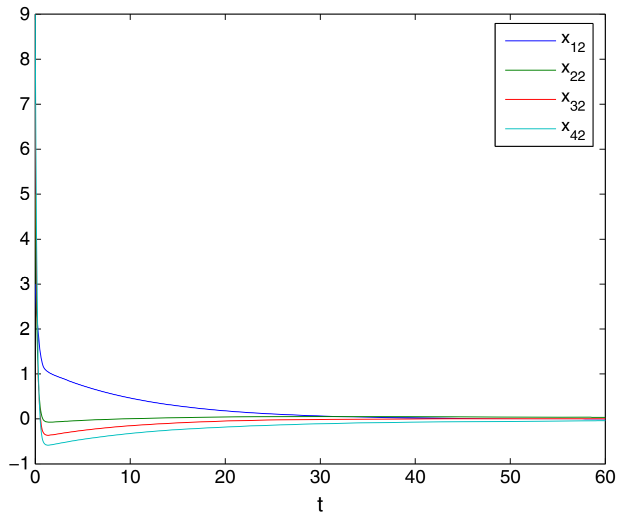

Figure 8. The second component of system (28).

Now, consider a network of four agents (Figure 6) with ith dynamics described by

$$\dot{x}_i = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_i, \quad i = 1, 2, 3, 4. \tag{28}$$

Apparently, (A, B) is stabilizable. Here, we assume that $\bar{\tau} = 0.5$. By calculating, we know that $\lambda_2(\mathcal{L}) = 1.1716$, $\lambda_4(\mathcal{L}) = 6.8284$. From Theorem 2, by using the LMI Toolbox of the MATLAB (MathWorks, Inc., Natick, MA, USA), we can obtain the results of $K = YX^{-1} = \begin{bmatrix} 0.0148 & 0.0463 \end{bmatrix}$.

Based on this result, we can obtain the simulation result of system (28) as Figures 7 and 8. They illustrate the first and second components of $x_i = [x_{i1}, x_{i2}]^T$, $i = 2, 3, 4$, and both of them take on the tendency of being asymptotically stable. Thus, x_i, $i = 1, 2, 3, 4$ reach consensus.

6. CONCLUSION

This paper studied the consensusability of high-order linear MASs with uniform constant communication delay in an undirected network. The consensusability problem of N agents can be turned into simultaneous stability of $N-1$ time-delay subsystems associated with the eigenvalues of Laplacian matrix by employing an appropriate linear transformation. We presented a sufficient condition for the MAS to reach consensus in the form of some linear matrix inequalities associated with the eigenvalues of Laplacian matrix of the network. We also considered the consensusability condition for first-order integrators by using frequency-domain analysis and showed that by choosing a relatively small (or large) gain, we can guarantee that the agents achieve consensus when the time delay is large (or small).

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