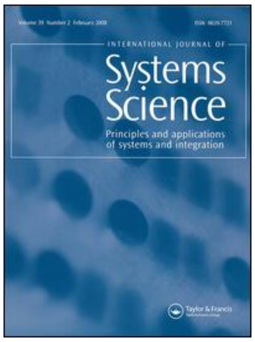


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# Constrained consensus of discrete-time multi-agent systems with time delay

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**ABSTRACT**
In this paper, we consider the consensus conditions for discrete-time multi-agent systems with communication delay between agents, subject to that each agent's state is constrained to lie in a given convex set. And we will present some consensus conditions for unconstrained multi-agent systems with time delay.

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Constrained consensus; stochastic matrix; time delay; multi-agent systems; projection

## 1. Introduction

Recently, there has been considerable interest in distributed control problems for multi-agent systems, in which several autonomous agents collectively try to achieve a global objective by using local measurements and local communications. One fundamental problem for distributed control is the well-studied consensus control problem (Cao, Morse, & Anderson, 2008; Hendrickx & Tsitsiklis, 2011; Hua, You, & Guan, 2016; Jadbabaie, Lin, & Morse, 2003; Kashyap, Başar, & Srikant, 2007; Moreau, 2005; Nedic & Liu, 2014; Olfati-Saber & Murry, 2004; Ren & Beard, 2005; Shi, Johansson, & Hong, 2013; Shi, Xia, & Johansson, 2015; Touri & Nedić, 2014), where the agents in a network all aim to agree on a common quantity via local information exchange only with their neighbouring agents. The consensus convergence property depends on the network topology, the form of consensus protocol and the dynamics of agents.

A special important consensus problem is called constrained consensus problem (Lee & Mesbahi, 2011; Lin & Ren, 2012; Liu & Chen, 2012; Nedić, Ozdaglar, & Parrilo, 2010; Qiu, Liu, & Xie, 2016; Sun, Ong, & White, 2013), where each agent's value (or state) is constrained to a given set. Such constraints are significant in a number of applications including motion planning and alignment problems (where each agent's position is limited to a certain region or range) and distributed constrained multi-agent optimisation problems. Nedic et al. (2010) presented a constrained consensus problem for a discrete-time system with the state of the $i$th agent restricted to lie in a closed convex sets $X_i$ only known to the $i$th agent. Lee and Mesbahi (2011) proposed a constrained consensus algorithm for continuous-time system by using logarithmic barrier functions.

In real applications, communications among agents are typically subject to time delays, due to, for example, limited communication bandwidth. The presence of time delays may cause degraded performance, poor robustness or even instability of a multi-agent system. Xie (2016), Hou, Fu, Zhang, and Wu (2017) and Jiang, Xie, and Cao (2017) only consider the consensus problem with time delay. It is thus important to consider the constrained consensus problem with time delay.

This paper considers a constrained consensus problem with time delay for discrete-time multi-agent systems. First, we gave some convergence conditions for a general consensus algorithm by using the a special property of stochastic matrices. Then, we connect the original system with the consensus algorithm and we can present some consensus conditions for the original system with time delay.

## 2. Algebraic graph theory basics

The communication topology between multi-agents will be denoted by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of vertices, and vertex $i$ denotes the $i$th agent. $\mathcal{E} \subset \{(i, j) : i, j \in \mathcal{V}\}$ is edges set. $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ is the weighted adjacency matrix. The edge $(i, j) = (j, i)$ denotes that the communication channel between agent

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$i$ and agent $j$ is bi-directional. The neighbours $\mathcal{N}_i$ of vertex $i$ is the set of the vertices that can communicate with the $i$th agent. For any $i$, $j \in \mathcal{V}$, and $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. $d_i = \sum_{j=1}^{N} a_{ij}$ is called the degree of the $i$th vertex. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We define the Laplacian matrix of graph $\mathcal{G}$ as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(d_1, d_2, \dots, d_N)$. All the eigenvalues of $\mathcal{L}$ are $\lambda_i$, $i = 1, 2, \dots, N$.

We say that a vector is *stochastic* if its entries are nonnegative and sum to 1. A square matrix is said to be *stochastic* if its entries are nonnegative and its row sums all equal 1 (see Richard, Seymour, & Hans, 1966). A matrix $A = [a_{ij}] \in R^{n\times n}$ is nonnegative, if all its entries are nonnegative. $\mathbf{1}_m$ denotes the $m$-dimensional column vector with all components 1. $X_i \subset R$ be a nonempty closed convex set, $i = 1, 2, \dots, N$. We use $\|\cdot\|$ for the Euclidean norm. The Euclidean projection (or projection, for short) of a point $x$ on a convex closed set $X$ is denoted by $P_X[x]$, i.e. $P_X[x] = \arg\min_{y\in X} \|y - x\|$.

**Definition 2.1** (Consensus)**:** A multi-agent system is said to achieve consensus if, for any initial conditions and $i \neq j$, $i, j = 1, 2, \dots, N$, $\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0$, where $x_i(k)$ is the state of $i$th agent.

We will use the following important non-expansiveness property of projection.

**Lemma 2.1** (Liu, Nedić, & Başar, 2014)**:** *Let $Y \subset R^n$ be a nonempty closed convex set. Then, for any $x \in R^n$ and $z \in Y$, there holds*

$$\|P_Y[x] - z\|^2 \leq \|x - z\|^2 - \|P_Y[x] - x\|^2.$$

Next, we present a property of stochastic matrices.

**Lemma 2.2** (Fang & Antsaklis, 2005)**:** *Let $F \in R^{n\times n}$ be a stochastic matrix. Then, $F$ has a unique eigenvalue at 1 with maximum modulus if and only if the graph associated with $F$ has a spanning tree. In this case, $\lim_{m\to\infty} F^m = \mathbf{1}\mu^T$, where $m \in N^+$, and $\mu = [\mu_1, \mu_2, \dots, \mu_n]^T \geq 0$ satisfies $\mu^T F = \mu^T$ and $\mathbf{1}^T\mu = 1$.*

## 3. Constrained consensus with time delay

Here, we consider the constrained consensus problem with constant communication time delay $\tau$ and the state of the $i$th agent is constrained to lie in a nonempty closed convex set $X_i$ known only to agent $i$. The objective is to cooperatively reach a consensus on a common vector through a sequence of local estimate updates and local information exchanges. We employ the following projected consensus algorithm:

$$x_i(k+1) = P_{X_i}\left[x_i(k-\tau) + \alpha \sum_{j=1}^{N} a_{ij}[x_j(k-\tau) - x_i(k-\tau)]\right], \quad (1)$$

where $x_i(k) \in R$ is the state of the $i$th agent at time $k$.

**Remark 3.1:** We note in the consensus protocol above that the same delay $\tau$ also applies to node $i$. This is to ensure that correct error signals are used in the feedback to guarantee the consensus. In applications where $x_i(t)$ is instantaneously known to node $i$, this signal needs to be delayed before being applied in controller. In applications where only relative information can be measured (e.g. $x_i(t)$ is not directly measured but only $x_i(t) - x_j(t)$ is measured) and time delay is involved in the measurement, taking the same time delay for node $i$ and node $j$ is natural. Note that relative measurements are common, including relative distance, relative velocity, etc.

First, we consider the following projected consensus algorithm:

$$x_i(k+1) = P_{X_i}\left[\sum_{j=1}^{m} b_{ij}x_j(k-\tau)\right], \quad k \geq 0. \quad (2)$$

In order to analyse the consensus conditions of (2), we introduce a technical lemma based on the next assumption.

**Assumption 3.1:** *$B = [b_{ij}] \in R^{m\times m}$ is a stochastic matrix. $T$ is the graph associated with adjacent matrix $B$, and $T$ has a spanning tree.*

Set

$$v_i(k) = \sum_{j=1}^{m} b_{ij}x_j(k-\tau), \quad e_i(k) = P_{X_i}[v_i(k)] - v_i(k).$$

Then, (2) is equivalent to

$$x_i(k+1) = P_{X_i}[v_i(k)] = v_i(k) + e_i(k).$$

We have the following result for the algorithm (2).

**Lemma 3.1:** *Suppose Assumption 3.1 holds and that the intersection $X = \bigcap_{i=1}^{m} X_i$ is nonempty. Then, for algorithm (2), there hold*

(*a*) *$\lim_{k\to+\infty} e_i(k) = 0$, $i = 1, 2, \dots, m$;*
(*b*) *$\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0$;*

(c) *there exists a unique* $x^* \in X$ *such that* $lim_{k\to+\infty} x_i(k) = x^*$, $i = 1, 2, \dots, m$.

***Proof:***

(a) Since $X$ is nonempty, we can take $z \in X$. Obviously, $z \in X_i$, $i = 1, 2, \dots, m$, then from Lemma 2.1 we have

$$\|P_{X_i}[v_i(k)] - z\|^2 \le \|v_i(k) - z\|^2 - \|P_{X_i}[v_i(k)] - v_i(k)\|^2.$$

i.e.

$$\|x_i(k+1) - z\|^2 \le \|v_i(k) - z\|^2 - \|e_i(k)\|^2. \tag{3}$$

So, we have

$$\begin{aligned}\|v_i(k) - z\|^2 &= \left\|\sum_{j=1}^m b_{ij} x_j(k-\tau) - z\right\|^2 \\ &= \left\|\sum_{j=1}^m b_{ij}\left[x_j(k-\tau) - z\right]\right\|^2 \\ &\le \sum_{j=1}^m b_{ij}\left\|x_j(k-\tau) - z\right\|^2.\end{aligned} \tag{4}$$

From Lemma 2.2 we know that $\lim_{k\to\infty} B^k = \mathbf{1}\mu^T$, where $k \in N^+$, and $\mu = [\mu_1, \mu_2, \dots, \mu_m]^T \ge 0$ satisfies $\mu^T B = \mu^T$ and $\mathbf{1}^T\mu = 1$. By setting $\hat{\mu}_i = \mu_i$, if $\mu_i > 0$, and $\hat{\mu}_i = \frac{1}{m}$, if $\mu_i = 0$. Then based on (4), we get

$$\begin{aligned}&\sum_{i=1}^m \hat{\mu}_i \|v_i(k) - z\|^2 \\ &\quad\le \sum_{i=1}^m \hat{\mu}_i \sum_{j=1}^m b_{ij}\left\|x_j(k-\tau) - z\right\|^2 \\ &\quad= \sum_{i=1}^m \hat{\mu}_i \|x_i(k-\tau) - z\|^2,\end{aligned} \tag{5}$$

so from (3) and (5)

$$\begin{aligned}&\sum_{i=1}^m \hat{\mu}_i \|x_i(k+1) - z\|^2 \\ &\quad\le \sum_{i=1}^m \hat{\mu}_i \|v_i(k) - z\|^2 - \sum_{i=1}^m \hat{\mu}_i \|e_i(k)\|^2 \\ &\quad\le \sum_{i=1}^m \hat{\mu}_i \|x_i(k-\tau) - z\|^2 - \sum_{i=1}^m \hat{\mu}_i \|e_i(k)\|^2.\end{aligned} \tag{6}$$

Thus,

$$\sum_{i=1}^m \hat{\mu}_i \|e_i(k)\|^2 \le \sum_{i=1}^m \hat{\mu}_i \|x_i(k-\tau) - z\|^2 - \sum_{i=1}^m \hat{\mu}_i \|x_i(k+1) - z\|^2.$$

So,

$$\begin{aligned}\sum_{k=\tau}^\infty \sum_{i=1}^m \hat{\mu}_i \|e_i(k)\|^2 &\le \sum_{k=\tau}^\infty \left[\sum_{i=1}^m \hat{\mu}_i \|x_i(k-\tau) - z\|^2 - \sum_{i=1}^m \hat{\mu}_i \|x_i(k+1) - z\|^2\right] \\ &= \sum_{l=0}^\tau \sum_{i=1}^m \hat{\mu}_i \|x_i(l) - z\|^2 < \infty.\end{aligned}$$

Since

$$\sum_{k=\tau}^\infty \sum_{i=1}^m \hat{\mu}_i \|e_i(k)\|^2 \ge \hat{\mu}_i \sum_{k=\tau}^\infty \|e_i(k)\|^2, \quad i = 1, 2, \dots, m,$$

then we have

$$\sum_{k=\tau}^\infty \|e_i(k)\|^2 < \infty, \quad i = 1, 2, \dots, m,$$

so we have $\lim_{k\to+\infty} e_i(k) = 0$, $i = 1, 2, \dots, m$.

(b) Since $\lim_{k\to+\infty} B^k = \mathbf{1}\mu^T$, so $\lim_{k\to+\infty} |[B^k]_{il} - [B^k]_{jl}| = 0$, $\forall i, j, l = 1, 2, \dots, m$. Then for $\forall \varepsilon > 0$, there exists some $K_1 > 0$ such that $|[B^k]_{il} - [B^k]_{jl}| \le \varepsilon$ if $k \ge K_1$. Also from (a) we know that there exists some $K_2 > 0$ such that $\|e_i(k)\| \le \varepsilon$ when $k \ge K_2$. Thus for $q \ge K_1(\tau+1) + K_2$, we have

$$\begin{aligned}&\sum_{r=1}^{q-1}\left(\sum_{l=1}^m \left|[B^r]_{il} - [B^r]_{jl}\right| \|e_l(k - r(\tau+1))\|\right) \\ &\quad= \sum_{l=1}^m \sum_{r=1}^{K_1} \left|[B^r]_{il} - [B^r]_{jl}\right| \|e_l(k - r(\tau+1))\| \\ &\qquad+ \sum_{l=1}^m \sum_{r=K_1+1}^{q-1} \left|[B^r]_{il} - [B^r]_{jl}\right| \|e_l(k - r(\tau+1))\|\end{aligned}$$

$$\leq \varepsilon \sum_{l=1}^{m} \sum_{r=1}^{K_1} \left|[B^r]_{il} - [B^r]_{jl}\right| + \varepsilon \sum_{l=1}^{m} \sum_{r=K_1+1}^{q-1} \|e_l(k - r(\tau + 1))\|.$$

Besides,

$$\begin{aligned} x_i(k+1) &= \sum_{l=1}^{m} b_{il} x_l(k-\tau) + e_i(k) \\ &= \sum_{l=1}^{m} [B^q]_{il} x_l(k+1-q(\tau+1)) \\ &\quad + \sum_{r=1}^{q-1} \left( \sum_{l=1}^{m} [B^r]_{il} e_l(k - r(\tau+1)) \right) \\ &\quad + e_i(k). \end{aligned}$$

Here, $q$ is chosen to be satisfying $k + 1 - q(\tau + 1) \in [0, 10)$, and $k \to +\infty$ is equivalent to $q \to +\infty$.
So, we have

$$\begin{aligned} &\|x_i(k+1) - x_j(k+1)\| \\ &= \Bigg\| \sum_{l=1}^{m} \left([B^q]_{il} - [B^q]_{jl}\right) x_l(k+1-q(\tau+1)) \\ &\quad + \sum_{r=1}^{q-1} \left( \sum_{l=1}^{m} \left([B^r]_{il} - [B^r]_{jl}\right) e_l(k - r(\tau+1)) \right) \\ &\quad + e_i(k) - e_j(k) \Bigg\| \\ &\leq \sum_{l=1}^{m} \left|[B^q]_{il} - [B^q]_{jl}\right| \|x_l(k+1-q(\tau+1))\| \\ &\quad + \sum_{r=1}^{q-1} \left( \sum_{l=1}^{m} \left|[B^r]_{il} - [B^r]_{jl}\right| \|e_l(k - r(\tau+1))\| \right) \\ &\quad + \|e_i(k)\| + \|e_j(k)\| \\ &\leq \varepsilon \sum_{l=1}^{m} \|x_l(k+1-q(\tau+1))\| \\ &\quad + \varepsilon \sum_{l=1}^{m} \sum_{r=1}^{K_1} \left|[B^r]_{il} - [B^r]_{jl}\right| \\ &\quad + \varepsilon \sum_{l=1}^{m} \sum_{r=K_1+1}^{q-1} \|e_l(k - r(\tau+1))\| + 2\varepsilon. \end{aligned}$$

Thus, $\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0$.

(c) First, we prove the existence of common accumulation point in $X$. From (6) we know that for any $z \in X$, $\{\sum_{i=1}^{m} \hat{\mu}_i \left\|x_i(k(\tau+1)+l) - z\right\|^2\}$, $l = 0, 1, \ldots, \tau$ are non-increasing in $k$, and also bounded. Then each $\{\hat{\mu}_i \|x_i(k(\tau+1)+l) - z\|^2\}$, $l = 0, 1, \ldots, \tau$, is bounded, so $\{x_i(k)\}$, $i = 1, 2, \ldots, m$ are also bounded and have an accumulation point. And from $\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0$ we can see that, the accumulation points of $\{x_i(k)\}$, $i = 1, 2, \ldots, m$ are the same. Since $\{x_i(k)\} \subset X_i$, $i = 1, 2, \ldots, m$, so the accumulation points belong to $X_i$, $i = 1, 2, \ldots, m$, thus the accumulation points belong to $X$.
Then, we employ reduction to absurdity to deduce the uniqueness of the accumulation point. Since if there exist two accumulation points $a_1$ and $a_2$, and without loss of generality, we can assume that $\{x_i(k_s(\tau+1)+l)\}$ and $\{x_i(\bar{k}_s(\tau+1)+l))\}$, $k_s < \bar{k}_s$, convergence to $a_1 \in X$, $a_2 \in X$, respectively. Then from (6), we have

$$\sum_{i=1}^{m} \hat{\mu}_i \left\|x_i(\bar{k}_s(\tau+1)+l)) - a_1\right\|^2 \leq \sum_{i=1}^{m} \hat{\mu}_i \|x_i(k_s(\tau+1)+l)) - a_1\|^2 .$$

Since $1 \geq \hat{\mu}_i \geq \beta =: \min_{j=1,2,\ldots,m} \{\hat{\mu}_j\} > 0$, $i = 1, 2, \ldots, m$, so

$$\begin{aligned} \beta \sum_{i=1}^{m} \left\|x_i(\bar{k}_s(\tau+1)+l)) - a_1\right\|^2 &\leq \sum_{i=1}^{m} \hat{\mu}_i \left\|x_i(\bar{k}_s(\tau+1)+l)) - a_1\right\|^2 \\ &\leq \sum_{i=1}^{m} \hat{\mu}_i \|x_i(k_s(\tau+1)+l)) - a_1\|^2 \\ &\leq \sum_{i=1}^{m} \|x_i(k_s(\tau+1)+l)) - a_1\|^2 . \end{aligned}$$

Let $s \to \infty$, then $\beta \sum_{i=1}^{m} \|a_2 - a_1\|^2 \leq 0$, thus $a_1 = a_2$. So $\{x_i(t)\}$, $i = 1, 2, \ldots, m$, have a unique accumulation point $x^* \in X$. □

**Remark 3.2:** Note that if we regard $\tau + 1$ as a period, then (2) is equivalent to

$$\begin{aligned} x_i(k+1) &= P_{X_i}\left[\sum_{j=1}^{m} b_{ij} x_j(k-\tau)\right], \\ x_i(k-l) &= x_i(k-\tau) \\ &= P_{X_i}\left[x_i(k-\tau)\right], \ l = 0, 1, \ldots, \tau - 1. \end{aligned}$$

That is to say, each agent $i$ only updates its state every $\tau + 1$ time slots, thus the convergence rate is slowed down.

For (1), if we set

$$\tilde{v}_i(k) := x_i(k-\tau) + \alpha \sum_{j=1}^{N} a_{ij}[x_j(k-\tau) - x_i(k-\tau)]$$
$$= \sum_{j=1}^{N} \tilde{a}_{ij} x_j(k-\tau),$$

with $\tilde{a}_{ii} = 1 - \alpha d_i$, and $\tilde{a}_{ij} = \alpha a_{ij}, j \neq i$.

$$\tilde{e}_i(k) := P_{X_i}[\tilde{v}_i(k)] - \tilde{v}_i(k).$$

Then,

$$x_i(k+1) = P_{X_i}[\tilde{v}_i(k)] = \tilde{v}_i(k) + \tilde{e}_i(k).$$

By taking $\tilde{\mathcal{A}} = [\tilde{a}_{ij}]$, we have $\tilde{\mathcal{A}}\mathbf{1} = \mathbf{1}$. Note that the spanning tree in $\mathcal{G}$ associated with $\mathcal{A}$ is also a spanning tree for the graph associated with $\tilde{\mathcal{A}}$, and they only have different weights for the same edge. So, $\tilde{\mathcal{A}}$ satisfies the Assumption 3.1 if and only if $\alpha \leq \min_{i=1,2,\ldots,N}\{\frac{1}{d_i}\}$ and $\mathcal{G}$ has a spanning tree. Then from the above analysis, we have the following result.

**Theorem 3.1:** *Suppose that $\mathcal{G}$ has a spanning tree and $\alpha \leq \min_{i=1,2,\ldots,N}\{\frac{1}{d_i}\}$. Then, for the projected consensus algorithm (1), there hold*

(a) $\lim_{k\to+\infty} \tilde{e}_i(k) = 0,\ i = 1, 2, \ldots, N,$
(b) $\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0,$
(c) *there exists a unique* $x^* \in X$ *such that* $\lim_{k\to+\infty} x_i(k) = x^*,\quad i = 1, 2, \ldots, N.$

## 4. Simulation

Without loss of generality, we assumed that there are four agents associated with

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Obviously, $\mathcal{G}$ is undirected connected graph. We know that $\lambda_2(\mathcal{L}) = 2, \lambda_3(\mathcal{L}) = \lambda_4(\mathcal{L}) = 4$.

We assumed that $X_1 = \{(x, y)|(x-2)^2 + (y-2)^2 \leq 4\}$, $X_2 = \{(x, y)|(x-3)^2 + (y-2)^2 \leq 1\}$, $X_3 = \{(x, y)|(x-4)^2 + (y-3)^2 \leq 4\}$, $X_4 = \{(x, y)|(x-3)^2 + (y-3)^2 \leq 1\}$. And $x_1(0) = [1, 2]^T$, $x_2(0) = [3, 4]^T$, $x_3(0) = [5, 6]^T$, $x_4(0) = [7, 8]^T$. From Theorem 3.1 we know that if $\alpha \leq \frac{1}{3}$, then projected consensus algorithm (1) can guarantee consensus. So we take $\alpha = 0.3$, and we obtain Figure 1 for the case of constrained consensus.

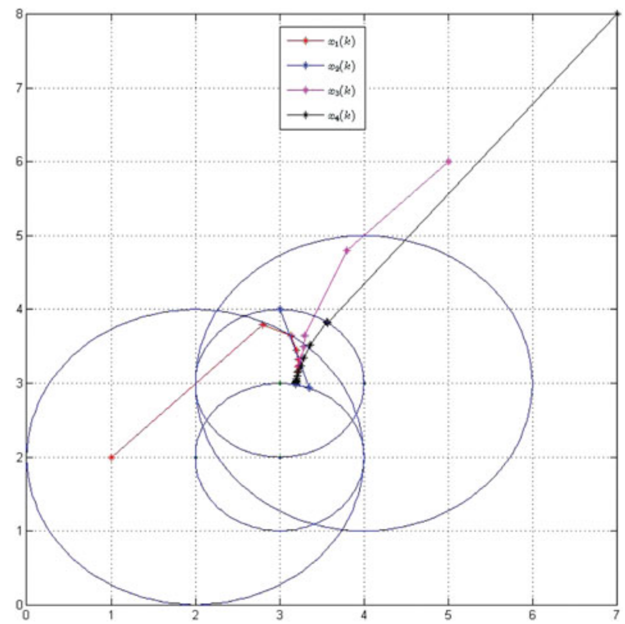


**Figure 1.** Constrained consensus-1.

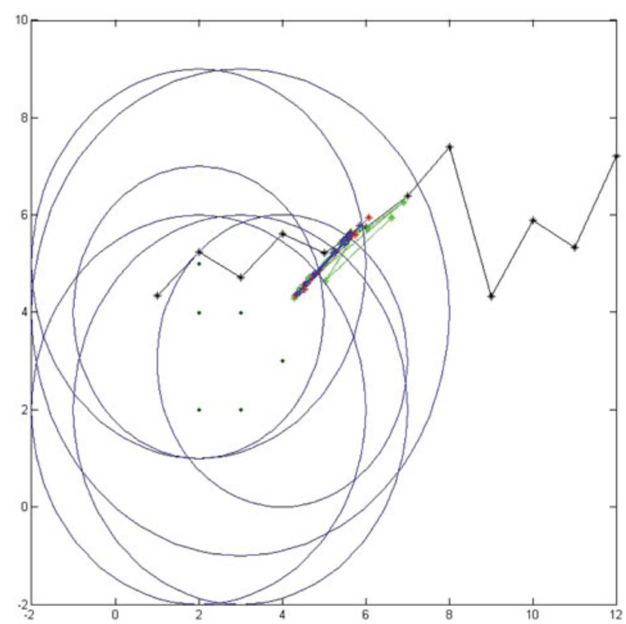

**Figure 2.** Constrained consensus-2.

Figure 2 displayed the result for constrained consensus with six agents according to Theorem 3.1.

## 5. Conclusion

We have studied the constrained and unconstrained consensus problems for discrete-time multi-agent systems with time delay. Based on a special property of stochastic matrices, we gave some consensus conditions. Then, we obtained some consensus conditions for the original system with time delay.

## Disclosure statement

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## Notes on contributors

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