0

On the Error Covariance Distribution for Kalman Filters with Packet Dropouts

Eduardo Rohr, Damián Marelli, and Minyue Fu
University of Newcastle
Australia

1. Introduction

The fast development of network (particularly wireless) technology have encouraged its use in control and signal processing applications. Under the control systems' perspective, this new technology has imposed new challenges concerning how to deal with the effects of quantisation, delays and loss of packets, leading to the development of a new networked control theory Schenato et al. (2007). The study of state estimators, when measurements are subject to random delays and losses, finds applications in both control and signal processing. Most estimators are based on the well-known Kalman filter Anderson & Moore (1979). In order to cope with network induced effects, the standard Kalman filter paradigm needs to undergo certain modifications.

In the case of missing measurements, the update equation of the Kalman filter depends on whether a measurement arrives or not. When a measurement is available, the filter performs the standard update equation. On the other hand, if the measurement is missing, it must produce open loop estimation, which as pointed out in Sinopoli et al. (2004), can be interpreted as the standard update equation when the measurement noise is infinite. If the measurement arrival event is modeled as a binary random variable, the estimator's error covariance (EC) becomes a random matrix. Studying the statistical properties of the EC is important to assess the estimator's performance. Additionally, a clear understanding of how the system's parameters and network delivery rates affect the EC, permits a better system design, where the trade-off between conflicting interests must be evaluated.

Studies on how to compute the expected error covariance (EEC) can be dated back at least to Faridani (1986), where upper and lower bounds for the EEC were obtained using a constant gain on the estimator. In Sinopoli et al. (2004), the same upper bound was derived as the limiting value of a recursive equation that computes a weighted average of the next possible error covariances. A similar result which allows partial observation losses was presented in Liu & Goldsmith (2004). In Dana et al. (2007); Schenato (2008), it is showed that a system in which the sensor transmits state estimates instead of raw measurements will provide a better error covariance. However, this scheme requires the use of more complex sensors. Most of the available research work is concerned with the expected value of the EC, neglecting higher order statistics. The problem of finding the complete distribution function of the EC has been recently addressed in Shi et al. (2010).

This chapter investigates the behavior of the Kalman filter for discrete-time linear systems whose output is intermittently sampled. To this end we model the measurement arrival event as an independent identically distributed (i.i.d.) binary random variable. We introduce a method to obtain lower and upper bounds for the cumulative distribution function (CDF) of the EC. These bounds can be made arbitrarily tight, at the expense of increased computational complexity. We then use these bounds to derive upper and lower bounds for the EEC.

2. Problem description

In this section we give an overview of the Kalman filtering problem in the presence of randomly missing measurements. Consider the discrete-time linear system:

$$\begin{cases} x_{t+1} &= Ax_t + w_t \\ y_t &= Cx_t + v_t \end{cases} \tag{1}$$

where the state vector $x_t \in \mathbb{R}^n$ has initial condition $x_0 \sim N(0, P_0)$, $y \in \mathbb{R}^p$ is the measurement, $w \sim N(0, Q)$ is the process noise and $v \sim N(0, R)$ is the measurement noise. The goal of the Kalman filter is to obtain an estimate $\hat{x}_t$ of the state x_t, as well as providing an expression for the covariance matrix P_t of the error $\tilde{x}_t = x_t - \hat{x}_t$.
We assume that the measurements y_t are sent to the Kalman estimator through a network subject to random packet losses. The scheme proposed in Schenato (2008) can be used to deal with delayed measurements. Hence, without loss of generality, we assume that there is no delay in the transmission. Let γ_t be a binary random variable describing the arrival of a measurement at time t. We define that $\gamma_t = 1$ when y_t was received at the estimator and $\gamma_t = 0$ otherwise. We also assume that γ_t is independent of γ_s whenever $t \neq s$. The probability to receive a measurement is given by

$$\lambda = \mathcal{P}(\gamma_t = 1). \tag{2}$$

Let $\hat{x}_{t|s}$ denote the estimate of x_t considering the available measurements up to time s. Let $\tilde{x}_{t|s} = x_t - \hat{x}_{t|s}$ denote the estimation error and $\Sigma_{t|s} = E\{(\tilde{x}_{t|s} - E\{\tilde{x}_{t|s}\})(\tilde{x}_{t|s} - E\{\tilde{x}_{t|s}\})'\}$ denote its covariance matrix. If a measurement is received at time t (i.e., if $\gamma_t = 1$), the estimate and its EC are recursively computed as follows:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - Cx_t) \tag{3}$$

$$\Sigma_{t|t} = (I - K_t C)\Sigma_{t|t-1} \tag{4}$$

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} \tag{5}$$

$$\Sigma_{t+1|t} = A\Sigma_{t|t}A' + Q, \tag{6}$$

with the Kalman gain K_t given by

$$K_t = \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C' + Q)^{-1}. \tag{7}$$

On the other hand, if a measurement is not received at time t (i.e., if $\gamma_t = 0$), then (3) and (4) are replaced by

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} \tag{8}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1}. \tag{9}$$

We will study the statistical properties of the EC $\Sigma_{t|t-1}$. To simplify the notation, we define $P_t = \Sigma_{t|t-1}$. Then, the update equation of P_t can be written as follows:

$$P_{t+1} = \begin{cases} \Phi_1(P_t), & \gamma_t = 1 \\ \Phi_0(P_t), & \gamma_t = 0 \end{cases} \tag{10}$$

with

$$\Phi_1(P_t) = AP_tA' + Q - AP_tC'(CP_tC' + R)^{-1}CP_tA' \tag{11}$$

$$\Phi_0(P_t) = AP_tA' + Q. \tag{12}$$

We point out that when all the measurements are available, and the Kalman filter reaches its steady state, the EC is given by the solution of the following algebraic Riccati equation

$$\underline{P} = A\underline{P}A' + Q - A\underline{P}C'(C\underline{P}C' + R)^{-1}C\underline{P}A'. \tag{13}$$

Throughout this chapter we use the following notation. For given $T \in \mathbb{N}$ and $0 \leq m \leq 2^T - 1$, the symbol S_m^T denotes the binary sequence of length T formed by the binary representation of m. We also use $S_m^T(i)$, $i = 1, \cdots, T$ to denote the i-th entry of the sequence, i.e.,

$$S_m^T = \{S_m^T(1),\ S_m^T(2), \ldots,\ S_m^T(T)\} \tag{14}$$

and

$$m = \sum_{k=1}^{T} 2^{k-1} S_m^T(k). \tag{15}$$

(Notice that S_0^T denotes a sequence of length T formed exclusively by zeroes.) We use $|S_m^T|$ to denote the number of ones in the sequence S_m^T, i.e.,

$$|S_m^T| = \sum_{k=1}^{T} S_m^T(k) \tag{16}$$

For a given sequence S_m^T, and a matrix $P \in \mathbb{R}^{n \times n}$, we define the map

$$\phi(P, S_m^T) = \Phi_{S_m^T(T)} \circ \Phi_{S_m^T(T-1)} \circ \ldots \Phi_{S_m^T(1)}(P) \tag{17}$$

where $\circ$ denotes the composition of functions (i.e. $f \circ g(x) = f(g(x))$). Notice that if m is chosen so that

$$S_m^T = \{\gamma_{t-1}, \gamma_{t-2}, \ldots, \gamma_{t-T}\}, \tag{18}$$

then the map $\phi(\cdot, S_m^T)$ updates P_{t-T} according to the measurement arrivals in the last T sampling times, i.e.,

$$P_t = \phi(P_{t-T}, S_m^T) = \Phi_{\gamma_{t-1}} \circ \Phi_{\gamma_{t-1}} \circ \ldots \Phi_{\gamma_{t-T}}(P_{t-T}). \tag{19}$$

3. Bounds for the cumulative distribution function

In this section we present a method to compute lower and upper bounds for the limit CDF $F(x)$ of the trace of the EC, which is defined by

$$F(x) = \lim_{T\to\infty} F^T(x) \tag{20}$$

$$F^T(x) = \mathcal{P}\left(\mathrm{Tr}\{P_T\} < x\right) \tag{21}$$

$$= \sum_{m=0}^{2^T-1} \mathcal{P}\left(S_m^T\right) H\left(x - \mathrm{Tr}\{\phi(P_0, S_m^T)\}\right), \tag{22}$$

where $H(\cdot)$ is the Heaviside step function, and the probability to observe the sequence S_m^T is given by

$$\mathcal{P}\left(S_m^T\right) = \lambda^{|S_m^T|}(1-\lambda)^{T-|S_m^T|}. \tag{23}$$

The basic idea is to start with either the lowest or the highest possible value of EC, and then evaluate the CDF resulting from each starting value after a given time horizon T. Doing so, for each T, we obtain a lower bound $\underline{F}^T(x)$ and an upper bound $\overline{F}^T(x)$ for $F(x)$, i.e.,

$$\underline{F}^T(x) \le F(x) \le \overline{F}^T(x), \text{ for all } T \in \mathbb{R}. \tag{24}$$

As we show in Section 3.3, both bounds monotonically approach $F(x)$ as T increases.
To derive these results we make use of the following lemma stating properties of the maps $\Phi_0(\cdot)$ and $\Phi_1(\cdot)$ defined in (11)-(12).

Lemma 3.1. *Let $X, Y \in \mathbb{R}^{n\times n}$ be two positive semi-definite matrices. Then,*

$$\Phi_1(X) < \Phi_0(X). \tag{25}$$

If $Y \ge X$,

$$\Phi_0(Y) \ge \Phi_0(X) \tag{26}$$

$$\Phi_1(Y) \ge \Phi_1(X). \tag{27}$$

Proof: The proof of (25) is direct from (11)-(12). Equation (26) follows straightforwardly since $\Phi_0(X)$ is affine in X. Using the matrix inversion lemma, we have that

$$\Phi_1(X) = A(X^{-1} + C'R^{-1}C)^{-1}A' + Q \tag{28}$$

which shows that $\Phi_1(X)$ is monotonically increasing with respect to X. ■

3.1 Upper bounds for the CDF

The smallest possible value of the EC is obtained when all the measurements are available, and the Kalman filter reaches its steady state. In this case, the EC $\underline{P}$ is given by (13). Now, fix T, and suppose that m is such that $S_m^T = \{\gamma_{T-1}, \gamma_{T-2}, \ldots, \gamma_0\}$ describes the measurement arrival sequence. Then, assuming that[1] $P_0 \geq \underline{P}$, from (26)-(27), it follows that $P_T \geq \phi(\underline{P}, S_m^T)$. Hence, from (22), an upper bound of $F(x)$ is given by

$$\overline{F}^T(x) = \sum_{m=0}^{2^T-1} \mathcal{P}\left(S_m^T\right) H\left(x - \mathrm{Tr}\{\phi(\underline{P}, S_m^T)\}\right). \tag{29}$$

3.2 Lower bounds for the CDF

A lower bound for the CDF can be obtained using an argument similar to the one we used above to derive an upper bound. To do this we need to replace in (22) $\mathrm{Tr}\{\phi(P_0, S_m^T)\}$ by an upper bound of $\mathrm{Tr}\{P_T\}$ given the arrival sequence S_m^T. To do this we use the following lemma.

Lemma 3.2. *Let m be such that $S_m^T = \{\gamma_{T-1}, \gamma_{T-2}, \cdots, \gamma_0\}$ and $0 \leq t_1, t_2, \cdots, t_I \leq T-1$ denote the indexes where $\gamma_i = 1$, $i = 0, \cdots, T-1$. Define*

$$O = \begin{bmatrix} CA^{t_1} \\ CA^{t_2} \\ \vdots \\ CA^{t_I} \end{bmatrix}, \ \Sigma_Q = \begin{bmatrix} \sum_{j=0}^{t_1-1} CA^j QA'^{T-t_1+j} \\ \sum_{j=0}^{t_2-1} CA^j QA'^{T-t_2+j} \\ \vdots \\ \sum_{j=0}^{t_I-1} CA^j QA'^{T-t_I+j} \end{bmatrix}', \tag{30}$$

and the matrix $\Sigma_V \in \mathbb{R}^{nI \times nI}$ whose $[\Sigma_V]_{i,j} \in \mathbb{R}^{n \times n}$ submatrix is given by

$$[\Sigma_V]_{i,j} = \sum_{k=0}^{\min\{t_i,t_j\}-1} CA^{t_i-1-k} QA'^{t_j-1-k} C' + R\delta(i,j) \tag{31}$$

where

$$\delta(i,j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{32}$$

If O has full column rank, then

$$P_T \leq \overline{P}(S_m^T), \tag{33}$$

where the S_m^T-dependant bound $\overline{P}(S_m^T)$ is given by

$$\begin{aligned} \overline{P}(S_m^T) = {} & A^T \left(O'\Sigma_V^{-1}O\right)^{-1} A'^T + \sum_{j=0}^{T-1} A^j QA'^j - A^T (\Sigma_V^{-\frac{1}{2}} O)^\dagger \Sigma_V^{-\frac{1}{2}} \Sigma_Q' + \\ & -\Sigma_Q \Sigma_V^{-\frac{1}{2}} (\Sigma_V^{-\frac{1}{2}} O)'^\dagger A'^T - \Sigma_Q \left(\Sigma_V^{-1} - \Sigma_V^{-1} O (O'\Sigma_V^{-1}O)^{-1} O' \Sigma_V^{-1}\right) \Sigma_Q', \end{aligned} \tag{34}$$

with $(\Sigma_V^{-\frac{1}{2}} O)^\dagger$ denoting the Moore-Penrose pseudo-inverse of $\Sigma_V^{-\frac{1}{2}} O$ Ben-Israel & Greville (2003).

[1] If this assumption does not hold, one can substitute $\underline{P}$ by P_0 without loss of generality.

Proof: Let Y_T be the vector formed by the available measurements

$$Y_T = \begin{bmatrix} y'_{t_1} \; y'_{t_2} \; \cdots \; y'_{t_I} \end{bmatrix}' \tag{35}$$
$$= O x_0 + V_T, \tag{36}$$

where

$$V_T = \begin{bmatrix} \sum_{j=0}^{t_1-1} C A^{t_1-1-j} w_j + v_{t_1} \\ \sum_{j=0}^{t_2-1} C A^{t_2-1-j} w_j + v_{t_2} \\ \vdots \\ \sum_{j=0}^{t_I-1} C A^{t_I-1-j} w_j + v_{t_I} \end{bmatrix}. \tag{37}$$

From the model (1), it follows that

$$\begin{bmatrix} x_T \\ Y_T \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_{xY} \\ \Sigma'_{xY} & \Sigma_Y \end{bmatrix} \right) \tag{38}$$

where

$$\Sigma_x = A^T P_0 A'^T + \sum_{j=0}^{T-1} A^j Q A'^j \tag{39}$$

$$\Sigma_{xY} = A^T P_0 O' + \Sigma_Q \tag{40}$$

$$\Sigma_Y = O P_0 O' + \Sigma_V. \tag{41}$$

Since the Kalman estimate $\hat{x}_T$ at time T is given by,

$$\hat{x}_T = E\left\{ x_T | Y_T \right\}, \tag{42}$$

it follows from (Anderson & Moore, 1979, pp. 39) that the estimation error covariance is given by

$$P_T = \Sigma_x - \Sigma_{xY} \Sigma_Y^{-1} \Sigma'_{xY}. \tag{43}$$

Substituting (39)-(41) in (43), we have

$$P_T = A^T P_0 A'^T + \sum_{j=0}^{T-1} A^j Q A'^j + \tag{44}$$
$$- \left(A^T P_0 O' + \Sigma_Q \right) \left(O P_0 O' + \Sigma_V \right)^{-1} \left(A^T P_0 O' + \Sigma_Q \right)'$$
$$= A^T \left(P_0 - P_0 O' \left(O P_0 O' + \Sigma_V \right)^{-1} O' P_0 \right) A^T + \sum_{j=0}^{T-1} A^j Q A'^j + \tag{45}$$
$$- A^T P_0 O' \left(O P_0 O' + \Sigma_V \right)^{-1} \Sigma'_Q - \Sigma_Q \left(O P_0 O' + \Sigma_V \right)^{-1} O' P_0 A^T +$$
$$- \Sigma_Q \left(O P_0 O' + \Sigma_V \right)^{-1} \Sigma'_Q.$$

Now, from (19),

$$P_T = \phi(P_0, S_m^T). \tag{46}$$

Since for any P_0 we can always find a k such that $kI \geq P_0$, from the monotonicity of $\phi(\cdot, S_m^T)$ (Lemma 3.1), it follows that

$$P_T \leq \lim_{k \to \infty} \phi(kI, S_m^T). \tag{47}$$

We then have that

$$P_T \leq P_{T,1} + P_{T,2} + P_{T,3} + P'_{T,3} + P_{T,4}, \tag{48}$$

with

$$\begin{aligned} P_{T,1} &= \lim_{k \to \infty} A^T \left(kI - k^2 O \left(kOO' + \Sigma_V \right)^{-1} O' \right) A'^T \qquad (49) \\ P_{T,2} &= \sum_{j=0}^{T-1} A^j Q A'^j \\ P_{T,3} &= -\lim_{k \to \infty} kA^T O \left(kOO' + \Sigma_V \right)^{-1} \Sigma'_Q \\ P_{T,4} &= -\lim_{k \to \infty} \Sigma_Q \left(kOO' + \Sigma_V \right)^{-1} \Sigma'_Q. \end{aligned}$$

Using the matrix inversion lemma, we have that

$$\begin{aligned} P_{T,1} &= A^T \lim_{k \to \infty} \left(k^{-1} I + O' \Sigma_V^{-1} O \right)^{-1} A'^T \qquad (50) \\ &= A^T \left(O' \Sigma_V^{-1} O \right)^{-1} A'^T. \qquad (51) \end{aligned}$$

It is straightforward to see that $P_{T,3}$ can be written as

$$\begin{aligned} P_{T,3} &= -\lim_{k \to 0} A^T O' \left(OO' + \Sigma_V k \right)^{-1} \Sigma'_Q \qquad (52) \\ &= -A^T \lim_{k \to 0} \Sigma_V^{-\frac{1}{2}} O' \left(\Sigma_V^{-\frac{1}{2}} OO' \Sigma_V^{-\frac{1}{2}} + kI \right)^{-1} \Sigma_V^{-\frac{1}{2}} \Sigma'_Q. \qquad (53) \end{aligned}$$

From (Ben-Israel & Greville, 2003, pp. 115), we know that $\lim_{k \to 0} \xi' \left(\xi \xi' + kI \right) = \xi^{\dagger}$. By making $\xi = \Sigma_V^{-\frac{1}{2}} O$, we have that

$$P_{T,3} = -A^T \left(\Sigma_V^{-\frac{1}{2}} O \right)^{\dagger} \Sigma_V^{-\frac{1}{2}} \Sigma'_Q. \tag{54}$$

Using the matrix inversion lemma, we have

$$\begin{aligned} P_{T,4} &= -\lim_{k \to \infty} \Sigma_Q \left(\Sigma_V^{-1} - \Sigma_V^{-1} O \left(O' \Sigma_V^{-1} O + k^{-1} I \right)^{-1} O' \Sigma_V^{-1} \right) \Sigma'_Q \qquad (55) \\ &= \Sigma_Q \left(\Sigma_V^{-1} - \Sigma_V^{-1} O \left(O' \Sigma_V^{-1} O \right)^{-1} O' \Sigma_V^{-1} \right) \Sigma'_Q \end{aligned}$$

and the result follows by substituting (51), (54) and (55) in (48). ■

In order to keep the notation consistent with that of Section 3.1, with some abuse of notation we introduce the following definition

$$\phi(\infty, S_m^T) \triangleq \begin{cases} \overline{P}(S_m^T), & \text{if } O \text{ has full column rank} \\ \infty I_n, & \text{otherwise} \end{cases} \tag{56}$$

where ∞I_n is an $n \times n$ diagonal matrix with ∞ on every entry of the main diagonal. Then, we obtain a lower bound for $F(x)$ as follows

$$\underline{F}^T(x) = \sum_{m=0}^{2^T-1} \mathcal{P}\left(S_m^T\right) H\left(x - \text{Tr}\{\phi(\infty, S_m^T)\}\right). \tag{57}$$

3.3 Monotonic approximation of the bounds to $F(x)$

In this section we show that the bounds $\underline{F}^T(x)$ and $\overline{F}^T(x)$ in (24) approach monotonically $F(x)$, as T tends to infinity. This is stated in the following theorem.

Theorem 3.1. *We have that*

$$\underline{F}^{T+1}(x) \geq \underline{F}^T(x) \tag{58}$$

$$\overline{F}^{T+1}(x) \leq \overline{F}^T(x). \tag{59}$$

Moreover, the bounds $\underline{F}^T(x)$ and $\overline{F}^T(x)$ approach monotonically the true CDF $F(x)$ as T tends to ∞.

Proof: Let S_m^T be a sequence of length T. From (17) and Lemma 3.1 and for any $P_0 > 0$, we have

$$\phi(P_0, \{S_m^T, 0\}) = \phi(\Phi_0(P_0), S_m^T) \leq \phi(\infty, S_m^T). \tag{60}$$

From the monotonicity of $\phi(\cdot, S_m^T)$ and $\Phi_0(\cdot)$, stated in Lemma 3.1 we have

$$\phi(P_0, \{S_m^T, 0\}) = \phi(\Phi_0(P_0), S_m^T) \geq \phi(P_0, S_m^T) \tag{61}$$

which implies that

$$\phi(\infty, \{S_m^T, 0\}) \geq \phi(\infty, S_m^T). \tag{62}$$

From (60) and (62), we have

$$\phi(\infty, \{S_m^T, 0\}) = \phi(\infty, S_m^T). \tag{63}$$

Also, if the matrix O (defined in Lemma 3.2) resulting from the sequence S_m^T has full column rank, then so has the same matrix resulting from the sequence $\{S_m^T, 1\}$. This implies that

$$\phi(\infty, \{S_m^T, 1\}) \leq \phi(\infty, S_m^T). \tag{64}$$

Now, from Lemma 3.1, $\Phi_0(\underline{P}) \geq \underline{P}$, and therefore,

$$\phi(\underline{P}, \{S_m^T, 0\}) = \phi(\Phi_0(\underline{P}), S_m^T) \tag{65}$$

$$\geq \phi(\underline{P}, S_m^T). \tag{66}$$

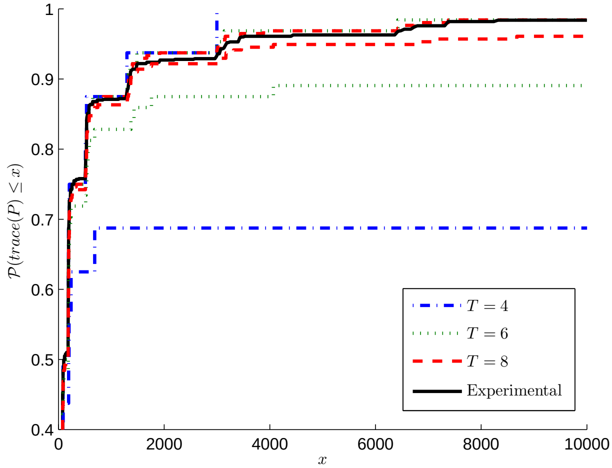

Fig. 1. Upper and lower bounds for the Error Covariance.

Also, since $\Phi_1(\underline{P}) = \underline{P}$, we have that

$$\phi(\underline{P}, \{S_m^T, 1\}) = \phi(\phi_1(\underline{P}), S_m^T) \tag{67}$$
$$= \phi(\underline{P}, S_m^T). \tag{68}$$

Hence, for any binary variable γ, we have that

$$\phi(\infty, \{S_m^T, \gamma\}) \leq \phi(\infty, S_m^T) \tag{69}$$
$$\phi(\underline{P}, \{S_m^T, \gamma\}) \geq \phi(\underline{P}, S_m^T). \tag{70}$$

Now notice that the bounds (29) and (57) only differ in the position of the step functions $H(\cdot)$. Hence, the result follows from (69) and (70). ■

3.4 Example

Consider the system below, which is taken from Sinopoli et al. (2004),

$$A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}' \qquad Q = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \quad R = 2.5, \tag{71}$$

with $\lambda = 0.5$. In Figure 1 we show the upper bound $\overline{F}^T(x)$ and the lower bound $\underline{F}^T(x)$, for $T = 3$, $T = 5$ and $T = 8$. We also show an estimate of the true CDF $F(x)$ obtained from a Monte Carlo simulation using 10,000 runs. Notice that, as T increases, the bounds become tighter, and for $T = 8$, it is hard to distinguish between the lower and the upper bounds.

4. Bounds for the expected error covariance

In this section we derive upper and lower bounds for the trace G of the asymptotic EEC, i.e.,

$$G = \lim_{t\to\infty} \mathrm{Tr}\{E\{P_t\}\}. \tag{72}$$

Since P_t is positive-semidefinite, we have that,

$$\mathrm{Tr}\{E\{P_t\}\} = \int_0^\infty (1 - F^t(x))dx. \tag{73}$$

Hence, [2]

$$G = \int_0^\infty (1 - \lim_{t\to\infty} F^t(x))dx \tag{75}$$

$$= \int_0^\infty (1 - F(x))dx \tag{76}$$

4.1 Lower bounds for the EEC

In view of (76), a lower bound for G, can be obtained from an upper bound of $F(x)$. One such bound is $\overline{F}^T(x)$, derived in Section 3.1. A limitation of $\overline{F}^T(x)$ is that $\overline{F}^T(x) = 1$, for all $x > \phi(\underline{P}, S_0^T)$, hence it is too conservative for large values of x. To go around this, we introduce an alternative upper bound for $F(x)$, denoted by $\overline{F}_\star(x)$.

Our strategy for doing so is to group the sequences S_m^T, $m = 0, 1, \cdots, 2^T - 1$, according to the number of consecutive lost measurements at its end. Then, from each group, we only consider the worst sequence, i.e., the one producing the smallest EEC trace.

Notice that the sequences S_m^T with $m < 2^{T-z}$, $0 \le z \le T$, are those having the last z elements equal to zero. Then, from (25) and (26), it follows that

$$\operatorname*{arg\,min}_{0\le m<2^{T-z}} \mathrm{Tr}\{\phi(X, S_m^T)\} = 2^{T-z} - 1, \tag{77}$$

i.e., from all sequences with z zeroes at its end, the one that produces the smallest EEC trace has its first $T - z$ elements equal to one. Using this, an upper bound for $F(x)$ is given by

$$F(x) \le \overline{F}_\star(x) \triangleq 1 - (1 - \lambda)^{k(x)} \tag{78}$$

where

$$k(x) = \begin{cases} 0, & x \le \underline{P} \\ \min\left\{j : \mathrm{Tr}\left(\phi(\underline{P}, S_0^j)\right) > x\right\}, & x > \underline{P}. \end{cases} \tag{79}$$

[2] Following the argument in Theorem 3.1, it can be verified that $(1 - F^t(x)) \le \overline{F}(x)$ with

$$\overline{F}(x) = \begin{cases} 1 & x \le \mathrm{Tr}\{P_0\} \\ F(x) & x > \mathrm{Tr}\{P_0\}. \end{cases} \tag{74}$$

Hence, using Lebesgue's dominated convergence theorem, the limit can be exchanged with the integral whenever $\int_0^\infty (1 - F(x))dx < \infty$, i.e., whenever the asymptotic EEC is finite.

We can now use both $\overline{F}^T(x)$ and $\overline{F}_\star(x)$ to obtain a lower bound $\underline{G}^T$ for G as follows

$$\underline{G}^T = \int_0^\infty 1 - \min\{\overline{F}^T(x), \overline{F}_\star(x)\} dx. \tag{80}$$

The next lemma states the regions in which each bound is less conservative.

Lemma 4.1. *The following properties hold true:*

$$\overline{F}^T(x) \leq \overline{F}_\star(x), \ \forall x \leq \operatorname{Tr}\left(\phi(\underline{P}, S_0^T)\right) \tag{81}$$

$$\overline{F}^T(x) > \overline{F}_\star(x), \ \forall x > \operatorname{Tr}\left(\phi(\underline{P}, S_0^T)\right). \tag{82}$$

Proof: Define

$$Z(i,j) \triangleq \operatorname{Tr}\left(\phi(\underline{P}, S_i^j)\right). \tag{83}$$

To prove (81), notice that $\overline{F}^T(x)$ can be written as

$$\overline{F}^T(x) = \sum_{\substack{j=0 \\ j:Z(j,T)\leq x}}^{2^T-1} \mathcal{P}(S_j^T). \tag{84}$$

Substituting $x = Z(0,K)$ we have for all $1 < K \leq T$

$$\overline{F}^T(Z(0,K)) = \sum_{\substack{j=0 \\ j:Z(j,T)\leq Z(0,K)}}^{2^T-1} \mathcal{P}(S_j^T) \tag{85}$$

$$= 1 - \sum_{\substack{j=0 \\ j:Z(j,T)>Z(0,K)}}^{2^T-1} \mathcal{P}(S_j^T) \tag{86}$$

Now, notice that the summation in (86) includes, but is not limited to, all the sequences finishing with K zeroes. Hence

$$\sum_{\substack{j=0 \\ j:Z(j,T)>Z(0,K)}}^{2^T-1} \mathcal{P}(S_j^T) \geq (1-\lambda)^K \tag{87}$$

and we have

$$\overline{F}^T(Z(0,K)) \leq 1 - (1-\lambda)^K \tag{88}$$

$$= \overline{F}_\star(Z(0,K)). \tag{89}$$

Proving (82) is trivial, since $\overline{F}^T(x) = 1, x > Z(0,T)$. ■

We can now present a sequence of lower bounds $\underline{G}^T$, $T \in \mathbb{N}$, for the EEC G. We do so in the next theorem.

Theorem 4.1. *Let E_j, $0 < j \leq 2^T$ denote the set of numbers* $\mathrm{Tr}\left(\phi(\underline{P}, S_m^T)\right)$, $0 \leq m < 2^T$, *arranged in ascending order, (i.e.,* $E_j = \mathrm{Tr}\left(\phi(\underline{P}, S_{m_j}^T)\right)$, *for some m_j, and $E_1 \leq E_1 \leq \cdots < E_{2^T}$). For each $0 < j \leq 2^T$, let $\pi_j = \sum_{k=0}^{m_j} \mathcal{P}(S_k^T)$. Also define $E_0 = \pi_0 = 0$. Then, $\underline{G}^T$ defined in* (80) *is given by*

$$\underline{G}^T = \underline{G}_1^T + \underline{G}_2^T \tag{90}$$

where

$$\underline{G}_1^T = \sum_{j=0}^{2^T-1} (1-\pi_j)(E_{j+1} - E_j) \tag{91}$$

$$\underline{G}_2^T = \sum_{j=T}^{\infty} (1-\lambda)^j \mathrm{Tr}\left\{A^j \left(A\underline{P}A' + Q - \underline{P}\right) A'^j\right\} \tag{92}$$

Moreover, if the following condition holds

$$\max|\mathrm{eig}(A)|^2(1-\lambda) < 1, \tag{93}$$

and A is diagonalizable, i.e., it can be written as

$$A = VDV^{-1}, \tag{94}$$

with D diagonal, then,

$$\underline{G}_2^T = \mathrm{Tr}\{\Gamma\} - \sum_{j=0}^{T-1} (1-\lambda)^j \mathrm{Tr}\left\{A^j \left(A\underline{P}A' + Q - \underline{P}\right) A'^j\right\} \tag{95}$$

where

$$\Gamma \triangleq \left(X^{1/2}V'^{-1} \otimes V\right) \Delta \left(X^{1/2}V'^{-1} \otimes V\right)' \tag{96}$$

$$X \triangleq A\underline{P}A' + Q - \underline{P}. \tag{97}$$

Also, the $n^2 \times n^2$ matrix Δ is such that its i,j-th entry $[\Delta]_{i,j}$ is given by

$$[\Delta]_{i,j} \triangleq \frac{1}{1-(1-\lambda)[\overrightarrow{D}]_i[\overrightarrow{D}]_j}, \tag{98}$$

where $\overrightarrow{D}$ denotes a column vector formed by stacking the columns of D, i.e.,

$$\overrightarrow{D} \triangleq \left[\, [D]_{1,1} \cdots [D]_{n,1}\ [D]_{1,2} \cdots [D]_{n,n} \,\right]' \tag{99}$$

Proof: In view of lemma 4.1, (90) can be written as

$$\underline{G}^T = \int_0^{Z(0,T)} (1-\overline{F}^T(x))dx + \int_{Z(0,T)}^{\infty} (1-\overline{F}_\star(x))dx \tag{100}$$

Now, $\overline{F}^T(x)$ can be written as

$$\overline{F}^T(x) = \pi_{i(x)}, \ i(x) = \max\{i: \ E_i < x\}. \tag{101}$$

In view of (101), it is easy to verify that

$$\int_0^{Z(0,T)} 1 - \overline{F}^T(x)dx = \sum_{j=1}^{2^T}(1-\pi_j)(E_j - E_{j-1}) = \underline{G}_1^T. \tag{102}$$

The second term of (90) can be written using the definition of $\overline{F}_\star(x)$ as

$$\int_{Z(0,T)}^{\infty} 1 - \tilde{F}(x)dx = \sum_{j=T}^{\infty}(1-\lambda)^j \left(Z(0,j+1) - Z(0,j)\right) \tag{103}$$

$$= \sum_{j=T}^{\infty}(1-\lambda)^j \mathrm{Tr}\left\{A^j \left(A\underline{P}A' + Q - \underline{P}\right) A'^j\right\} \tag{104}$$

$$= \underline{G}_2^T. \tag{105}$$

and (90) follows from (100), (102) and (105).
To show (95), we use Lemma 7.1 (in the Appendix), with $b = (1-\lambda)$ and $X = A\underline{P}A' + Q - \underline{P}$, to obtain

$$\sum_{j=0}^{\infty}(1-\lambda)^j \mathrm{Tr}\left\{A^j \left(A\underline{P}A' + Q - \underline{P}\right) A'^j\right\} = \mathrm{Tr}\{\Gamma\}. \tag{106}$$

The result then follows immediately. ■

4.2 Upper bounds for the EEC

Using an argument similar to the one in the previous section, we will use lower bounds of the CDF to derive a family of upper bounds $\overline{G}^{T,N}$, $T \leq N \in \mathbb{N}$, of G. Notice that, in general, there exists $\delta > 0$ such that $1 - \underline{F}^T(x) > \delta$, for all x. Hence, using $\underline{F}^T(x)$ in (76) will result in $\overline{G}$ being infinite valued. To avoid this, we will present two alternative lower bounds for $F(x)$, which we denote by $\underline{F}_\star^{T,N}(x)$ and $\underline{F}_\diamond^N(x)$.
Recall that $A \in \mathbb{R}^{n\times n}$, and define

$$N_0 \triangleq \min\left\{k \ : \mathrm{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}\right) = n\right\}. \tag{107}$$

The lower bounds $\underline{F}_\star^{T,N}(x)$ and $\underline{F}_\diamond^N(x)$ are stated in the following two lemmas.

Lemma 4.2. *Let $T \leq N \in \mathbb{N}$, with $N_0 \leq T$ and N satisfying*

$$|S_m^N| \geq N_0 \Rightarrow \mathrm{Tr}\{\phi(\infty, S_m^N)\} < \infty. \tag{108}$$

For each $T \leq n \leq N$, let

$$\overline{P}^*(n) \triangleq \max_{m:|S_m^n|=N_0} \phi(\infty, S_m^n) \quad (109)$$

$$\overline{p}^*(n) \triangleq \mathrm{Tr}(\overline{P}^*(n)). \quad (110)$$

Then, for all $\overline{p}^*(T) \leq x \leq \overline{p}^*(N)$,

$$F(x) \geq \underline{F}_\star^{T,N}(x), \quad (111)$$

where, for each $T \leq n < N$ and all $\overline{p}^*(n) \leq x \leq \overline{p}^*(n+1)$,

$$\underline{F}_\star^{T,N}(x) = 1 - \sum_{l=0}^{N_0-1} \lambda^l (1-\lambda)^{n-l} \frac{n!}{l!(n-l)!}. \quad (112)$$

Remark 4.1. *Lemma 4.2 above requires the existence of an integer constant N satisfying* (108)*. Notice that such constant always exists since* (108) *is trivially satisfied by N_0.*

Proof: We first show that, for all $T \leq n < N$,

$$\overline{p}^*(n) < \overline{p}^*(n+1). \quad (113)$$

To see this, suppose we add a zero at the end of the sequence used to generate $\overline{p}^*(n)$. Doing so we have

$$\overline{P}^*(n) < \Phi_0\left(\overline{P}^*(n)\right) \leq \overline{P}^*(n+1). \quad (114)$$

Now, for a given n, we can obtain a lower bound for $F^n(x)$ by considering in (57) that $\mathrm{Tr}(\phi(\infty, S_m^n)) = \infty$, whenever $|S_m^n| < N_0$. Also, from (25) we have that if $|S_m^n| \geq N_0$, then $\mathrm{Tr}(\phi(\infty, S_m^n)) < \overline{p}^*(n)$. Hence, a lower bound for $F(x)$ is given by $\mathcal{P}(|S_m^n| < N_0)$, for $x \geq \overline{p}^*(n)$.
Finally, the result follows by noting that the probability to observe sequences S_m^n with m such that $|S_m^n| < N_0$ is given by

$$\mathcal{P}(|S_m^n| < N_0) = 1 - \sum_{l=0}^{N_0-1} \lambda^l (1-\lambda)^{n-l} \frac{n!}{l!(n-l)!}, \quad (115)$$

since $\lambda^l(1-\lambda)^{n-l}$ is the probability to receive a given sequence S_m^n with $|S_m^n| = l$, and the number of sequences of length n with l ones is given by the binomial coefficient

$$\binom{n}{l} = \frac{n!}{l!(n-l)!}. \quad (116)$$

■

Lemma 4.3. *Let N, $\overline{P}^*(N)$ and $\overline{p}^*(N)$ be as defined in Lemma 4.2, and let $L = \sum_{n=0}^{N_0-1} \binom{N}{n}$. Then, for all $x \geq \overline{p}^*(N)$,*

$$F(x) \geq \underline{F}_\diamond^N(x), \quad (117)$$

where, for each $n \in \mathbb{N}$ and all $\phi(\overline{P}^*(N), S_0^{n-1}) \leq x < \phi(\overline{P}^*(N), S_0^n)$,

$$\underline{F}_\diamond^N(x) = 1 - u' M^n z \tag{118}$$

with the vectors $u, z \in \mathbb{R}^L$ defined by

$$u = \begin{bmatrix} 1 \; 1 \cdots 1 \end{bmatrix}' \tag{119}$$

$$z = \begin{bmatrix} 1 \; 0 \cdots 0 \end{bmatrix}'. \tag{120}$$

The i, j-th entry of the matrix $M \in \mathbb{R}^{L \times L}$ is given by

$$[M]_{i,j} = \begin{cases} \lambda, & Z_i^N = U_+(Z_j^N, 1) \\ 1 - \lambda, & Z_i^N = U_+(Z_j^N, 0) \\ 0, & \text{otherwise.} \end{cases} \tag{121}$$

where Z_m^N, $m = 0, \cdots, L-1$ denotes the set of sequences of length N with less than N_0 ones, with $Z_0^N = S_0^N$, but otherwise arranged in any arbitrary order (i.e.,

$$|Z_m^N| < N_0 \textit{ for all } m = 0, \cdots, L-1. \tag{122}$$

and $Z_m^N = S_{n_m}^N$, for some $n_m \in \{0, \cdots, 2^N - 1\}$). Also, for $\gamma \in \{0, 1\}$, the operation $U_+(Z_m^T, \gamma)$ is defined by

$$U_+(Z_m^T, \gamma) = \{Z_m^T(2), Z_m^T(3), \cdots, Z_m^T(T), \gamma\}. \tag{123}$$

Proof: The proof follows an argument similar to the one used in the proof of Lemma 4.2. In this case, for each n, we obtain a lower bound for $\underline{F}^n(x)$ by considering in (57) that $\text{Tr}(\phi(\infty, S_m^n)) = \infty$, whenever S_m^n does not contain a subsequence of length N with at least N_0 ones. Also, if S_m^n contains such a subsequence, the resulting EC is smaller that or equal to

$$\phi(\infty, \{S_{m^*}^N, S_0^n\}) = \phi(\phi(\infty, S_{m^*}^N), S_0^n) \tag{124}$$

$$= \phi(\overline{P}^*(N), S_0^n), \tag{125}$$

where $S_{m^*}^N$ denotes the sequence required to obtain $\overline{P}^*(N)$.
To conclude the proof we need to compute the probability $p_{N,n}$ of receiving a sequence of length $N + n$ that does not contain a subsequence of length N with at least N_0 ones. This is done in Lemma 7.2 (in the Appendix), where it is shown that

$$p_{N,n} = u' M^n z. \tag{126}$$

■

Now, for a given T and N, we can obtain an upper bound $\overline{G}^{T,N}$ for G using the lower bounds $\underline{F}^T(x)$, $\underline{F}_\star^{T,N}(x)$ and $\underline{F}_\diamond^N(x)$, as follows

$$\overline{G}^{T,N} = \int_0^\infty 1 - \max\{\underline{F}^T(x), \underline{F}_\star^{T,N}(x), \underline{F}_\diamond^N(x)\} dx. \tag{127}$$

We do so in the next theorem.

Theorem 4.2. *Let T and N be two given positive integers with $N_0 \leq T \leq N$ and such that for all $0 \leq m < 2^N$, $|S_m^N| \geq N_0 \Rightarrow \phi(\infty, S_m^N) < \infty$. Let J be the number of sequences such that $O(S_m^T)$ has full column rank. Let $E_0 \triangleq 0$ and E_j, $0 < j \leq J$ denote the set of numbers $\operatorname{Tr}\left(\phi(\infty, S_m^T)\right)$, $0 < m \leq J$, arranged in ascending order, (i.e., $E_j = \operatorname{Tr}\left(\phi(\infty, S_{m_j}^T)\right)$, for some m_j, and $E_0 \leq E_1 \leq \cdots \leq E_f$). For each $0 \leq j < J$, let $\pi_j = \sum_{k=0}^{m_j} \mathcal{P}(S_k^T)$, and let M, u and v be as defined as in Lemma 4.3. Then, an upper bound for the EEC is given by*

$$G \leq \overline{G}^{T,N}, \tag{128}$$

where

$$\overline{G}^{T,N} = \operatorname{Tr}(\overline{G}_1^T + \overline{G}_2^{T,N} + \overline{G}_3^N), \tag{129}$$

and

$$\overline{G}_1^T = \sum_{j=0}^{J} (1 - \pi_j)(E_{j+1} - E_j) \tag{130}$$

$$\overline{G}_2^{T,N} = \sum_{j=T}^{N-1} \sum_{l=0}^{N_0-1} \lambda^l (1-\lambda)^{j-l} \frac{j!}{l!(j-l)!} \left(\overline{P}^*(j+1) - \overline{P}^*(j)\right) \tag{131}$$

$$\overline{G}_3^N = \sum_{j=0}^{\infty} u' M^{N+j} z \{A^j (A\overline{P}^*(N)A' + Q - \overline{P}^*(N))A'^j\}. \tag{132}$$

Moreover, if A is diagonalizable, i.e.

$$A = VDV^{-1}, \tag{133}$$

with D diagonal, and

$$\max|\operatorname{eig}(A)|^2 \rho < 1, \tag{134}$$

where

$$\rho = (\max|\operatorname{sv} M|), \tag{135}$$

then the EEC is finite and

$$\overline{G}_3^N \leq u' M^N z \operatorname{Tr}(\Gamma^\star), \tag{136}$$

where

$$\Gamma^\star \triangleq \left(X^{1/2} V'^{-1} \otimes V\right) \Delta \left(X^{1/2} V'^{-1} \otimes V\right)' \tag{137}$$

$$X \triangleq A\underline{P}A' + Q - \underline{P}. \tag{138}$$

Also, the i,j-th entry $[\Delta]_{i,j}$ of the $n^2 \times n^2$ matrix Δ is given by

$$[\Delta]_{i,j} \triangleq \frac{\sqrt{2^{N_0} - 1}}{1 - \rho[\overrightarrow{D}]_i [\overrightarrow{D}]_j}. \tag{139}$$

Proof: First, notice that $\underline{F}^T(x)$ is defined for all $x > 0$, whereas $\underline{F}_\star^T(x)$ is defined on the range $\overline{P}^\star(T) < x \leq \overline{P}^\star(N)$ and $\underline{F}_\diamond^T(x)$ on $\overline{P}^\star(N) < x$. Now, for all $x \geq \overline{p}^*(T)$, we have

$$\underline{F}^T(x) = \sum_{j:|S_j^T|\geq N_0} \mathcal{P}(S_j^T) = 1 - \sum_{l=0}^{N_0-1} \lambda^l (1-\lambda)^{T-l} \frac{T!}{l!(T-l)!}, \tag{140}$$

which equals the probability of receiving a sequence of length T with N_0 or more ones. Now, for each integer $1 < n < N - T$, and for $\overline{p}^*(T+n) \leq x < \overline{p}^*(T+n+1)$, $\underline{F}_\star^{T,N}(x)$ represents the probability of receiving a sequence of length $T+n$ with more than or exactly N_0 ones. Hence, $\underline{F}_\star^{T,N}(x)$ is greater than $\underline{F}^T(x)$ on the range $\overline{P}^\star(T) < x \leq \overline{P}^\star(N)$. Also, $\underline{F}_\diamond^N(x)$ measures the probability of receiving a sequence of length N with a subsequence of length T with N_0 or more ones. Hence, it is greater than $\underline{F}^T(x)$ on $\overline{P}^\star(N) < x$. Therefore, we have that

$$\max\{\underline{F}^T(x), \underline{F}_\star^{T,N}(x), \underline{F}_\diamond^N(x)\} = \begin{cases} \underline{F}^T(x), & x \leq \overline{p}^*(T) \\ \underline{F}_\star^{T,N}(x), & \overline{p}^*(T) < x \leq \overline{p}^*(N) \\ \underline{F}_\diamond^N(x), & \overline{p}^*(N) < x. \end{cases} \tag{141}$$

We will use each of these three bounds to compute each term in (129). To obtain (130), notice that $\underline{F}^T(x)$ can be written as

$$\underline{F}^T(x) = \pi_{i(x)}, \; i(x) = \max\{i : E_i < x\}. \tag{142}$$

In view of the above, we have that

$$\int_0^{\overline{p}^*(T)} (1 - \underline{F}^T(x))dx = \sum_{j=0}^{J} (1-\pi_j)(E_{j+1} - E_j) = \overline{G}_1^T. \tag{143}$$

Using the definition of $\underline{F}_\star^{T,N}(x)$ in (112) we obtain

$$\int_{\overline{p}^*(T)}^{\overline{p}^*(N)} (1 - \underline{F}_\star^{T,N}(x))dx = \sum_{j=T}^{N-1} \sum_{l=0}^{N_0-1} \lambda^l (1-\lambda)^{j-l} \frac{j!}{l!(j-l)!} \left(\overline{P}^*(j+1) - \overline{P}^*(j)\right) \tag{144}$$

$$= \overline{G}_2^{T,N}. \tag{145}$$

Similarly, the definition of $\underline{F}_\diamond^N(x)$ in (118) can be used to obtain

$$\int_{\overline{p}^*(N)}^{\infty} (1 - \underline{F}_\diamond^N(x))dx = \sum_{j=0}^{\infty} u' M^j z \mathrm{Tr}\{A^j (A\overline{P}^*(N)A' + Q - \overline{P}^*(N))A'^j\} = \overline{G}_3^{T,N}. \tag{146}$$

To conclude the proof, notice that

$$\begin{aligned} uM^j z &= < u, M^j z > && (147) \\ &\leq \|u\|_2 \|M^j z\|_2 && (148) \\ &\leq \|u\|_2 \|M^j\| \|z\|_2 && (149) \\ &\leq \|u\|_2 \|M\|^j \|z\|_2 && (150) \\ &= \|u\|_2 (\max \mathrm{sv} M)^j \|z\|_2 && (151) \\ &= \sqrt{2^{N_0} - 1} (\max \mathrm{sv} M)^j. && (152) \end{aligned}$$

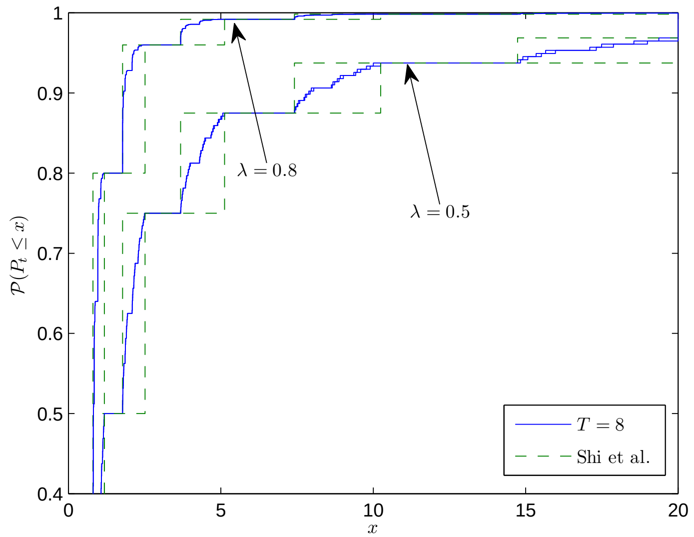

Fig. 2. Comparison of the bounds of the Cumulative Distribution Function.

where $\max \text{sv} M$ denotes the maximum singular value of M. Then, to obtain (136), we use the result in Lemma 7.1 (in the Appendix) with $b = \max \text{sv} M$ and $X = A\overline{P}^*(N)A' + Q - \overline{P}^*(N)$. ■

5. Examples

In this section we present a numerical comparison of our results with those available in the literature.

5.1 Bounds on the CDF

In Shi et al. (2010), the bounds of the CDF are given in terms of the probability to observe missing measurements in a row. Consider the scalar system below, taken from Shi et al. (2010).

$$A = 1.4, \; C = 1, \; Q = 0.2, \; R = 0.5 \tag{153}$$

We consider two different measurement arrival probabilities (i.e., $\lambda = 0.5$ and $\lambda = 0.8$) and compute the upper and lower bounds for the CDF. We do so using the expressions derived in Section 3, as well as those given in Shi et al. (2010). We see in Figure 2 how our proposed bounds are significantly tighter.

5.2 Bounds on the EEC

In this section we compare our proposed EEC bounds with those in Sinopoli et al. (2004) and Rohr et al. (2010).

Bound	Lower	Upper
From Sinopoli et al. (2004)	4.57	11.96
From Rohr et al. (2010)	-	10.53
Proposed	10.53	11.14

Table 1. Comparison of EEC bounds using a scalar system.

Bound	Lower	Upper
From Sinopoli et al. (2004)	2.15×10^4	2.53×10^5
From Rohr et al. (2010)	-	1.5×10^5
Proposed	9.54×10^4	3.73×10^5

Table 2. Comparison of EEC bounds using a system with a single unstable eigenvalue.

5.2.1 Scalar example

Consider the scalar system (153) with $\lambda = 0.5$. For the lower bound (90) we use $T = 14$, and for the upper bound (129) we use $T = N = 14$. Notice that in the scalar case $N_0 = 1$, that is, whenever a measurement is received, an upper bound for the EC is promptly available and using $N > T$ will not give any advantage. Also, for the upper bound in Rohr et al. (2010), we use a window length of 14 sampling times (notice that no lower bound for the EEC is proposed in Rohr et al. (2010)).

In Table 1 we compare the bounds resulting from the three works. We see that although the three upper bounds are roughly similar, our proposed lower bound is significantly tighter than that resulting from Sinopoli et al. (2004).

5.2.2 Example with single unstable eigenvalue

Consider the following system, taken from Sinopoli et al. (2004), where $\lambda = 0.5$ and

$$\begin{aligned} A &= \begin{bmatrix} 1.25 & 1 & 0 \\ 0 & 0.9 & 7 \\ 0 & 0 & 0.6 \end{bmatrix} & C' &= \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \\ R &= 2.5 & Q &= 20I. \end{aligned} \tag{154}$$

Table 2 compares the same bounds described above, with $T = 10$ and $N = 40$. The same conclusion applies.

6. Conclusion

We considered a Kalman filter for a discrete-time linear system, whose output is intermittently sampled according to an independent sequence of binary random variables. We derived lower and upper bounds for the CDF of the EC, as well as for the EEC. These bounds can be made arbitrarily tight, at the expense of increased computational complexity. We presented numerical examples demonstrating that the proposed bounds are tighter than those derived using other available methods.

7. Appendix

Lemma 7.1. *Let $0 \leq b \leq 1$ be a scalar, $X \in \mathbb{R}^{n\times n}$ be a positive-semidefinite matrix and $A \in \mathbb{R}^{n\times n}$ be diagonalizable, i.e., it can be written as*

$$A = VDV^{-1}, \tag{155}$$

with D diagonal. If

$$\max \ \mathrm{eig}(A)^2 b < 1, \tag{156}$$

then,

$$\mathrm{Tr}\left(\sum_{j=0}^{\infty} b^j A^j X A'^j\right) = \mathrm{Tr}(\Gamma) \tag{157}$$

where

$$\Gamma \triangleq \left(X^{1/2}V'^{-1} \otimes V\right) \Delta \left(X^{1/2}V'^{-1} \otimes V\right)' \tag{158}$$

with $\otimes$ denoting the Kronecker product. The $n^2 \times n^2$ matrix Δ is such that its i,j-th entry $[\Delta]_{i,j}$ is given by

$$[\Delta]_{i,j} \triangleq \frac{1}{1 - b[\overrightarrow{D}]_i[\overrightarrow{D}]_j}, \tag{159}$$

where $\overrightarrow{D}$ denotes a column vector formed by stacking the columns of D, i.e.,

$$\overrightarrow{D} \triangleq \left[\, [D]_{1,1} \ \cdots \ [D]_{n,1} \ [D]_{1,2} \ \cdots \ [D]_{n,n} \,\right]'. \tag{160}$$

Proof: For any matrix

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,n} \\ B_{2,1} & B_{2,2} & \dots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \dots & B_{n,n} \end{bmatrix} \tag{161}$$

with $B_{i,j} \in \mathbb{R}^{n\times n}$, we define the following linear transformation

$$\mathcal{D}_n(B) = \sum_{j=1}^{n} B_{j,j}. \tag{162}$$

Now, substituting (155) in (157), and using the vectorization operation $\overrightarrow{\cdot}$ defined above we have

$$\sum_{j=0}^{\infty} b^j A^j X A'^j = \sum_{j=0}^{\infty} b^j V D^j V^{-1} X^{1/2} \left(VD^jV^{-1}X^{1/2}\right)' \tag{163}$$

$$= \sum_{j=0}^{\infty} \mathcal{D}_n \left[b^j \overrightarrow{VD^jV^{-1}X^{1/2}} \left(\overrightarrow{VD^jV^{-1}X^{1/2}}\right)' \right] \tag{164}$$

$$= \mathcal{D}_n \left[\left(X^{1/2}V'^{-1} \otimes V\right) \sum_{j=0}^{\infty} b^j \overrightarrow{D}^j \overrightarrow{D}^{j\prime} \left(X^{1/2}V'^{-1} \otimes V\right)' \right], \tag{165}$$

where the last equality follows from the property

$$\overrightarrow{ABC} = (C' \otimes A)\,\overrightarrow{B}. \tag{166}$$

Let $\delta_{i,j}$ denote the i,j-th entry of $b\overrightarrow{D}\,\overrightarrow{D}'$, and $\text{pow}(Y,j)$ denote the matrix obtained after elevating each entry of Y to the j-th power. Then, if every entry of $b\overrightarrow{D}\,\overrightarrow{D}'$ has magnitude smaller than one, we have that

$$\left[\sum_{j=0}^{\infty} b^j \overrightarrow{(D)^j}\,\overrightarrow{(D)^j}'\right]_{i,j} = \left[\sum_{j=0}^{\infty} \text{pow}(b\overrightarrow{D}\,\overrightarrow{D}', j)\right]_{i,j} \tag{167}$$

$$= \frac{1}{1-\delta_{i,j}}. \tag{168}$$

where $[Y]_{i,j}$ denotes the i,j-th entry of Y. Notice that $\overrightarrow{D}\,\overrightarrow{D}'$ if formed by the products of the eigenvalues of A, so the series will converge if and only if

$$\max\ \text{eig}(A)^2 b < 1. \tag{169}$$

Putting (168) into (165), we have that

$$\sum_{j=0}^{\infty} b^j A^j X A'^j = \mathcal{D}_n \left[\left(X^{1/2} V'^{-1} \otimes V \right) \Delta \left(X^{1/2} V'^{-1} \otimes V \right)' \right] \tag{170}$$

$$= \mathcal{D}_n\left(\Gamma\right) \tag{171}$$

and the result follows since $\text{Tr}\{\mathcal{D}_n\{Y\}\} = \text{Tr}\{Y\}$. ■

Lemma 7.2. *Let u, z, N_0, L and M be as defined in Lemma 4.3. The probability $p_{N,n}$ of receiving a sequence of length $N+n$ that does not contain a subsequence of length N with at least N_0 ones is given by*

$$p_{N,n} = uM^{N+n}z. \tag{172}$$

Proof:
Let Z_m^N, $m = 0, \cdots, L-1$, and $U_+(Z_m^T, \gamma)$ be as defined in Lemma (4.3). Also, for each $N, t \in \mathbb{N}$, define the random sequence $V_t^N = \{\gamma_t, \gamma_{t-1}, \cdots, \gamma_{t-N+1}\}$. Let W_t be the probability distribution of the sequences Z_m^N, i.e.

$$W_t = \begin{bmatrix} \mathcal{P}(V_t^N = Z_0^N) \\ \mathcal{P}(V_t^N = Z_1^N) \\ \cdots \\ \mathcal{P}(V_t^N = Z_{L-1}^N) \end{bmatrix}. \tag{173}$$

One can write a recursive equation for W_{t+1} as

$$W_{t+1} = MW_t. \tag{174}$$

Hence, for a given n, the distribution W_n of V_n^N is given by

$$W_n = M^n W_0. \tag{175}$$

To obtain the initial distribution W_0, we make $V_{-N}^N = Z_0^N$, which gives

$$W_{-N} = z. \tag{176}$$

Then, applying (175), we obtain

$$W_0 = M^N z. \tag{177}$$

Finally, to obtain the probability $p_{N,n}$, we add all the entries of the vector W_n by pre-multiplying W_n by u. Doing so, and substituting (177) in (175), we obtain

$$p_{N,n} = uM^{N+n}z. \tag{178}$$

■

8. References

Anderson, B. & Moore, J. (1979). *Optimal filtering*, Prentice-Hall Englewood Cliffs, NJ.

Ben-Israel, A. & Greville, T. N. E. (2003). *Generalized inverses*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15, second edn, Springer-Verlag, New York. Theory and applications.

Dana, A., Gupta, V., Hespanha, J., Hassibi, B. & Murray, R. (2007). Estimation over communication networks: Performance bounds and achievability results, *American Control Conference, 2007. ACC '07* pp. 3450 –3455.

Faridani, H. M. (1986). Performance of kalman filter with missing measurements, *Automatica* 22(1): 117–120.

Liu, X. & Goldsmith, A. (2004). Kalman filtering with partial observation losses, *IEEE Control and Decision* .

Rohr, E., Marelli, D. & Fu, M. (2010). Statistical properties of the error covariance in a kalman filter with random measurement losses, *Decision and Control, 2010. CDC 2010. 49th IEEE Conference on*.

Schenato, L. (2008). Optimal estimation in networked control systems subject to random delay and packet drop, *IEEE Transactions on Automatic Control* 53(5): 1311–1317.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K. & Sastry, S. (2007). Foundations of control and estimation over lossy networks, *Proc .IEEE* 95(1): 163.

Shi, L., Epstein, M. & Murray, R. (2010). Kalman filtering over a packet-dropping network: A probabilistic perspective, *Automatic Control, IEEE Transactions on* 55(3): 594 –604.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M. & Sastry, S. (2004). Kalman filtering with intermittent observations, *IEEE Transactions on Automatic Control* 49(9): 1453–1464.