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ALGORITHMS FOR OPTIMAL MULTIRATE FILTER BANK DESIGN

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ABSTRACT

This paper is concerned with the problem of designing optimal multirate filter banks. Three design criteria are considered for linear phase FIR filter banks: H_2 measure, H_∞ and the mixture of them. A closed-form solution is presented for optimal H_2 filter bank design. For optimal H_∞ filter bank design, two algorithms are given which involve solving the design problem using positive semi-definite programming techniques. The mixed H_2/H_∞ design problem is also solved. The results of this paper are excepted to be useful in applications such as speech coding, sampling rate conversion and multi-stage filtering.

1. INTRODUCTION

The technique of multirate filtering has been studied by many researchers in recent years. The application of this technique in digital signal processing is wide spread, including filter banks for coding, sampling rate changers, and multi-stage filtering; see, e.g., [9, 4].

Using the standard polyphase representation, most multirate filter bank design problems can be converted into a multi-input multi-output filter design problem depicted in Figure 1. In this configuration, $\mathcal{T}$ represents a target system. For example, in a multistage filter design problem, $\mathcal{T}$ may be specified by passband and stopband specifications. The two stages, $G(z)$ and $H(z)$, are transfer matrices in polyphase representation. For simplicity, we only consider a two-stage system, but the concept in our paper can be extended to systems with many stages. The signal input $u(n) \in \mathcal{R}^m$, the target output $y_T(n) \in \mathcal{R}^r$, the filter output $y_H(n) \in \mathcal{R}^r$, and the filter error

$$\epsilon(n) = y_H(n) - y_T(n) \in \mathcal{R}^r \qquad (1)$$

The goal is to design $G(z)$ and $H(z)$ such that some given specifications, denoted by $\mathcal{T}$, are met.

Although there are many design methods available for various applications, most of them separate the multirate system into different parts and each part is designed separately according to its separate design specifications. Such separation often occurs on two levels. Frist, different stages of a multirate system are designed separately. Secondly, different channels of a given stage are further separated. The advantage of this "local" design approach is its simplicity in design specifications and in the computation required for design. In particular, conventional single rate, single channel filter design methods are all applicable. The main drawback, however, is the lack of "global" optimality of the design. For example, a two-stage implementation of a low-pass filter with δ-tolerance level in the passband is often designed by specifying a $\delta/2$-tolerance level in each stage[9, 4]. The resulting multirate filter, although meeting the desired specification, typically has its total degree higher than necessary.

The purpose of this paper is to study a "global" design approach for multirate systems which intends to overcome the weakness of the "local" design approach. The idea of this method is that the design specifications for stage i are given in terms of the cascaded subsystem consisting of stage 1 to stage i. That is, the design of stage i should be done to optimize the performance of the subsystem consisting stage 1 to stage i. This kind of specifications are very natural in many multirate filtering problems. For example, if filter banks are used for coding and decoding, the specifications for the analysis bank may be such that the maximum data compression can be achieved. The specifications for the synthesis filter may be such that the cascaded system (analysis filter and the synthesis filter) is more or less lossless and linear phase. For the two-stage low-pass filter example mentioned earlier, the first stage can be done with some δ_1-tolerance level ($\delta_1 < \delta$), and then the second stage should be designed such that the tolerance level of the cascaded system is less than δ while keeping the order of stage 2 as small as possible.

Using the design approach mentioned above, the design of each stage is done in the same fashion, and it can be illustrated in Figure 1. The transfer function $G(z)$ is given, representing the cascaded subsystem from stage 1 to stage $i-1$. For $i = 1$, $G(z) = I$. The function

$H(z)$ represents stage i. The design specifications for stage i are symbolically represented by $\mathcal{T}$.

To accommodate different applications, we consider H_2 and H_∞ measures and combinations of them for filter design. The ability to cope with different measures is important in multi-objective applications. For example, H_2 filters are commonly ued for processing signals involving Gaussian random noises. H_∞ filters are a natural generalization of the equirriple filter in the single rate single channel case. Our focus will be given to linear phase FIR filters.

Our main contributions of the paper are efficient algorithms for designing multirate H_2 and H_∞ filters. In the H_2 case, we present a closed form solution to the design of optimal filters. In the H_∞ case, two convex optimization algorithms are presented. Mixed H_2/H_∞ problems are considered too.

2. OPTIMAL H_2 FILTER DESIGN

The optimal H_2 filter design problem solved in this section has a very general setting as described below.

Let Ω_p and Ω_s be two disjoint compact set in $[-\pi, \pi]$ representing the passband and stopband of desired multirate filter. Denote by Ω the union of Ω_p and Ω_s and by $\Omega_d = [-\pi, \pi]\backslash\Omega$ (the don't care region). Let $T(\omega)$ be a real matrix function defined on Ω, which specifies the linear phase ideal filter. Let $W(\omega)$ be a real matrix function defined on $[-\pi, \pi]$, which specifies the weightings (corresponding to tolerance levels) in the passband and stopband regions. In particular, $W(\omega) = 0, \quad \forall\omega \in \Omega_d$. It is assumed that $T(\omega) = T(-\omega)$ and $W(\omega) = W(-\omega)$. Usually, both $T(\omega)$ and $W(\omega)$ are piecewise constant. Also given is a linear phase filter $G(z)$. We only consider the following $H(z)$ (Other types of linear phase filters are considered similarly):

$$H(z) = z^{-m}\left(H_0 + \sum_{i=1}^{m} H_i \frac{z^i + z^{-i}}{2}\right) \qquad (2)$$

Equivalently,

$$H(e^{j\omega}) = e^{-jm\omega}\left(H_0 + \sum_{i=1}^{m} H_i \cos(i\omega)\right) \qquad (3)$$

Two design problems are considered:

P1 Given m, design a $(2m+1)$-tap linear phase filter $H(z)$ as in (2) such that the following H_2-norm square is minimized:

$$J(H) = \int_\Omega \mathrm{tr}\left[X(\omega)X^*(\omega)\right] d\omega \qquad (4)$$

where

$$X(\omega) = (T(\omega) - H(e^{j\omega})G(e^{j\omega}))W(\omega) \qquad (5)$$

and $\mathrm{tr}[\cdot]$ denotes the trace.

P2 Given an upper bound $\bar{J}$, find the minimum-tap filter $H(z)$ in (2) as follows:

$$\min m \ \text{ subject to } J(H) \leq \bar{J}. \qquad (6)$$

We show that **P1** has an analytical solution and **P2** can be solved using a simple bisection algorithm.

To deal with **P1**, we define

$$J_{TT} = \int_\Omega T(\omega)W(\omega)W^*(\omega)T^*(\omega)d\omega \qquad (7)$$

$$J_{TG,i} = \int_\Omega TWW^*G^*e^{jm\omega}\cos(i\omega)d\omega \qquad (8)$$

$$J_{GG,ik} = \int_\Omega GWW^*G^*\cos(i\omega)\cos(k\omega)d\omega \qquad (9)$$

for $i, k = 0, 1, \cdots, m$, and

$$\mathbf{H} = \begin{pmatrix} H_0' \\ \vdots \\ H_m' \end{pmatrix}, \quad J_{TG} = \begin{pmatrix} J_{TG,0}^* \\ \vdots \\ J_{TG,m}^* \end{pmatrix} \qquad (10)$$

$$J_{GG} = \{J_{GG,ik}, \ i, k = 0, \cdots, m\} \qquad (11)$$

Then, J_{TT}, J_{TG} and J_{GG} are all real and $J_{GG} = J_{GG}^* \geq 0$ (symmetric and positive-semidefinite). We also note that these integrals are usually easy to compute because $W(\omega)$ and $T(\omega)$ are usually piecewise constant and $G(z)$ is also an FIR filter. Subsquently, we have

$$J(H) = \mathrm{tr}\left[J_{TT} - J_{TG}^*\mathbf{H} - \mathbf{H}'J_{TG} + \mathbf{H}'J_{GG}\mathbf{H}\right] \qquad (12)$$

Assume $J_{GG} > 0$. Rewriting the above, we obtain

$$\begin{aligned} J(H) &= \mathrm{tr}\left[J_{TT} - J_{TG}^* J_{GG}^{-1} J_{TG}\right. \\ &+ \left.(\mathbf{H} - J_{GG}^{-1}J_{TG})^* J_{GG}(\mathbf{H} - J_{GG}^{-1}J_{TG})\right] \end{aligned} \qquad (13)$$

Obviously, an optimal $J(H)$ is given by

$$J_{\min} = \min J(H) = \mathrm{tr}\left[J_{TT} - J_{TG}^* J_{GG}^{-1} J_{TG}\right] \qquad (14)$$

with the minimizer given by

$$\mathbf{H}_{\min} = J_{GG}^{-1}J_{TG} \qquad (15)$$

This result is summarized below.

Theorem 1. *The optimal solution to* **P1** *is given by (14)-(15), provided that* $J_{GG} > 0$.

To see how **P2** is solved, we assume that some upper bound $\bar{m}$ and lower bound $\underline{m}$ are given. Note that the upper bound can either be given from hardware limitations or be estimated using degree estimates of single channel linear phase FIR filters [4]. If no lower bound is given, one can set $\underline{m} = 0$. Then, the following bisection algorithm, which takes at most $\log_2(\bar{m} - \underline{m})$ iterations, will provide an optimal m to meet the design specifications.

Step 1: Set $m = (\underline{m} + \bar{m})/2$. Solve $J_{\min}$ as in (14).

Step 2: If $J_{\min} > \bar{J}$, set $\underline{m} = m$; else set $\bar{m} = m$. If $\bar{m} > \underline{m}$, go to Step 1; else $m_{\min} = \underline{m}$.

We note that J_{TT} and the elements of J_{TG} and J_{GG} are repeatedly used in the iterations above, indicating that the algorithm can be implemented very efficiently. However, we do not conduct a detailed analysis here due to page limitation.

3. H_∞ FILTER DESIGN–METHOD 1

The problem settings for H_∞ filter design are the same in the H_2 case except that an H_∞ measure is used here. That is, we consider the following problems:

P3 Given m, design a $(2m + 1)$-tap linear phase filter $H(z)$ as in (2) such that the following H_∞-norm is minimized:

$$\gamma(H) = \max_{\omega \in \Omega} \sigma_{\max} \left[(T(\omega) - G(e^{j\omega})H(e^{j\omega}))W(\omega) \right] \tag{16}$$

where $\sigma_{\max}[\cdot]$ denotes the maximum singular value.

P4 Given an upper bound $\bar{\gamma}$, find the minimum-tap filter $H(z)$ in (2) as follows:

$$\min m \;\; \text{subject to } \gamma(H) \leq \bar{\gamma}. \tag{17}$$

We will consider **P3** only because **P4** can be solved using a bisection algorithm, as for **P2**.

We note that in the single channel case, an H_∞ filter is nothing but an equiripple filter, which can be designed efficiently using the well-known Parks-McClellan algorithm [6]. Further, good estimates of filter orders exist [6, 4]. However, this algorithm does not have an multichannel counterpart.

It is also well-known that single-channel equiripple filters can be designed using linear programming techniques. This is simply done by taking a "dense" set of frequencies $\{w_1, \cdots, w_N\} \subset \Omega$, and modify **P3** to the following:

$$\begin{array}{ll} \min & \gamma \\ \text{subject to} & |X(\omega_k)| \leq \gamma, \; k = 1, \cdots, N \end{array} \tag{18}$$

where $X(\omega)$ is defined in (5). This method is substantially slower but has the advantage of being able to incorporate additional convex constraints in time domain and frequency domains; see [7].

The design method introduced above is readily extended to filter banks. We take a "dense" set of frequencies $\{w_1, \cdots, w_N\} \subset \Omega$, and modify **P3** to the following:

$$\begin{array}{ll} \min & \gamma^2 \\ \text{subject to} & \sigma^2_{\max}[X(\omega_k)] \leq \gamma^2, \; k = 1, \cdots, N \end{array} \tag{19}$$

Using the Schur complement, the above can be rewritten as

$$\begin{array}{ll} \min & \gamma^2 \\ \text{subject to} & \begin{pmatrix} \gamma^2 I & X(\omega_k) \\ X^*(\omega_k) & I \end{pmatrix} \geq 0, \\ & k = 1, \cdots, N \end{array} \tag{20}$$

Note that the matrix inequality above is linear in γ^2 and $H_i, i = 0, 1, \cdots, m$. Hence the problem can be solved efficiently using semidefinite programming techniques; see [1] for introduction and [5] for a Matlab toolbox.

4. H_∞ FILTER DESIGN–METHOD 2

The method introduced here requires a somewhat different setting. We assume that an ideal (high order) filter $T(z)$ and a weighting function $W(z)$ are expressed by either FIR or IIR matrices. Also given is $G(z)$ as before. The problem of interest is to design $H(z)$ of a given degree such that the H_∞-norm of the following function is minimized:

$$E(z) = (T(z) - G(z)H(z))W(z) \tag{21}$$

To motivate the design problem above, we consider an application of filter banks in coding. In this case, $G(z)$ represents the analysis filter bank and $H(z)$ the synthesis filter bank. The ideal filter $T(z)$ can simply be a pure delay function, i.e. $T(z) = z^{-d}I$ for some d. The design problem is to design a synthesis filter bank so that the decoded signal will match the original signal optimally in the H_∞ sense, module a pure delay. Note that the synthesis filter bank is not designed independently from the analysis filter bank.

Another example where T is an FIR or IIR filter is the case of multistage filtering. Here $T(z)$ is given and the design goal is to use a multistage configuration to reduce the overall order of the filter; see [4].

Without loss of generality, we will assume $W(z) = I$ because it can always be absorbed into $T(z)$ and $G(z)$.

Express the transfer matrices in state space realizations, i.e.

$$T(z) = D_T + C_T(zI - A_T)^{-1}B_T \quad (22)$$
$$G(z) = D_G + C_G(zI - A_G)^{-1}B_G \quad (23)$$
$$H(z) = D_H + C_H(zI - A_H)^{-1}B_H \quad (24)$$

The filtering error transfer function is given by

$$E(z) = D + C(zI - A)^{-1}B \quad (25)$$

where

$$A = \begin{bmatrix} A_G & 0 & 0 \\ 0 & A_T & 0 \\ B_H C_G & 0 & A_H \end{bmatrix}; \quad B = \begin{bmatrix} B_G \\ B_T \\ B_H D_G \end{bmatrix}; \quad (26)$$

$$C = \begin{bmatrix} D_H C_G & -C_T & C_H \end{bmatrix}; \quad D = D_H D_G - D_T \quad (27)$$

Since $H(z)$ is FIR, we can assume that A_H and B_H are constant and that C_H and D_H are design parameters depending on $H_i, i = 0, \cdots, m$ linearly. Therefore, A and B are known and that C and D are linear in H_i.

Theorem 2. *The optimal H_∞ filtering problem is equivalent to the following semidefinite problem:*

$$\gamma_{\min}^2 = \min \gamma^2 \quad \text{subject to}$$

$$\begin{bmatrix} A^T X A - X & A^T X B & C^T \\ B^T X A & B^T X B - I & D^T \\ C & D & -\gamma^2 I \end{bmatrix} \leq 0 \quad (28)$$

$$X = X^T > 0, \ \gamma \geq 0, \ H_i, i = 0, \cdots, m. \quad (29)$$

That is, $\min\{\|E(z)\|_\infty : H_i, i = 0, 1, \cdots, m\} = \gamma_{\min}$.

As for the first method, the inequalities in (28) are linear in X and $H_i, i = 0, 1, \cdots, m$. So semidefinite programming techniques can be applied to compute $\gamma_{\min}$.

5. MIXED H_2/H_∞ FILTER BANK DESIGN

The mixed H_2/H_∞ filter bank design criteria involve both H_2 constraints and H_∞ constraints. Suppose we are given $T(z)$, $G(z)$ and $W(z)$ as in the last section and some upper bound $\bar{J}$ for $J(H)$ in (12), we need to find a $H(z)$ (of a given degree) as in (2) such that the H_∞-norm of the weighted error transfer function $E(z)$ in (21) is minimized subject to $J(H) \leq \bar{J}$.

Combining the results in Sections 2 and 4, we have the following result:

Theorem 3. *The mixed H_2/H_∞ optimal filter bank design problem is equivalent to the following:*

$$\gamma_{\min}^2 = \min \gamma^2 \quad \text{subject to (28) – (29) and}$$

$$\begin{bmatrix} J_{TT} - J_{TG}^* \mathbf{H} - \mathbf{H}' J_{TG} - Y & \mathbf{H}' \\ \mathbf{H} & -J_{GG}^{-1} \end{bmatrix} \leq 0$$

$$\mathrm{tr}[Y] \leq \bar{J} \quad (30)$$

provided that $J_{GG} > 0$. That is,

$$\min\{\|E(z)\|_\infty : J(H) \leq \bar{J}, H_i, i = 0, 1, \cdots, m\} = \gamma_{\min}$$

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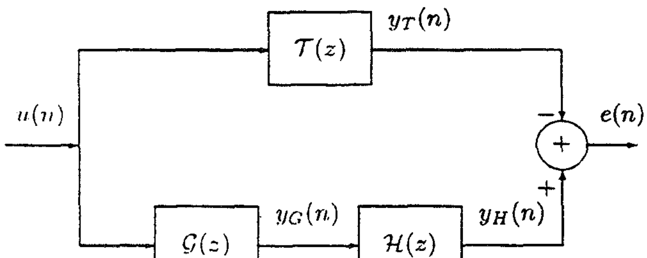

Figure 1: Multirate Filter Bank