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Improved Upper Bounds of the Structured Singular Value

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Abstract

In this paper, we take a new look at the mixed structured singular value problem. Several new upper bounds are proposed using a very simple approach which we call the multiplier approach. These new bounds are convex, computable by using the linear matrix inequality (LMI) method, and numerically efficient in many robust stability problems. We show, most importantly, that these upper bounds are actually lower bounds of a well-known upper bound which involves the so-called D-scaling (for complex perturbations) and G-scaling (for real perturbations).

1 Introduction

This paper addresses the problem of the mixed structured singular value. The notion of structured singular value, or μ for short, was initially proposed by Doyle [4] for studying the robust stability of linear systems which are subject to certain structured complex perturbations. It has also been extended to the case where there exist both real parameters and complex uncertainty, i.e., the so-called mixed perturbations; see Fan, Tits and Doyle [5]. The exact definition of the mixed structured singular value is given in section 2.

The computational issue of μ has been studied in numerous papers; see Young, Newlin and Doyle [13] for a summary. There is also a Matlab Toolbox available; see Balas, *et. al.* [1]. The best upper bound of μ known so far is given by Fan, Tits and Doyle [5] for a general μ problem where both real and complex perturbations are allowed. This bound is generalized from an earlier result of Doyle [4]. A nice feature of this upper bound is that it is convex.

In this paper, we propose several new upper bounds for the structured singular value. The first new upper bound is derived by using the following simple fact: *A matrix family $\mathcal{A}$ is nonsingular if there exists another matrix C (multiplier) such that the Hermitian part of the product AC is negative-definite for all $A \in \mathcal{A}$.* An interesting point regarding this upper bound is that it is convex. This upper bound holds an important conceptual value because it is extremely easy to derive and serves as a lower bound for other upper bounds. Because this upper bound is not easy to compute when there exist complex perturbations, we apply the well-known S-procedure on it and derive a second new upper bound which is loser than the first one but convex, computable by using the linear matrix inequality method, and numerical efficient in many robust stability problems. In fact, this bound is numerically efficient for cases where there are only a small or medium number of real perturbation blocks. For cases where the number of real perturbation blocks is large, we derive another new upper bound which is even looser but more efficient to compute. This upper bound is still convex and computable by using the LMI method. *But most surprisingly, we find that all these new upper bounds are actually lower bounds of the upper bound of [5] mentioned above.*

The numerical efficiency of our new upper bounds is interesting and deserves a bit explanation. People who have worked with the μ-theory may have the following experience: In order to fit a robust stability problem into the standard form $I - \Delta M$, a substantial dimension lifting is often required. For example, the robust stability of a 10th order linear system with 5 uncertain parameters results in a 50×50 matrix M (see Section 5.1 for detailed description of the conversion procedure). Consequently, a high dimensional convex optimization task needs to be solved. For the same example above, the optimization task is to solve a 50×50 linear matrix inequality involving roughly 2500 complex parameters (counting both D- and G-scalings)! However, dimension lifting is not needed with our approach. The resulting convex optimization is formulated using the original system, involving 32 number of 10×10 LMIs with only 100 complex parameters. Examples for which our new upper bound is very easy to compute include the so-called rank-one perturbations; robust nonsingularity and stability of a polytope of matrices; and low-rank perturbations in general.

The rest of this paper is organized as follows: Section 2 reviews the mixed structured singular value problem and the upper bound in [5]. Section 3 provides three new upper bounds. The relationships among these upper bounds and the upper bound in [5] are given in Section 4. The computational issue and an important example are discussed in section 5. Finally, the conclusions are drawn in section 6.

2 Review of μ

2.1 Notation and Definition

The notation needed for defining the mixed structured singular value is standard, and we simply duplicate it from [5].

Given $M \in \mathcal{C}^{n\times n}$ and nonnegative integers m_r, m_c and m_C, with

$$m = m_r + m_c + m_C \leq n \tag{1}$$

a block structure $\mathcal{K}$ is defined to be an m-tuple of positive integers:

$$\mathcal{K} = [k_1, \cdots, k_m] \tag{2}$$

subject to $\sum_{i=1}^{m} k_i = n$. Also define the family of block-diagonal $n \times n$ matrices

$$\begin{aligned}\mathcal{X} = \{&\text{block diag}\left(\delta_1^r I_{k_1}, \cdots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \cdots,\right.\\ &\left.\delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \cdots, \Delta_{m_C}^C\right) : \delta_i^r \in \mathcal{R}, \delta_i^c \in \mathcal{C},\\ &\Delta_i^C \in \mathcal{C}^{(k_{m_r+m_c+i})\times(k_{m_r+m_c+i})}\}\end{aligned} \tag{3}$$

In the above, I_k denotes the $k \times k$ identity matrix.

Definition 1 *[4, 5] Given a complex $n \times n$ matrix M and a block-structure $\mathcal{K}$, the associated structured singular value $\mu_{\mathcal{K}}(M)$ of is defined to be 0 if $I_n - \Delta M$ is nonsingular for all $\Delta \in \mathcal{X}$, and*

$$\mu_{\mathcal{K}}(M) = \left(\min_{\Delta\in\mathcal{X}} \{\bar{\sigma}(\Delta) : \det(I_n - \Delta M) = 0\}\right)^{-1} \tag{4}$$

otherwise, where $\bar{\sigma}()$ denotes the largest singular value.

2.2 A Known Upper Bound

The exact computation of $\mu_{\mathcal{K}}(M)$ is very difficult. In fact, its computational complexity is known to be NP-hard; see Poljak and Rohn [11] and Braatz, *et. al.* [3]. In practice, an upper bound and a lower bound are computed to approximate the exact value. The upper bound is very important because it serves as an guaranteed margin for robust nonsingularity. In contrast, the lower bound is used mainly for checking the tightness of the upper bound.

Many papers have been devoted to finding a good upper bound for the structured singular value. Here we quote a result from Fan, Tits and Doyle [5], the best one known so far which is computable by a convex program. To this end, we introduce some more notation:

$$\begin{aligned}\mathcal{D} = \{&\text{block diag}\left(D_1, \cdots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}}, \cdots,\right.\\ &\left.d_{m_c+m_C} I_{k_m}\right) : 0 < D_i = D_i^H \in \mathcal{C}^{k_i\times k_i}, d_i > 0\}\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{G} = \{&\text{block diag}\left(G_1, \cdots, G_{m_r}, 0_{k_{m_r+1}}, \cdots, 0_{k_m}\right) :\\ &G_i = G_i^H \in \mathcal{C}^{k_i\times k_i}\}\end{aligned} \tag{6}$$

where 0_k denotes a $k \times k$ zero matrix. For any $a \in \mathcal{R}$, define

$$\Phi_a(D, G) = M^H DM + j(GM - M^H G) - a^2 D \tag{7}$$

Lemma 1 *[5] Given $M \in \mathcal{C}^{n\times n}$ and an associated block structure $\mathcal{K}$, the following holds:*

$$\mu_{\mathcal{K}}(M) \leq \nu_{\mathcal{K}}(M) \tag{8}$$

where

$$\nu_{\mathcal{K}}(M) = \inf_{a>0}\{a : \exists D \in \mathcal{D}, G \in \mathcal{G} : \Phi_a(D, G) < 0\} \tag{9}$$

Although the upper bound above is very popularly used and its computational issues have been extensively studied, it is known that its gap to the exact value of $\mu_{\mathcal{K}}(M)$ can be very large, especially when there are real perturbation blocks; see Young, Newlin and Doyle [13], for example.

3 New Upper Bounds

We present three new upper bounds for the mixed structured singular value. The first one is derived by using an extremely simple approach which we call the multiplier approach. This upper bound is convex but is numerically computable only for cases without real perturbation blocks. However, it holds an important conceptual value due to its simplicity. Subsequently, a loser upper bound is derived from the first one by applying the well-known $\mathcal{S}$-procedure of Yakubovich [12]. The resulting upper bound is convex and numerical efficient for many robust stability problems where the number of real perturbation blocks is small or medium. For treating the cases where the number of real perturbation blocks is large, we provide a third yet looser new upper bound.

3.1 The First New Upper Bound

The first new upper bound is proposed by using the following trivial fact: *A matrix family $\mathcal{A} \subset \mathcal{C}^{n\times n}$ is nonsingular if there exists another matrix $C \in \mathcal{C}^{n\times n}$ (multiplier) such that the Hermitian part of the product AC (or CA) is negative-definite for all $A \in \mathcal{A}$.*

When applied to the μ problem, the result above says that $\mu_{\mathcal{K}}(M) \leq a$ $(a > 0)$ if there exists a multiplier $C \in \mathcal{C}^{n\times n}$ such that

$$\begin{aligned}&C(I - \Delta M) + (I - \Delta M)^H C^H < 0,\\ &\forall \Delta \in \mathcal{X} \text{ with } \bar{\sigma}(\Delta) \leq a^{-1}\end{aligned} \tag{10}$$

When M has the following special structure:

$$M = AB^H \tag{11}$$

where $A, B \in \mathcal{C}^{n\times q}, q \le n$, the following equation becomes useful:

$$\det(I_n - \Delta M) = \det(I_n - M\Delta) = \det(I_q - B^H \Delta A) \quad (12)$$

An important special case of the above is $A = M$ and $B^H = I_n$.

With the rewriting of M in (11), the sufficient condition for μ above reduces to finding the multiplier $C \in \mathcal{C}^{q\times q}$ and checking

$$(I_q - B^H \Delta A)^H C + C^H (I_q - B^H \Delta A) < 0, \quad \forall \Delta \in \mathcal{X} \text{ with } \bar{\sigma}(\Delta) \le a^{-1} \quad (13)$$

Re-denoting Δ^H by Δ, the above is equivalent to

$$C^H (I_q - A^H \Delta B)^H + (I_q - A^H \Delta B) C < 0, \quad \forall \Delta \in \mathcal{X} \text{ with } \bar{\sigma}(\Delta) \le a^{-1} \quad (14)$$

Remark 1 *The condition (13) or (14) has an obvious advantage over (10) because the size of the multiplier in (13) or (14)is only $q \times q$ rather than $n \times n$.*

Based on the analysis above, we propose a simple new upper bound as follows: Given a matrix $M \in \mathcal{C}^{n\times n}$ of the form (11) and an associated block-structure $\mathcal{K}$, define

$$\hat{\mu}_{\mathcal{K}}(M) = \inf_{a>0} \{a : \exists C \in \mathcal{C}^{q\times q} : E(C,\Delta) > 0, \quad \forall \Delta \in \mathcal{X}, \bar{\sigma}(\Delta) \le a^{-1}\} \quad (15)$$

where

$$E(C,\Delta) = C^H (I - A^H \Delta B)^H + (I - A^H \Delta B) C \quad (16)$$

Obviously, we have

$$\mu_{\mathcal{K}}(M) \le \hat{\mu}_{\mathcal{K}}(M) \quad (17)$$

Further, we observe that $\hat{\mu}_{\mathcal{K}}(M)$ is convex. By this we mean that the inequality $E(C,\Delta) < 0$ is a convex property in terms of C. This property follows trivially from the fact that $E(C,\Delta)$ is a linear function of C.

3.2 The Second New Upper Bound

The computation of the first new upper bound is inconvenient in cases where there are complex perturbation blocks. To simplify the computation, we obtain a looser upper bound by applying the well-known $\mathcal{S}$-procedure [12] on the complex blocks (i.e., replacing them with D-scaling). To this end, we denote by Δ_R and Δ_C the real and complex sub-blocks of Δ, respectively. That is,

$$\Delta_R = \text{block diag}\left(\delta_1^r I_{k_1}, \cdots, \delta_{m_r}^r I_{k_{m_r}}\right) \quad (18)$$

$$\Delta_C = \text{block diag}\left(\delta_1^c I_{k_{m_r+1}}, \cdots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \cdots, \Delta_{m_C}^C\right) \quad (19)$$

$\mathcal{K}_R, \mathcal{K}_C, \mathcal{X}_R, \mathcal{X}_C, \mathcal{D}_R, \mathcal{D}_C, D_R, D_C, G_R$ are similarly defined (note that $G_C = 0$). We also rewrite (11) as follows:

$$M = AB^H = \begin{bmatrix} A_R \\ A_C \end{bmatrix} [B_R^H \;\; B_C^H] \quad (20)$$

Then, $E(C,\Delta)$ can be rewritten as

$$E(C,\Delta) = E(C,\Delta_R) - C^H B_C^H \Delta_C^H A_C - A_C^H \Delta_C B_C C \quad (21)$$

where

$$E(C,\Delta_R) = C^H (I - A_R^H \Delta_R B_R)^H + (I - A_R^H \Delta_R B_R) C \quad (22)$$

Applying the $\mathcal{S}$-procedure on the complex blocks in (21), we obtain the following result:

Theorem 1 *Let $E(C,\Delta)$ be given in (21) and $a > 0$. Then, $E(C,\Delta) < 0$ for all $\Delta \in \mathcal{X}$ with $\bar{\sigma}(\Delta) \le a^{-1}$ if there exists scalars $d_i, i = 1, \cdots, m_c + m_C$ to form $D_C \in \mathcal{D}_{\mathcal{K}_C}$ such that the following 2^{m_r} inequalities hold:*

$$F(C, D_C, \Delta_R) = E(C,\Delta_R) + a^{-2} C^H B_C^H D_C^{-1} B_C C + A_C^H D_C A_C < 0 \quad \forall \delta_i^r = \pm a^{-1}, 1 \le i \le m_r \quad (23)$$

Furthermore, $F(C, D_C, \Delta_R) < 0$ is equivalent to the following LMI:

$$\hat{F}(C, D_C, \Delta_R) = \begin{bmatrix} E(C,\Delta_R) + A_C^H D_C A_C & C^H B_C^H \\ B_C C & -a^2 D_C \end{bmatrix} < 0 \quad (24)$$

implying that $F(C, D_C, \Delta_R)$ is jointly convex in C and D_C.

Proof: *Let us suppose (23) holds for some D_C. Then, $F(C, D_C, \Delta_R) < 0$ for all $\Delta_R \in \mathcal{X}_R$ with $\bar{\sigma}(\Delta_R) \le a^{-1}$ because $F(C, D_C, \Delta_R)$ is linear in Δ_R. Thus, for any $\Delta \in \mathcal{X}$ with $\bar{\sigma}(\Delta) \le a^{-1}$, we have*

$$E(C,\Delta) \le E(C,\Delta_R) + A_C^H D_C^{1/2} D_C^{1/2} A_C + C^H B_C^H D_C^{-1/2} \Delta_C^H \Delta_C D_C^{-1/2} B_C C \le F(C, D_C, \Delta_R) < 0 \quad (25)$$

The equivalence between (23) and (24) is obvious. Consequently, $F(C, D_C, \Delta_R)$ is convex in C and D_C because $\hat{F}(C, D_C, \Delta_R)$ is linear in them.

Based on Theorem 1, we define our second new upper bound as follows:

$$\bar{\mu}_{\mathcal{K}}(M) = \inf_{a>0} \{a : \exists C \in \mathcal{C}^{q\times q}, D_C \in \mathcal{D}_C : F(C, D_C, \Delta_R) < 0, \forall \delta_i^r = \pm a^{-1}, 1 \le i \le m_r\} \quad (26)$$

where $F(C, D_C, \Delta_R)$ is given in (23).

The equivalence between (23) and (24) suggests that the new upper bound can be computed using the LMI method. More precisely, we can diagonally stack up all the 2^{m_r} LMIs in (24) to form a single LMI which is linear in C and D_C. However, the use of (24) will lead to a large dimensional LMI problem because the dimension of $\hat{F}(C, D_C, \Delta_R)$ is substantially larger than that of $F(C, D_C, \Delta_R)$. Alternatively, we can significantly reduce the dimensions by introducing an additional variable matrix. This point is made precise in the following result.

Theorem 2 *The set of inequalities (23) hold if and only if there exists $K = K^H \in \mathcal{C}^{q\times q}$ such that the following $2^{m_r} + 1$ LMIs hold:*

$$F_1(C, K, \Delta_R) = E(C, \Delta_R) + K < 0, \quad \forall \delta_i^r = \pm a^{-1}, 1 \le i \le m_r \qquad (27)$$

$$F_2(C, K, D_C) = \begin{bmatrix} -K + A_C^H D_C A_C & C^H B_C^H \\ B_C C & -a^2 D_C \end{bmatrix} \le 0 \qquad (28)$$

Consequently, $\bar{\mu}_K(M)$ can be alternatively expressed by

$$\bar{\mu}_K(M) = \inf_{a>0}\{a : \exists C, K = K^H \in \mathcal{C}^{q\times q}, D_C \in \mathcal{D}_C : F_2(C, K, D_C) \le 0, F_1(C, K, \Delta_R) < 0, \forall \delta_i^r = \pm a^{-1}, 1 \le i \le m_r\} \qquad (29)$$

Proof: *We first note that the inequality in (28) can be alternatively expressed by*

$$K - A_C^H D_C A_C - a^{-2} C^H B_C^H D_C^{-1} B_C C \ge 0 \qquad (30)$$

Suppose the inequalities in (23) hold for some D_C. Then, we simply take

$$K = A_C^H D_C A_C + a^{-2} C^H B_C^H D_C^{-1} B_C C \qquad (31)$$

Obviously, both (27) and (30) hold.

Conversely, suppose (27) and (30) hold for some K. Then, (23) follows trivially too.

Remark 2 *Obviously, $F_1(\cdot)$ and $F_2(\cdot)$ are jointly linear in C, K and D_C. The collective dimension of the LMIs in (27)-(28) is substantially less than what obtained by using (24). The tradeoff is the additional variable matrix. Fortunately, this tradeoff is not very costly for many robust stability problems because the matrix is $q \times q$ only and it is Hermitian.*

3.3 The Third New Upper Bound

We now derive another new upper bound which is even looser than $\bar{\mu}_K(M)$. However, as we will see later, this new upper bounds serves two purposes: 1) It bridges the gap between $\bar{\mu}_K(M)$ and $\nu_K(M)$, i.e. it is in between the two; and 2) It is numerically more efficient than $\bar{\mu}_K(M)$ when the number of real perturbation blocks becomes large.

Our first step is to partition Δ_R into two parts:

$$\Delta_R = \mathrm{diag}\{\Delta_{R_1}, \Delta_{R_2}\} \qquad (32)$$

where Δ_{R_1} consists of the first m_1 repeated real blocks while Δ_{R_2}, the remaining $m_r - m_1$ ones. Also, $A_R, B_R, D_R, \mathcal{K}_R$ and $\mathcal{X}_R$ are partitioned similarly. In particular, we can rewrite $E(C, \Delta)$ as follows:

$$E(C, \Delta) = E(C, \Delta_{R_1}) - C^H B_{R_2}^H \Delta_{R_2}^H A_{R_2} - A_{R_2}^H \Delta_{R_2} B_{R_2} C - C^H B_C^H \Delta_C^H A_C - A_C^H \Delta_C B_C C \qquad (33)$$

where

$$E(C, \Delta_{R_1}) = C^H (I - A_{R_1}^H \Delta_{R_1} B_{R_1})^H + (I - A_{R_1}^H \Delta_{R_1} B_{R_1}) C \qquad (34)$$

The motivation for the partition above stems from the fact that the number of LMIs involved in computing $\bar{\mu}_K(M)$ is 2^{m_r} which is an exponential number. With this partition, we will reduce this number to 2^{m_1} by converting the Δ_{R_2} block into additional scaling variables D_{R_2} and G_{R_2} (which are a part of D_R and G for $\nu_K(M)$). The tradeoff between the number of LMIs and the additional variables will be discussed in Section 5.

With the partition above, we obtain the following result:

Theorem 3 *Let $F(C, D_C, \Delta_R)$ be given in (23) and $a > 0$. Then, $F(C, D_C, \Delta_R) < 0$ for all $\Delta_R \in \mathcal{X}_R$ with $\bar{\sigma}(\Delta) \le a^{-1}$ if there exist $D_{R_2} \in \mathcal{D}_{R_2}$ and $G_{R_2} \in \mathcal{G}_{R_2}$ such that the following LMIs hold:*

$$L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) = \begin{bmatrix} L_{11} & C^H B_C^H & L_{13} \\ B_C C & -a^2 D_C & 0 \\ L_{13}^H & 0 & -a^2 D_{R_2} \end{bmatrix} < 0, \qquad \forall \delta_i^r = \pm a^{-1}, i = 1, \cdots, m_1 \qquad (35)$$

where

$$L_{11} = E(C, \Delta_{R_1}) + A_C^H D_C A_C + A_{R_2}^H D_{R_2} A_{R_2} \qquad (36)$$

$$L_{13} = C^H B_{R_2}^H + j A_{R_2}^H G_{R_2} \qquad (37)$$

Further, the conditions in (35) are equivalent to the following: there exist $D_{R_2} \in \mathcal{D}_{R_2}$, $G_{R_2} \in \mathcal{G}_{R_2}$ and $K = K^H \in \mathcal{C}^{q\times q}$ such that the following LMIs hold:

$$L_1(C, K, \Delta_{R_1}) = E(C, \Delta_{R_1}) + K < 0, \quad \forall \delta_i^r = \pm a^{-1}, i = 1, \cdots, m_1 \qquad (38)$$

$$L_2(C, K, D_C, D_{R_2}, G_{R_2}) = \begin{bmatrix} -\hat{K} & C^H B_C^H & L_{13} \\ B_C C & -a^2 D_C & 0 \\ L_{13}^H & 0 & -a^2 D_{R_2} \end{bmatrix} \leq 0 \quad (39)$$

where

$$\hat{K} = K - A_C^H D_C A_C + A_{R_2}^H D_{R_2} A_{R_2} \quad (40)$$

Proof: *Suppose* $L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) < 0$ *for all* $\Delta_{R_1} \in \mathcal{X}_{R_1}$ *with* $\bar{\sigma}(\Delta_{R_1}) \leq a^{-1}$. *Define*

$$Y = \begin{bmatrix} I_q & 0 \\ 0 & I_q \\ -\Delta_{R_2} A_{R_2} & 0 \end{bmatrix} \quad (41)$$

It follows that

$$\begin{aligned} & Y^H L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) Y \\ &= F(C, D_C, \Delta_R) \\ &+ \mathrm{diag}\{A_{R_2}^H D_{R_2}^{1/2}(I - a^2 \Delta_{R_2}^H \Delta_{R_2}) D_{R_2}^{1/2} A_{R_2}, 0\} \end{aligned} \quad (42)$$

Therefore, $F(C, D_C, \Delta_R) < 0$ *for all* $\Delta_R \in \mathcal{K}_R$ *with* $\bar{\sigma}(\Delta_R) \leq a^{-1}$.

The proof for the equivalence between (35) and (38)-(39) is similar to the proof of Theorem 2.

Using the result in Theorem 3, we can define the third new upper bound as follows:

$$\begin{aligned} \tilde{\mu}_{\mathcal{K}}(M) = \inf_{a>0} \{ a : \exists C \in \mathcal{C}^{q\times q}, D_{R_2} \in \mathcal{D}_{R_2}, G_{R_2} \in \mathcal{G}_{R_2}, \\ D_C \in \mathcal{D}_C : L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) < 0, \\ \forall \delta_i^r = \pm a^{-1}, i = 1, \cdots, m_1 \} \end{aligned} \quad (43)$$

Alternatively,

$$\begin{aligned} \tilde{\mu}_{\mathcal{K}}(M) = \inf_{a>0} \{ a : \exists C, K = K^H \in \mathcal{C}^{q\times q}, D_{R_2} \in \mathcal{D}_{R_2}, \\ G_{R_2} \in \mathcal{G}_{R_2}, D_C \in \mathcal{D}_C : L_2(C, K, D_C, D_{R_2}, G_{R_2}) \leq 0; \\ L_1(C, K, \Delta_{R_1}) < 0, \ \forall \delta_i^r = \pm a^{-1}, i = 1, \cdots, m_1 \} \end{aligned} \quad (44)$$

4 Relationships of Upper Bounds

The relationships among the structured singular value $\mu_{\mathcal{K}}(M)$ and their upper bounds $\hat{\mu}_{\mathcal{K}}(M), \bar{\mu}_{\mathcal{K}}(M), \tilde{\mu}_{\mathcal{K}}(M)$ and $\nu_{\mathcal{K}}(M)$ are simply expressed in the following:

Theorem 4 *Given a matrix* $M \in \mathcal{C}^{n\times n}$ *of the form (20) and an associated block-structure* $\mathcal{K}$, *we have*

$$\mu_{\mathcal{K}}(M) \leq \hat{\mu}_{\mathcal{K}}(M) \leq \bar{\mu}_{\mathcal{K}}(M) \leq \tilde{\mu}_{\mathcal{K}}(M) \leq \nu_{\mathcal{K}}(M) \quad (45)$$

The proof of Theorem 4 is given in the full version of the paper [7]. Further, in a follow-up paper [9], we provide an example to demonstrate a gap between our upper bounds and $\nu_{\mathcal{K}}(M)$.

5 Computational Issue and Example

Now let us address the computational issue of $\bar{\mu}_{\mathcal{K}}(M)$ and $\tilde{\mu}_{\mathcal{K}}(M)$. For each given $a > 0$, the computation of the $\bar{\mu}_{\mathcal{K}}(M)$ boils down to the following: Find $C, K = K^H \in \mathcal{C}^{q\times q}$ and $D_C \in \mathcal{X}_C$ such that the LMIs (27)-(28) hold. Note that K is void if the complex or real perturbation blocks do not exist. Similarly, the computation of $\tilde{\mu}_{\mathcal{K}}(M)$ involves finding $C, K = K^H \in \mathcal{C}^{q\times q}$, $D_C \in \mathcal{D}_C$, $D_{R_2} \in \mathcal{D}_{R_2}$ and $G_{R_2} \in \mathcal{G}_{R_2}$ such that the LMIs (38)-(39) are satisfied. See [2, 8] for solutions to LMIs.

The computational complexity of the upper bounds above depends critically on the number of real perturbation blocks, i.e., m_r. An obvious numerical advantage of $\bar{\mu}_{\mathcal{K}}(M)$ over $\nu_{\mathcal{K}}(M)$ comes from the dimension reduction when q is substantially lower than n. Fortunately, this is often the case in robust stability and control problems. However, when there are many real parameters, the upper bound $\tilde{\mu}_{\mathcal{K}}(M)$ can be used. By choosing an appropriate partition of Δ_R, we can control the number of LMIs while keeping the size of additional variables D_{R_2} and G_{R_2} reasonably small.

When comparing the computational complexities of the different upper bounds, we should not forget that the tightness of them is perhaps more important. Although the upper bound $\nu_{\mathcal{K}}(M)$ may be numerically more efficient than $\bar{\mu}_{\mathcal{K}}(M)$ when the number of real perturbation blocks becomes very large, additional computation, such as branch and bound technique, is needed to refine $\nu_{\mathcal{K}}(M)$ so that it will reach the same tightness of $\bar{\mu}_{\mathcal{K}}(M)$.

In the rest of this section, we demonstrate an important robust stability example where $\bar{\mu}_{\mathcal{K}}(M)$ is numerically efficient. More examples are given in [7].

5.1 Robust Stability of A Polytope of Matrices

Given a polytope of matrices $\mathcal{A}$ of the following form:

$$\mathcal{A} = \{A_0 + \sum_{i=1}^{m} \delta_i A_i : |\delta_i| \leq 1, i = 1, \cdots, m\} \quad (46)$$

where $A_i \in \mathcal{C}^{q\times q}, i = 0, 1, \cdots, m$. It is well-known that $\mathcal{A}$ is robustly Hurwitz stable if and only if A_0 is Hurwitz stable and in addition, the following holds

$$\det(I - \sum_{i=1}^{m} \delta_i A_i (j\omega I_q - A_0)^{-1}) \neq 0, \ \ \forall |\delta_i| \leq 1 \quad (47)$$

is nonsingular for all $\omega \in \mathcal{R}$. Obviously, this problem is a special case of (11) with $B_i^H = (j\omega I_q - A_0)^{-1}$, $\mathcal{K} = (q, \cdots, q)$, $m_r = m$, $m_c = m_C = 0$.

Note that the associated matrix M is $qm \times qm$, a much larger one compared to A_i. The upper bounds

$\hat{\mu}_{\mathcal{K}}(M)$ and $\bar{\mu}_{\mathcal{K}}(M)$ are the same, and they are equivalent to the following: $\mathcal{A}$ is robustly Hurwitz stable if A_0 is Hurwitz stable and in addition, there exists some multiplier $C_\omega \in \mathcal{C}^{q\times q}$ for every $\omega \in \mathcal{R}$ such that

$$C_\omega^H(I - \sum_{i=1}^m \delta_i B_i^H A_i) + (I - \sum_{i=1}^m \delta_i B_i^H A_i)^H C_\omega > 0, \ \forall|\delta_i| \pm 1 \quad (48)$$

Equivalently, $\mathcal{A}$ is robustly Hurwitz stable if for every $\omega \in \mathcal{R}$, there exists some (different) $C_\omega \in \mathcal{C}^{q\times q}$ such that

$$(A_0 + \sum_{i=1}^m \delta_i A_i)^H C_\omega^H + C_\omega(A_0 + \sum_{i=1}^m \delta_i A_i) + j\omega(C_\omega^H - C_\omega) < 0, \ \forall \delta_i = \pm 1 \quad (49)$$

6 Conclusions

In this paper, we have provided several new upper bounds for the mixed structured singular value. Despite of the fact that these upper bounds are derived based on a very simple multiplier approach, we have shown that they are indeed lower bounds of an upper bound given in [5]. We must stress that these new upper bounds still appear to be very coarse. In other words, the news of this paper is somewhat disappointing as it points out that our knowledge about μ is still very primitive despite of years of research. It is a challenging task to find better upper bounds which are also convex and efficiently computable. The computational issue of μ remains wide open.

The type of uncertainty analyzed by the μ framework is somewhat restrictive, i.e, it must be of the structure (3) and ℓ_∞ norm-bounded. As we have shown that the multiplier approach can deal with a much larger class of uncertainty, namely, it allows any convex or even nonconvex set of uncertainty. This observation provides a simple connection between the absolute stability theory and μ theory. The connection lies in the use of a multiplier.

Finally, we point out that many more nice properties of the multiplier approach are reported in a follow-up paper [9]. In particular, a simple explicit relationship between our first new upper bound and (D, G)-scaling is provided; and the continuity issue of the upper bounds is addressed.

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