Automatica, Vol. 29, No. 6, pp. 1547–1550, 1993
Printed in Great Britain.

0005-1098/93 \$6.00 + 0.00
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Brief Paper

Achieving Diagonal Interactor Matrix for Multivariable Linear Systems with Uncertain Parameters*

PETER W. GIBBENS,† CARLA A. SCHWARTZ‡ and MINYUE FU†

Key Words—Adaptive control; interactor matrix; linear systems; robustness; vector relative degree.

Abstract—The notion of interactor matrix or equivalently the Hermite normal form, is a generalization of relative degree to multivariable systems, and is crucial in problems such as decoupling, inverse dynamics, and adaptive control. In order for a system to be input–output decoupled using static state feedback, the existence of a diagonal interactor matrix must first be established. For a multivariable linear system which does not have a diagonal interactor matrix, dynamic precompensation or dynamic state feedback is required for achieving a diagonal interactor matrix for the compensated system. Such precompensation often depends on the parameters of system, and is thus difficult to implement with accuracy when the system is subject to parameter uncertainty. In this paper we characterize a class of linear systems which can be precompensated to achieve a diagonal interactor matrix without the exact knowledge of the system parameters. More precisely, we present necessary and sufficient conditions on the transfer matrix of the system under which there exists a *diagonal* dynamic precompensator such that the compensated system has a diagonal interactor matrix. These conditions are associated with the so-called (non)generic singularity of certain matrix related to the system structure but independent of the system parameters. The result of this paper is expected to be useful in robust and adaptive designs.

1. *Introduction*

THE NOTION of interactor matrix is a generalization of relative degree to multivariable systems. It is defined for every system with a nonsingular square transfer matrix $T(s)$. More precisely, an interactor matrix $\xi_T(s)$ is a unique lower triangular polynomial matrix of certain structure which makes the product $\xi_T(s)T(s)$ bicausal; see Wolovich and Falb (1976). The importance of the interactor matrix lies in its necessity for many problems such as decoupling, inverse dynamics, and adaptive control; see Wolovich and Falb (1976), Morse and Wonham (1971), Descusse and Dion (1982), Hautus and Heymann (1983), Morse (1976), and Singh and Narendra (1984). Diagonal interactor matrices are of significant interest because they provide a solution to the static state feedback decoupling problem, among others. It is unfortunate that many systems do not have a diagonal interactor matrix. For this class of systems, a dynamic precompensator can be applied such that the resulting system has a diagonal interactor matrix. However, the design of such a precompensator usually requires the exact knowledge of the system parameters. Therefore, the robustness issue arises for this type of precompensator.

In this paper we are concerned with the task of achieving a diagonal interactor matrix by dynamic compensation for multivariable linear systems with uncertain parameters. We specifically consider the class of diagonal dynamics compensators comprised of integrators or linear filters of suitably selected order. We seek diagonal precomposition for the simplicity of its implementation, and provide a necessary and sufficient condition for the existence of such diagonal dynamic precompensation which achieves a diagonal interactor matrix. As we will see, one advantage of such a diagonal dynamic compensator is that it is related only to the structure of the system and is independent of the specific values of uncertain parameters. Hence, the compensated system and the resulting diagonal interactor matrix possess a certain degree of robustness against parameter variations in the system. This robustness property is important in problems such as adaptive control, as explained in Singh and Narendra (1984). In addition to characterizing its existence, we devise an algorithm to construct a diagonal dynamic precompensator which provides a diagonal interactor. A result of this analysis is that the question of diagonal interactor matrix, after possible diagonal precompensation, is essentially associated with the nonsingularity or (non)generic singularity of a certain matrix related to the system. The condition for achieving a diagonal interactor matrix given here is similar to a condition in Singh and Narendra (1984), but we provide a clear derivation and an algorithm for computing the precompensator.

2. *Preliminaries*

We begin by giving some preliminary definitions for a linear multivariable system modeled by an $m \times m$ rational transfer matrix $T(s)$.

Definition 2.1. A square transfer matrix $T(s)$ is called nonsingular if the determinant of the matrix exists and is nonzero for almost all finite complex numbers s. $T(s)$ is called bicausal if it is nonsingular and all entries of both $T(s)$ and $T^{-1}(s)$, the inverse of $T(s)$, are proper. A nonsingular system $y = T(s)u$ is called decoupled if and only if $T(s)$ is a diagonal matrix or is diagonalizable by an elementary row/column interchange operation.

Definition 2.2 (Wolovich and Falb, 1976). Given an $m \times m$ nonsingular system $y = T(s)u$, its diagonal interactor matrix is defined (when it exists), to be a diagonal polynomial matrix $\bar{D}(s)$ of the form

$$\bar{D}(s) = \mathrm{diag}\,\{s^{\bar{d}_i}\}, \quad \bar{d}_i \geq 0, \quad 1 \leq i \leq m \qquad (1)$$

such that the product $\bar{D}(s)T(s)$ is bicausal.

* Received 17 September 1991; revised 11 August 1992; revised 8 December 1992; received in final form 13 February 1993. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor R. V. Patel under the direction of Editor H. Kwakernaak. Corresponding author: M. Fu. Tel. (61) 49 216026; Fax (61) 49 601712.

† Department of Electrical and Computer Engineering, The University of Newcastle, N.S.W. 2308, Australia.

‡ Department of Computer Science and Electrical Engineering, University of Vermont, Burlington, VT 05405, U.S.A.

Remark 2.1. The indices $\bar{d}_i$ are referred to as decoupling indices (Falb and Wolovich, 1967) and the set $\{\bar{d}_1, \bar{d}_1, \ldots, \bar{d}_m\}$ is also called vector relative degree (Isidori, 1989). It should be noted that many multivariable linear systems do not have a diagonal interactor matrix. However, every system with a nonsingular transfer matrix $T(s)$ can be precompensated to achieve a diagonal interactor matrix. A simply constructed precompensator is $T^{-1}(s)s^{-\delta}$ for any integer δ sufficiently large to make the precompensator proper, assuming that we do not worry about possible unstable zero/pole cancellation between the system and the precompensator.

But such a precompensator requires the exact knowledge of $T(s)$, or the system parameters, and is therefore difficult to apply to systems with uncertain parameters. In particular, a small change in the coefficients of $T(s)$ may result in a precompensator of different structure.

The following definitions describe the concepts of generic and nongeneric singularity of transfer matrices. These concepts are also mentioned, but not precisely defined in (Singh and Narendra, 1984).

Definition 2.3. Linearly dependent row/column vectors $v_1, v_2, \ldots, v_n$ of the same dimension are called generically linearly dependent if for any other vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ of the same dimension such that v_i and $\bar{v}_i$ have the same zero elements (*i.e.* $\bar{v}_i$ is obtained by varying the nonzero elements of v_i), $i = 1, 2, \ldots, n$, the linear dependence property remains. Otherwise, the vectors are called nongenerically linearly dependent.

Remark 2.2. In other words, generic linear dependence occurs irrespective of specific values of the nonzero elements in a set of linearly dependent vectors. Definition 2.3 also implies that the linear dependence of a set of nongenerically linearly dependent vectors can be lost by arbitrarily small perturbations on the values of the nonzero elements of the vectors. In particular, the number of vectors must exceed one in order to have nongeneric linear dependence. Furthermore, a set of row (resp. column) vectors are generically linearly dependent if and only if either of the following cases happens:

(i) there is a zero row (resp. column); or
(ii) there exists a subset of $\{v_1, v_2, \ldots, v_n\}$ such that by forming them as a matrix, the number of nonzero columns (resp. rows) in the matrix is strictly less than the number of rows (resp. columns) in the matrix, *i.e.* the nonzero columns (resp. rows) form a 'tall' (resp. 'wide') submatrix.

Definition 2.4. A singular constant matrix is called nongenerically (resp. generically) singular if the singularity depends on (resp. is independent of) the particular values of the nonzero elements, *i.e.* all (resp. some of) its rows/columns are nongenerically (resp. generically) linearly dependent.

As an illustration of these ideas, consider the following two singular constant matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{2}$$

The first matrix is generically singular because the first two rows are generically linearly dependent. The second matrix is nongenerically singular because the second and third rows are nongenerically linearly dependent.

3. *Main results*

In this section, we consider the problem of finding a diagonal precompensator for a given system to achieve a diagonal interactor matrix. We seek diagonal precompensation for the simplicity of its implementation as well as robustness. Given an $m \times m$ nonsingular transfer matrix $T(s)$, we wish to define conditions under which there exists a diagonal dynamic precompensator $D(1/s)$ of the form

$$D(1/s) = \mathrm{diag}\,\{s^{-d_i}\}, \quad d_i \geq 0, \quad 1 \leq i \leq m \tag{3}$$

so that $T(s)D(1/s)$ has a diagonal interactor matrix, *i.e.* the rational matrix

$$K(s) := \bar{D}(s)T(s)D(1/s) \tag{4}$$

is a bicausal matrix for some polynomial matrix $\bar{D}(s)$ of the form (1).

It is possible however that for some transfer matrices, the aforementioned problem is not solvable. For example, for the following transfer matrix

$$T(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{-1}{s} \\ \dfrac{1}{s} + \dfrac{1}{s^2} & \dfrac{-1}{s} + \dfrac{1}{s^2} \end{bmatrix}, \tag{5}$$

it can be shown that no diagonal precompensator can achieve a diagonal interactor for $T(s)$ (see Example 3.2).

The main results of this paper are presented in Theorem 3.1 and Algorithm 3.2 below and are proven in Appendix A.

Theorem 3.1. Given a nonsingular transfer matrix $T(s)$, one of the following two cases must occur and they are mutually exclusive:

(a) There exists a pair $D(1/s)$ *and* $\bar{D}(s)$ of the forms (3) and (1), respectively, such that

$$K_0 := \lim_{s\to\infty} \{\bar{D}(s)T(s)D(1/s)\} \tag{6}$$

is nongenerically singular. In this case, there does not exist any other diagonal precompensator of the form (3) to achieve a diagonal interactor matrix.

(b) There exists a pair $D(1/s)$ and $\bar{D}(s)$ of the forms (3) and (1), respectively, such that K_0 in (6) is nonsingular. In this case, the compensated system $T(s)D(1/s)$ has a diagonal interactor $\bar{D}(s)$.

Based on this theorem, the following algorithm is developed for determining $\bar{D}(s)$ and $D(1/s)$ which ensures that K_0 is either nonsingular or nongenerically singular.

Algorithm 3.2. Initialize $D(1/s) = I$, *i.e.* $d_i = 0$, $\forall i$.

Step 1. Find $\bar{D}(s)$ with the minimal number of differentiations in each entry such that every row of $K(s) = \bar{D}(s)T(s)D(1/s)$ has an entry with zero relative degree, and take $K_0 = \lim_{s\to\infty} K(s)$. There are three cases:

1. K_0 is nongenerically singular: no diagonal compensator exists which will give a diagonal interactor.
2. K_0 is nonsingular: $D(1/s)$ is a diagonal precompensator for $T(s)$, and $\bar{D}(s)$ is the associated diagonal interactor.
3. K_0 is generically singular: proceed to Step 2. (Note that $\bar{D}(s)$ guarantees that K_0 has no zero rows.)

Step 2. Extract the maximum set i of rows for which the nonzero columns form a tall matrix. Denote the set of those nonzero columns by j and the set of remaining columns by $j^{\perp}$ for which all elements in the rows in set i are zero. Then determine γ, the minimum relative degree of any element contained in the set i of rows and the set $j^{\perp}$ of columns of $K(s)$. Then for all $l \in j$, increment d_l by γ. Return to step 1.

The algorithm is complete when either case 1 or case 2 occurs.

In case 2, $D(1/s)$ found is minimal order because we only introduce the minimum number of integrators required to remove generic singularities, and not more.

We now illustrate Algorithm 3.2 with two examples.

Example 3.1. Let

$$T(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{2}{s^3} & \dfrac{1}{s^4} \\ \dfrac{1}{s^2} & \dfrac{1}{s^2} & \dfrac{1}{s^2} \\ \dfrac{1}{s^2} & \dfrac{1}{s^4} & \dfrac{1}{s^6} \end{bmatrix}. \tag{7}$$

Initialize $D(1/s) = I$, *i.e.* $d_1 = d_2 = d_3 = 0$.

Iteration 1. In order to ensure that every row of $K(s)$ has some entry with zero relative degree, Step 1 gives $\bar{d}_1 = 1$,

$\bar{d}_2 = 2$ and $\bar{d}_3 = 2$ resulting in

$$K(s) = \begin{bmatrix} 1 & \frac{2}{s^2} & \frac{1}{s^3} \\ 1 & 1 & 1 \\ 1 & \frac{1}{s^2} & \frac{1}{s^4} \end{bmatrix}, \quad \text{with} \quad K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{8}$$

where K_0 is generically singular. We now apply Step 2 to $K(s)$. We find the set $i = \{1, 3\}$, *i.e.* rows 1 and 3 of K_0 are linearly dependent. We also find $j = \{1\}$ and accordingly $j^\perp = \{2, 3\}$. The minimum relative degree of any element of $K(s)$ which belongs to both sets i and $j^\perp$ is $\gamma = 2$. For all $l \in j$ we choose $d_l \triangleq d_l + \gamma$, which gives $d_1 = 2$, $d_2 = 0$ and $d_3 = 0$.

Iteration 2. We now return to Step 1 and formulate a new $K(s)$ with our new $D(1/s)$ by updating $\bar{d}_1 = 3$, $\bar{d}_2 = 2$ and $\bar{d}_3 = 4$ such that each row of the new $K(s)$ has some entry with zero relative degree.

$$K(s) = \begin{bmatrix} 1 & 2 & \frac{1}{s} \\ \frac{1}{s^2} & 1 & 1 \\ 1 & 1 & \frac{1}{s^2} \end{bmatrix}, \quad \text{with} \quad K_0 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{9}$$

Note that K_0 is now nonsingular, hence the new $\bar{D}(s)$ is the diagonal interactor for the precompensated system $T(s)D(1/s)$.

Example 3.2. Consider the matrix $T(s)$ in (5). Application of Step 1 of Algorithm 3.2 gives

$$K(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{-1}{s} \\ \frac{1}{s} + \frac{1}{s^2} & \frac{-1}{s} + \frac{1}{s^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 + \frac{1}{s} & -1 + \frac{1}{s} \end{bmatrix}. \tag{10}$$

As a result,

$$K_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

is nongenerically singular. By Theorem 3.1, there does not exist any diagonal dynamic precompensation which will give a diagonal interactor. Notice however that if any one of the constant coefficients of $T(s)$ in (5) were perturbed, then K_0 would be nonsingular, and the $\bar{D}(s)$ and $D(1/s)$ found would be the diagonal interactor and precompensator respectively which give a diagonal interactor matrix to the perturbed $T(s)$.

We end this section by considering the particular case of triangular transfer matrices.

Corollary 3.1. For any lower (or upper) triangular nonsingular proper transfer matrix $T(s)$, there exists a diagonal dynamic precompensator $D(1/s)$ of form (3) such that $T(s)D(1/s)$ has a diagonal interactor matrix.

Proof. Let $T(s)$ be any triangular transfer matrix. Using Theorem 3.1, there exist some $D(1/s)$ and $\bar{D}(s)$ of the form (3) and (1), respectively, such that

$$K_0 = \lim_{s\to\infty} \bar{D}(s)T(s)D(1/s) \tag{11}$$

is either nonsingular or nongenerically singular. Note that $K(s)$ is triangular and so is K_0. Therefore, K_0 must be nonsingular and $\bar{D}(s)$ is the diagonal interactor for $T(s)D(1/s)$. ∇∇∇

4. *Conclusion*

Necessary and sufficient conditions have been given to characterize those systems for which there exists a diagonal dynamic precompensation to achieve a diagonal interactor matrix. More specifically we have shown that the decouplability is determined by the generic/nongeneric singularity of a certain matrix associated with the system. In support of this cause, we have provided an algorithm for the construction of a minimal order diagonal dynamic precomposition required to achieve a diagonal interactor matrix.

Acknowledgements—The authors wish to thank Professor G. C. Goodwin for his inspiration and support. The work of the first author is supported by the Aeronautical Research Laboratories of the Defence Science and Technology Organisation, Australia; that of the second author was partially supported by NSF contract ECS-9111643, U.S.A.; and that of the third author is supported by the Australian Research Council. The authors also appreciate the constructive comments made by the reviews on this paper and its preversion (Gibbens *et al.*, 1991).

Appendix A: *Proof of Theorem* 3.1 *and Algorithm* 3.2

Proof. To solve Theorem 3.1, it is sufficient to show that given any nonsingular transfer matrix $T(s)$, cases 1 and 2 of Step 1 of Algorithm 3.2 are exclusive, and that the application of Algorithm 3.2 will result in either case 1 or case 2, *i.e.* we will end up with either a nonsingular or a nongenerically singular K_0.

In the following, part (i) implies the exclusiveness of the two possibilities, and part (ii) shows that one or the other can always be achieved.

(i) Suppose $\lim_{s\to\infty} \bar{D}(s)T(s)D(1/s)$ is nongenerically singular for some pair $\bar{D}(s)$ and $D(1/s)$, then for any other pair $\bar{D}_1(s)$ and $D_1(1/s)$, the matrix $\lim_{s\to\infty} \bar{D}_1(s)T(s)D_1(1/s)$ cannot be nonsingular. In other words, if $\lim_{s\to\infty} \bar{D}(s)T(s)D(1/s)$ is nonsingular, then there does not exist any other pair $\bar{D}_1(s)$ and $D_1(1/s)$ such that $\lim_{s\to\infty} \bar{D}_1(s)T(s)D_1(1/s)$ is nongenerically singular.

(ii) Given any $T(s)$, the application of Algorithm 3.2 will lead to a pair $\bar{D}(s)$ and $D(1/s)$ such that $\lim_{s\to\infty} \bar{D}(s)T(s)D(1/s)$ is either nonsingular or nongenerically singular.

For (i) it is sufficient to show the following:

(iii) Given a matrix $K(s)$ for which $\lim_{s\to\infty} K(s)$ is nongenerically singular, there do not exist any $\bar{D}_2(s)$ and $D_2(1/s)$ such that $\lim_{s\to\infty} \bar{D}_2(s)K(s)D_2(1/s)$ is nonsingular.

Indeed, suppose (iii) holds, let $K(s) = \bar{D}(s)T(s)D(1/s)$ and for any given $\bar{D}_1(s)$ and $D_1(1/s)$ define

$$D_2(1/s) = D(s)D_1(1/s)s^{-\delta}$$
$$\bar{D}_2(s) = s^{\delta}\bar{D}_1(s)\bar{D}(1/s)$$

where $\delta > 0$ is the minimum integer which keeps $D_2(1/s)$ and $\bar{D}_2(1/s)$ proper. Then

$$\bar{D}_1(s)T(s)D_1(1/s) = \bar{D}_2(s)K(s)D_2(1/s).$$

Hence, if $\lim_{s\to\infty} \bar{D}_2(s)K(s)D_2(1/s)$ cannot be nonsingular, then neither can $\lim_{s\to\infty} \bar{D}_1(s)T(s)D_1(1/s)$ for any $\bar{D}_1(s)$ and $D_1(1/s)$, which implies (i).

To show (iii) we suppose that $\lim_{s\to\infty} K(s)$ is nongenerically singular, and assume on the contrary that there exist matrices $\bar{D}_2(s)$ and $D_2(1/s)$ such that

$$\lim_{s\to\infty} \bar{D}_2(s)K(s)D_2(1/s) = \lim_{s\to\infty} \bar{K}(s)$$

is nonsingular. We denote $\bar{D}_2(s)K(s)D_2(1/s)$ by $K_2(s)$. Without loss of generality we assume that we can choose

$$D_2(1/s) = \begin{bmatrix} I_1 & 0 \\ 0 & D_{22}(1/s) \end{bmatrix}, \qquad \bar{D}_2(s) = \begin{bmatrix} I_2 & 0 \\ 0 & \bar{D}_{22}(s) \end{bmatrix}$$

where I_1, I_2 are identity matrices. This can be achieved by using row/column interchanges which place $K(s)$ in the following form,

$$K(s)=\begin{bmatrix}K_{11}(s) & K_{21}(s)\\ K_{21}(s) & K_{22}(s)\end{bmatrix} \tag{12}$$

where the dimensions of the submatrices correspond to those of $\bar{D}_{22}(s)$, $D_{22}(1/s)$ and I_1, I_2. We now assume that either (or both) I_1 or I_2, will have at least dimension one. Otherwise, replace $\bar{D}_2(s)$ and $D_2(1/s)$ by $\bar{D}_2(s)s^{-\epsilon}$ and $D_2(1/s)s^{\epsilon}$, where ϵ is the minimum order of the diagonal terms of $\bar{D}_2(s)$ and $D_2(s)$. Note that for the new $\bar{D}_2(s)$ and $D_2(1/s)$ either I_1 or I_2 will not be empty. With the given choice of $\bar{D}_2(s)$ and $D_2(1/s)$, we have

$$\bar{K}_2(s)=\begin{bmatrix}I_2 & 0\\ 0 & \bar{D}_{22}(s)\end{bmatrix}\begin{bmatrix}K_{11}(s) & K_{12}(s)\\ K_{21}(s) & K_{22}(s)\end{bmatrix}\begin{bmatrix}I_1 & 0\\ 0 & D_{22}(1/s)\end{bmatrix}$$
$$=\begin{bmatrix}K_{11}(s) & K_{12}(s)D_{22}(1/s)\\ \bar{D}_{22}(s)K_{21}(s) & \bar{D}_{22}(s)K_{22}(s)D_{22}(1/s)\end{bmatrix}.$$

In order for $\lim_{s\to\infty}\bar{K}_2(s)$ to be nonsingular then $\bar{D}_{22}(s)K_{21}(s)$ must be proper, which implies that $K_{21}(s)$ is strictly proper, that is $\lim_{s\to\infty}K_{21}(s)=0$. This implies that $K_{22}(s)$ cannot be a tall matrix, otherwise $\lim_{s\to\infty}K(s)$ would be generically singular (Remark 2.1). Similarly, $\lim_{s\to\infty}K_{12}(s)D_{22}(1/s)=0$ because $K_{12}(s)$ is proper, which implies that $K_{11}(s)$ cannot be tall, otherwise $\lim_{s\to\infty}\bar{K}_2(s)$ would be generically singular, thus contradicting the initial assumption that $K_2(s)$ is nonsingular. Therefore both $K_{11}(s)$ and $K_{22}(s)$ must be square matrices. This implies that I_1 and I_2 are of the same dimension.

Due to the nonsingularity of $\lim_{s\to\infty}K_2(s)$ and since $\lim_{s\to\infty}K_{12}(s)D_{22}(1/s)$ is zero, we know that $\lim_{s\to\infty}K_{11}(s)$ and $\lim_{s\to\infty}\bar{D}_{22}(s)K_{22}(s)D_{22}(1/s)$ are both nonsingular.

Using the nongeneric singularity of $\lim_{s\to\infty}K(s)$, nonsingularly of $\lim_{s\to\infty}K_{11}(s)$, and the fact that $\lim_{s\to\infty}K_{21}(s)=0$ then $\lim_{s\to\infty}K_{22}(s)$ must be nongenerically singular.

In conclusion, our initial assumption leads to two observations,

- $\lim_{s\to\infty}K_{22}(s)$ must be nongenerically singular,
- $\exists\bar{D}_{22}(s)$, $D_{22}(1/s)$ such that $\lim_{s\to\infty}\bar{D}_{22}(s)K_{22}(s)D_{22}(1/s)$ is nonsingular.

Noticing however that the dimension of $K_{22}(s)$ is lower than that of $K(s)$, the problem described in (iii) is repeated in a lower dimension, with $K_{22}(s)$, $\bar{D}_{22}(s)$, $D_{22}(1/s)$ replacing $K(s)$, $\bar{D}_2(s)$, $D_2(1/s)$. We can repeat the above argument until in the limit we have a 1×1 nongenerically singular matrix $K_{22}(s)$, which is not possible. This conclusion contradicts the initial assumption and therefore $\lim_{s\to\infty}K_2(s)$ cannot be made nonsingular by any choice of $\bar{D}_2(s)$, $D_2(1/s)$ if $\lim_{s\to\infty}K(s)$ is a nongenerically singular matrix.

To show (ii) we suppose that $D(1/s)$ is initialized to be I, $\bar{D}(s)$ is chosen according to Step 1 of Algorithm 3.2, and the resulting K_0 is generically singular. Without loss of generality, let the set i of rows and the set j of columns described in Step 2 be the first i rows and the first j columns of K_0. Then we can write $K(s)$ in the form (12), where $K_{11}(s)$ contains the first $i\times j$ elements of $K(s)$ and $K_{12}(s)$, $K_{21}(s)$, $K_{22}(s)$ are suitably dimensioned. Note that $K_{12}(s)$ is strictly proper, and $K_{11}(s)$ is tall by the definition of generic linear dependence. Also, the relative degree of every row of $K_{22}(s)$ is zero and $\lim_{s\to\infty}K_{22}(s)$ must be nongenerically singular [any row for which this is not the case should have been included in $K_{11}(s)$, $K_{12}(s)$].

Choose γ according to Step 2 of Algorithm 3.2 and formulate $D_1(1/s)=\mathrm{diag}\{I_1s^{-\gamma}, I_2\}$, where I_1, I_2 are identity matrices of dimensions $j\times j$ and $(m-j)\times(m-j)$ respectively. The new $D(1/s)$ matrix, denoted by $D^{\text{new}}(1/s)$, becomes $D^{\text{new}}(1/s)=D(1/s)D_1(1/s)$. The application of Step 1 of the algorithm will lead to $\bar{D}^{\text{new}}(s)=\bar{D}_1(1/s)\bar{D}(1/s)$, where $\bar{D}_1(s)=\mathrm{diag}\{s^{\gamma}I_1, I_2\}$. Here, the lower block of $\bar{D}_1(s)$ is an identity because the diagonal interactor matrix of each row of $K_{22}(s)$ is zero. The resulting $K(s)$ matrix becomes

$$K^{\text{new}}(s)=\bar{D}^{\text{new}}(s)T(s)D^{\text{new}}(1/s)$$
$$=\begin{bmatrix}K_{11}(s) & s^{\gamma}K_{12}(s)\\ K_{21}(s)s^{-\gamma} & K_{22}(s)\end{bmatrix}.$$

Note now that $K_{21}(s)s^{-\gamma}$ is strictly proper and at least one column of $s^{\gamma}K_{21}(s)$ has zero relative degree.

The new matrix $K^{\text{new}}(s)$, if its limit is still generically singular, can again be reorganized in the form (12), *i.e.*

$$K^{\text{new}}(s)=\begin{bmatrix}K_{11}^{\text{new}}(s) & K_{12}^{\text{new}}(s)\\ K_{21}^{\text{new}}(s) & K_{22}^{\text{new}}(s)\end{bmatrix}.$$

The conclusion that we make from the above is that the number of columns of $K_{12}^{\text{new}}(s)$ will be reduced by at least one.

We can apply the algorithm as many times as is required until the number of columns of $K_{12}^{\text{new}}(s)$ is reduced to zero, in which case $K^{\text{new}}(s)$ is either nonsingular or nongenerically singular. ∇∇∇

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