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Mean field stochastic linear quadratic games for continuum-parameterized multi-agent systems☆

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Abstract

In this paper, we consider a stochastic linear quadratic mean field game for the continuum-parameterized multi-agent systems with multiplicative noise. Based on the Nash certain equivalence principle, we obtain a series of decentralized control laws. Then, Dynkin's formula and comparison principle are employed to prove the boundedness of the state of the closed loop system in the mean square sense. Finally, we show that the set of decentralized controls has an ϵ-Nash equilibrium property.

1. Introduction

Recently, the control and optimization problems of multi-agent systems (MASs) have attracted a lot of attention due to its wide application background in many fields, such as engineering, economics, biology and communication networks [1–3]. In these areas, the optimization problem of MAS in the game theoretic framework, though is not investigated extensively, has catched the attention of many researchers. The areas investigated by the mean field approaches have been extended to game theory, economics and finance. Basar and Olsder [4] gave a good survey for noncooperative game models, which were widely

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used in the flow control and routing of networks [5,6]. Papers such as [7–13] investigated the stochastic differential games for large-population multi-agent systems (LPMSs) and gave ϵ-Nash equilibrium strategies by the Nash certainty equivalence principle methodology. This approach involves the following steps: Firstly, use the state aggregation method to approximate the population state average (PSA) by a deterministic signal v; Then, solve the tracking-like quadratic optimal control problem with the deterministic reference signal; At last, prove the existence and uniqueness of v, and construct the decentralized control law by using the unique solution v^*. Intuitively, v^* should possess the following property: if every agent views it as an approximation of the PSA, and according to which, makes the optimal decision, then the expectation of the closed-loop PSA should approach to v^* when the number of agents tends to infinity. This methodology for construction of the decentralized control law is called Nash certainty equivalence principle (NCEP) [14]. In addition, [15] expounded the mean field game and mean field type, and [16] discussed for minimizing a social cost.

As is well known, the geometric Brownian motion has many applications in physical sciences, biology and financial mathematics, such as the equations of the water heating and cooling model, the drug absorption model, the population model and the stock price model [17,18]. Motivated by these, we consider the linear quadratic (LQ) mean field game problem of continuum parameterized multi-agent systems with multiplicative noise, which is different from the previous models. We assume the dynamic systems of MASs are continuum parameterized, for example, [19] considered a sensor network applied to environment monitoring, where a large number of micro-senors are scattered randomly on a bounded monitoring area (for example, the parameter takes values in $[\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}]$). The position (x, y) for each sensor can be regarded as a realization of a random variable with uniform distribution $F(x, y)$. For another example, the population growths of different fish groups (the same race) located in the same sea area (e.g. $[\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [\underline{c}, \overline{c}]$) are different due to environmental differences, their relative rates of growth vary from place to place, see references [20,21] for more examples.

The rest of this paper is organized as follows. In Section 2, the mean field stochastic linear-quadratic games is formulated. In Section 3, stochastic LQ tracking problem with a known reference signal is studied, then the approximation of the PSA by state aggregation and the decentralized control laws based on NCEP are discussed. In Section 4, the asymptotic optimality of the decentralized control law is analyzed. In Section 5, a scalar model is calculated. In Section 6, we conclude our paper. Appendix contains the proofs of our propositions and lemmas.

The following notation will be used in the paper. $\|.\|$ denotes the Euclidean norm. I denotes the identity matrix with proper dimension. For a given matrix A, A' denotes its transpose. For any vector with proper dimensions and symmetric matrix $Q \geq 0$, $\|x\|_Q = (x'Qx)^{1/2}$. $C_n^b([0, \infty); \|\cdot\|_\infty)$ denotes the class of $\mathbb{R}^n$-valued continuous functions in $[0, \infty)$ with finite norm, the norm is defined by $\|f\|_\infty \triangleq \sup_{t\in[0,\infty)} \|f(t)\|$. In addition, $C_0, C_1, \ldots, C_k$, etc., are used to denote a series of positive constants which are independent of the number N of all agents.

2. Problem formulation

In this paper, we consider the following model:

$$dx_i(t) = [A(\theta_i)x_i(t) + Bu_i(t)]dt + \sum_{j=1}^{n_w}[D^j x_i(t) + E^j]dw_i^j(t), \tag{1}$$

where $i = 1, \dots, N$, and $x_i(t) \in \mathbb{R}^n$ is the state of agent $\mathcal{A}_i$, $u_i(t) \in \mathbb{R}^m$ is its control input, $\theta_i = (\theta_i^1, \dots, \theta_i^{n_\theta}) \in \Theta \subseteqq \mathbb{R}^{n_\theta}$ is its dynamic parameter vector. $A(\cdot) : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and D^j, E^j are $n \times n$ and $n \times 1$ dimensional matrix respectively, for each $j = 1, \dots, n_w$. Denote $w_i(t) \triangleq (w_i^1(t), \dots, w_i^{n_w}(t))'$, and $\{(w_i(t), \mathcal{F}_t^i), t \geq 0, i \geq 1\}$ is a sequence of independent n_w-dimensional standard Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{S}^N$ be the system comprised of N dynamic equations of (1). For each agent $i = 1, \dots, N$, the control set of agent $\mathcal{A}_i$ is defined by

$$\mathcal{U}_{loc,i} = \Big\{u_i \,\Big|\, u_i(t) \in \sigma(x_i(0), w_i(s);\ s \leq t), \quad \mathbb{E}\|x_i(T)\| = o(\sqrt{T}),$$
$$\mathbb{E}\int_0^T \|x_i(t)\|^2 dt = O(T), \quad T \to \infty\Big\}.$$

Here, a control group of $\mathbf{S}^N$ is $\mathcal{U}^N = \{u_1, \dots, u_N\}$, and the cost function of agent $\mathcal{A}_i$ has the coupled quadratic form

$$J_i(u_i, \mathbf{u}_{-i}) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \Big[\|x_i(t) - \gamma x^{(N)}(t)\|_Q^2 + \|u_i(t)\|_R^2\Big]dt, \tag{2}$$

where $Q > 0$ and $R > 0$. $\mathbf{u}_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ and $x^{(N)}(t) = (1/N)\sum_{j=1}^N x_j(t)$ is the population state average.

Remark 1. Regarding the model (1), see [8] for the case of the subset Θ being a single point set i.e., the dynamic equations of all agents are the same and [22] for the case of the subset Θ only containing finite points. In addition, suppose the state variable in the diffusion term disappears, our model will degenerate to [23].

The parameter vector θ_i in model (1) has the following property: $\{\theta_i,\ i \geq 1\}$ are independently sampled from the statistical structure $(\mathbb{R}^{n_\theta}, F(\theta))$, where $F(\cdot) : \mathbb{R}^{n_\theta} \to [0, 1]$ is a distributed function on the parameter vector space $\mathbb{R}^{n_\theta}$, called prior distribution. We can construct the empirical functions by $\{\theta_i,\ i \geq 1\}$, $F_N(\theta) \triangleq \frac{1}{N}\sum_{i=1}^N \chi_{\{\theta_i<\theta\}}$, $N \geq 1$, where $\theta = (\theta^1, \dots, \theta^{N_\theta})$, $\{\theta_i < \theta\} \triangleq \{\theta_i^1 < \theta^1, \dots, \theta_i^{N_\theta} < \theta^{N_\theta}\}$.

In the following, for the model considered, we need some assumptions.

Basic assumptions:

(A1). The support of $F(\cdot)$ denoted also by Θ is a compact set of $\mathbb{R}^{n_\theta}$.

(A2). $\{F_N,\ N \geq 1\}$ converge to F weakly [24], i.e., for any bounded and continuous function $h(\theta)$ on $\mathbb{R}^{n_\theta}$, $\lim_{N\to\infty}\int_\Theta h(\theta)dF_N(\theta) = \int_\Theta h(\theta)dF(\theta)$.

(A3). $A(\cdot)$ are is matrix-valued functions of $\theta \in \Theta$.

(A4). $\{x_i(0),\ 1 \leq i \leq N\}$ are independent and identically distributed stochastic variables and independent of $\sigma(w_i(t), t \geq 0, 1 \leq i \leq N)$ and satisfy $\mathbb{E}x_1(0) = x_0$, $\sup_{1\leq i\leq N}\|x_i(0) - x_0\|^2 < \infty$.

3. Construction of decentralized strategies

3.1. Stochastic LQ tracking problem with a known reference signal

In this subsection, we discuss the optimal tracking problem with a known reference signal for a stochastic LQ system.

Proposition 1. *Consider the following optimal control problem*

$$dx(t) = [Ax(t) + Bu(t)]dt + \sum_{j=1}^{n_w}[D^j x(t) + E^j]dw^j(t),$$

$$J(u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [\|x(t) - \gamma h(t)\|_Q^2 + \|u(t)\|_R^2]dt, \tag{3}$$

where $A, D^j \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ *and* $E^j \in \mathbb{R}^n$. $h \in \mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$ *is a known reference signal and* $(w^1(t), \ldots, w^{n_w}(t))'$ *is* n_w*-dimensional standard Brownian motion. The admissible control set is defined by*

$$\mathcal{U} = \Big\{ u | u(t) \in \sigma(x(0), w(s);\ s \le t),\quad \mathbb{E}\|x(T)\| = o(\sqrt{T}),$$
$$\mathbb{E}\int_0^T \|x(t)\|^2 dt = O(T),\quad T \to \infty \Big\}.$$

Assume $Q > 0$, $R > 0$, *then the following results hold:*

(a). The algebraic Riccati equation

$$A'P + PA - PBR^{-1}B'P + \sum_{j=1}^{n_w} D^{j'}PD^j + Q = 0$$

has a unique positive definite solution.

(b). All the eigenvalues of $G \triangleq A - BR^{-1}B'P$ *have negative real parts.*

(c). The backward linear ordinary differential equation

$$\dot{g}(t) + G'g(t) + \sum_{j=1}^{n_w} D^{j'}PE^j - \gamma Qh(t) = 0.$$

admits a unique solution in $\mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$

$$g(t) = \int_t^\infty \exp\{-G'(t-s)\}\left[\sum_{j=1}^{n_w} D^{j'}PE^j - \gamma Qh(s)\right]ds.$$

(d). The optimal control law $u^o = \arg\inf_{u\in\mathcal{U}} J(u)$ *is given by*

$$u^o(t) = -R^{-1}B'(Px(t) + g(t))$$

and the corresponding minimal cost value is

$$J(u^o) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T [\gamma^2 h'(t)Qh(t) - g'(t)BR^{-1}B'g(t)]dt$$
$$+ \sum_{j=1}^{n_w} trace(E^{j'}PE^j).$$

Proof. We put its proof into Appendix. □

Remark 2. Here, the condition $Q > 0$ is asked to conveniently verify the state process of the closed-loop of the continuum parameter multi-agent systems satisfy $\sup_{\theta\in\Theta}\mathbb{E}\int_0^\infty \|x_\theta(t)\|^2 dt < \infty$ and to show our Lemma 1 below. In addition, paper [25] discussed the case of $g(\cdot) \in L^2_{\mathcal{F}}(\mathbb{R}^n)$. Instead, we let the function $g(\cdot) \in \mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$, it seems that to verify the boundedness of a function is easier than to check its square integrability.

3.2. Approximation of the PSA based on the NCEP

Proposition 1 solves the stochastic LQ optimal tracking problem with a known reference signal. However, the reference signal $h(t)$ in the cost function is unknown and cannot be used directly to design control strategies. Thus, we first try to estimate the reference signal and then construct decentralized strategies by using Proposition 1 and the NCEP.

Denote $A_i = A(\theta_i)$ and in the cost function (2), if $x^{(N)}$ is replaced by some $v \in C_n^b([0, \infty); \|\cdot\|_\infty)$, then Eq. (2) changes into

$$J_i(u_i, \mathbf{u}_{-i}) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[\|x_i(t) - \gamma v(t)\|_Q^2 + \|u_i(t)\|_R^2\right]dt. \tag{4}$$

By Proposition 1, we may obtain the optimal control for agent $\mathcal{A}_i$

$$u_i^o(t) = -R^{-1}B'(P_i x_i(t) + g_i(t)), \tag{5}$$

where $P_i = P_\theta|_{\theta=\theta_i}$ and $g_i(t) = g_\theta(t)|_{\theta=\theta_i}$.

From the term (a) of Proposition 1, we have

$$A_i'P_i + P_iA_i - P_iBR^{-1}B'P_i + Q + \sum_{j=1}^{n_w} D^{j'}P_iD^j = 0. \tag{6}$$

Let $h(t) \equiv v(t)$. Proposition 1 also yields

$$g_i(t) = \int_t^\infty \exp\{-G_i'(t-s)\}\left[\sum_{j=1}^{n_w} D^{j'}P_iE^j - \gamma Qv(s)\right]ds, \tag{7}$$

where $G_i \triangleq A_i - BR^{-1}B'P_i$.

Substituting Eq. (5) into Eq. (1), we obtain the closed-loop equation for agent $\mathcal{A}_i$

$$dx_i(t) = [G_ix_i(t) - BR^{-1}B'g_i(t)]dt + \sum_{j=1}^{n_w}[D^jx_i(t) + E^j]dw_i^j(t). \tag{8}$$

Taking the integral on both sides of above equation, along with the Fubini's theorem, we have

$$\mathbb{E}x_i(t) = x_0 + \int_0^t [G_i\mathbb{E}x_i(s) - BR^{-1}B'g_i(s)]ds. \tag{9}$$

Taking the differential on both sides of Eq. (9) yields

$$d\mathbb{E}x_i(t) = [G_i\mathbb{E}x_i(t) - BR^{-1}B'g_i(t)]dt. \tag{10}$$

Now, we approximate the PSA by state aggregation. To do so, we construct an auxiliary system

$$\begin{cases} d\mathbb{E}x_\theta^v(t) = [G_\theta \mathbb{E}x_\theta^v(t) - BR^{-1}B'g_\theta(t)]dt, \\ \mathbb{E}x_\theta^v(0) = x_0, \\ g_\theta(t) = \int_t^\infty \exp\{-G_\theta'(t-s)\}\Big[\sum_{j=1}^{n_w} D^{j'}P_\theta E^j - \gamma Qv(s)\Big]ds, \\ v(t) = \int_\Theta \mathbb{E}x_\theta^v(t)dF(\theta), \end{cases} \tag{11}$$

where $G_\theta = A(\theta) - BR^{-1}B'P_\theta$.

This describes the limiting system of $\mathbf{S}^N$ when $N \to \infty$, which is a continuum of agents, each agent marked by a parameter vector θ.

By Eq. (11), we have

$$\mathbb{E}x_\theta^v(t) = \exp\{G_\theta t\}x_0 + \int_0^t \int_{s_1}^\infty \exp\{G_\theta(t-s_1)\}BR^{-1}B' \times \exp\{G_\theta'(s_2-s_1)\}\left[\gamma Qv(s_2) - \sum_{j=1}^{n_w} D^{j'}P_\theta(s_2)E^j\right]ds_2ds_1. \tag{12}$$

Define an operator $\mathcal{T}$ on $\mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$ as

$$(\mathcal{T}v)(t) \triangleq \int_\Theta \mathbb{E}x_\theta^v(t)dF(\theta), \tag{13}$$

where $\mathbb{E}x_\theta^v(t)$ is given by Eq. (12).

In the following, we want to prove the existence and uniqueness of v by Banach's fixed-point theorem. To do it, we need the following proposition and we put its proof into Appendix.

Under (A1)–(A4), it is easy to check from Eq. (12) that $(\mathcal{T}v)(t)$ is continuous in $t \in [0, \infty)$ and $\|(\mathcal{T}v)(t)\|_\infty < \infty$. Thus, $\mathcal{T}$ is indeed an operator from the Banach space $\mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$ to itself. From the definition of the operator $\mathcal{T}$, the auxiliary system (11) can be rewritten as

$$v = \mathcal{T}v. \tag{14}$$

The following theorem shows under some conditions, the linear operator $\mathcal{T}$ is a contraction operator on the Banach space $\mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$.

Theorem 1. *Under (A1)–(A4), if*

$$|\gamma|\|Q\|\|B\|^2\|R^{-1}\| \int_\Theta \left[\int_0^\infty \|\exp\{G_\theta t\}\|dt\right]^2 dF(\theta) < 1 \tag{15}$$

holds. Then there exists a unique solution v^* for Eq. (14).

Proof. For $v_1^*, v_2^* \in \mathcal{C}_n^b([0,\infty); \|\cdot\|_\infty)$, Eqs. (12) and (13) yield

$$\begin{aligned} \|(\mathcal{T}v_1^*)(t) - (\mathcal{T}v_2^*)(t)\| &= \Bigg\| \int_\Theta \int_0^t \int_{s_1}^\infty \exp\{G_\theta(t-s_1)\}BR^{-1}B' \\ &\quad \times \exp\{G_\theta'(s_2-s_1)\}\gamma Q[v_1^*(t) - v_2^*(t)]ds_2ds_1dF(\theta)\Bigg\| \\ &\le |\gamma|\|Q\|\|B\|^2\|R^{-1}\|\|v_1^*(t) - v_2^*(t)\|_\infty \\ &\quad \times \int_\Theta \left[\int_0^\infty \|\exp\{G_\theta t\}\|dt\right]^2 dF(\theta). \end{aligned}$$

The theorem holds by the contractive mapping theorem. □

Remark 3. Li and Zhang [26] discussed the linear quadric mean field game problem, where the multi-agent systems are additive noise. It appears that the form of the condition (15) is the same as theirs, except for P_θ in G_θ. In addition, suppose all eigenvalues of G have negative real part, [27] provides a kind of method to estimate $\|\exp\{Gt\}\|$. For detail, there are two real numbers r, $k>0$, such that $\|\exp\{Gt\}\| \le r\exp\{-kt\}$, where k satisfies that $k+\lambda_j<0$ for each eigenvalue λ_j of G. It is not hard to choose a real number $k>0$, satisfying the conditions above, in the case of the compact support set Θ containing only finite sample points, i.e., $\|\exp\{G_\theta t\}\| \le r\exp\{-kt\}$. In the case of continuum-parameter, the method of a sample space partition can be used to approximate Θ by finite sample points and this gives a numerical method. The theory method to calculate k is still open, we take it as our future work.

4. Asymptotic equilibrium analysis

4.1. Stability of closed-loop system

From the analysis and conclusion of the previous subsection, we obtain a series of decentralized strategies:

$$u_i^*(t) = -R^{-1}B'\Big(P_i x_i^*(t) + g_i^*(t)\Big), \quad 1 \le i \le N. \tag{16}$$

where P_i is the solution of Eq. (6) and $g_i^*(t)$ is given by Eq. (7), just substituting v with v^* in there.

Applying the control strategy Eq. (16) into Eq. (1), for $1\le i\le N$, we have the closed-loop system equation

$$dx_i^*(t) = [G_i x_i^*(t) - BR^{-1}B' g_i^*(t)]dt + \sum_{j=1}^{n_w}[D^j x_i^*(t) + E^j]dw_i^j(t). \tag{17}$$

In the following, we study the stability of the closed-loop system. Here, we use Dynkin's formula and the comparison principle to estimate it.

Lemma 1. *Suppose (A1)–(A4) hold, then there exists a constant C_1 independent N such that the strategies (16) and the corresponding closed-loop systems (17) satisfy*

$$\sup_{N\ge 1}\max_{1\le i\le N}\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T [\|x_i^*(t)\|^2 + \|u_i^*(t)\|^2]dt \le C_1.$$

Proof. From Eq. (7), it follows that

$$\begin{aligned}\|g_i^*(t)\|_\infty &\le \left(\sum_{j=1}^{n_w}(\|D^j\|\|E^j\|\|P_i\|) + |\gamma|\|Q\|\|v\|_\infty\right)\int_t^\infty \|\exp\{-G_i'(t-s)\}\|ds \\ &\le \left(\sum_{j=1}^{n_w}(\|D^j\|\|E^j\|M_P) + |\gamma|\|Q\|\|v\|_\infty\right)\int_t^\infty r\exp\{-\alpha(s-t)\}ds \triangleq M_g,\end{aligned}$$

where r, α are positive constants and $M_P = \sup_{\theta\in\Theta}\|P_\theta\|$. □

Letting $V(x) = x' P_i x$, Dynkin formula yields

$$\begin{aligned}\frac{d\mathbb{E}V(x_i^*(t))}{dt} &= \mathbb{E}\mathcal{L}^i V(x_i^*(t)) \\ &= \mathbb{E}\big[-x_i^*(t)' N_i x_i^*(t) - 2x_i^*(t)'\Lambda_i(t)\big] + \sum_j E^{j'} P_i E^j, \end{aligned} \tag{18}$$

where $\mathcal{L}^i$ is the infinitesimal generator of the diffusion process $x_i(\cdot)$. $N_i \triangleq Q + P_i B R^{-1} B' P_i$, $\Lambda_i(t) \triangleq P_i B R^{-1} B' g_i^*(t) - \sum_j D^{j'} P_i E^j$.

By Young's inequality, and for any $\lambda > 0$, we have

$$2\mathbb{E}[x_i^*(t)'\Lambda_i(t)] \le \frac{\lambda}{2}\mathbb{E}[x_i^*(t)' P_i x_i(t)] + \frac{2}{\lambda}\|P_i^{-\frac{1}{2}}\Lambda_i(t)\|^2. \tag{19}$$

From Eqs. (18) and (19), it follows that

$$\frac{d\mathbb{E}V(x_i^*(t))}{dt} \le -\frac{1}{2}\underline{\lambda}\mathbb{E}V(x_i^*(t)) + \Gamma + \sum_j M_P \|E^j\|^2,$$

where $\Gamma \triangleq 2M_P\|BR^{-1}B'\|M_g/\underline{\lambda} + \sum_j M_P \|D^{j'}E^j\|$ and $\underline{\lambda} = \inf_{\theta\in\Theta}\lambda_n(P_\theta^{-\frac{1}{2}} N_\theta P_\theta^{-\frac{1}{2}})$, $\lambda_n(\cdot)$ denotes the smallest eigenvalue of symmetric matrices.

From the comparison principle [28], we have

$$\mathbb{E}V(x_i^*(t)) \le \exp\left\{-\frac{1}{2}\underline{\lambda}\right\}\mathbb{E}V(x_i^*(0)) + \frac{2}{\underline{\lambda}}\Gamma\left(1 - \exp\left\{-\frac{\lambda}{2}t\right\}\right),$$

which together with the definition of $\mathbb{E}V(x_i^*(t))$ result in

$$\begin{aligned}\mathbb{E}\|x_i^*(t)\|^2 &\le \frac{\mathbb{E}V(x_i^*(t))}{\underline{\zeta}} \\ &= \exp\left\{-\frac{1}{2}\underline{\lambda}\right\}\frac{\mathbb{E}V(x_i^*(0))}{\underline{\zeta}} + \frac{2}{\underline{\lambda\zeta}}\Gamma\left(1 - \exp\left\{-\frac{\lambda}{2}t\right\}\right),\end{aligned} \tag{20}$$

where $\underline{\zeta} = \inf_{\theta\in\Theta}\lambda_n(P_\theta)$.

Then, from Eq. (20), we obtain

$$\begin{aligned}\|u_i^*(t)\|^2 &= \| - R^{-1}B'(P_i x_i^*(t) + g_i^*(t))\|^2 \\ &\le \|R^{-1}\|^2\|B\|^2(2M_P^2\|x_i^*(t)\|^2 + 2M_g^2).\end{aligned} \tag{21}$$

The conclusion is thus given by Eqs. (20) and (21).

Lemma 2. *Suppose (A1)–(A4) hold, then under the strategies Eq. (16), the closed-loop system (17) satisfies*

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \|\hat{x}^{(N)}(t) - v^*(t)\|^2 dt = O\left(\epsilon_N^2 + \frac{1}{N}\right).$$

where $\hat{x}^{(N)}(t) = (1/N)\sum_{i=1}^N x_i^*(t)$ *and* $\lim_{N\to\infty}\epsilon_N = 0$.

Proof. See Appendix. □

4.2. Analysis of optimality

In this section, we discuss the asymptotic equilibrium property of the associated decentralized control. To start, let us first introduce the definition of ϵ-Nash equilibrium. Let

$$\begin{aligned}&\mathcal{U}_{glo,i} = \{u_i | u_i(t) \in \sigma(x(0), w(s);\ \ s \le t), \quad \mathbb{E}\|x_i(T)\| = o(\sqrt{T}),\\ &\mathbb{E}\int_0^T \|x_i(t)\|^2 dt = O(T), \quad T \to \infty\}.\end{aligned}$$

Definition. Given a series of strategies $\{u_i \in \mathcal{U}_{loc,i},\ 1 \le i \le N\}$, if there exists $\epsilon \ge 0$ such that for any i, $1 \le i \le N$,

$$J_i(u_i, \mathbf{u}_{-i}) \le \inf_{u_i^\dagger \in \mathcal{U}_{glo,i}} J_i(u_i^\dagger, \mathbf{u}_{-i}) + \epsilon,$$

where $\mathbf{u}_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$, then we call this series of strategies an ϵ-Nash equilibrium with respect to the series of cost functions $\{J_i,\ 1 \le i \le N\}$.

Let

$$\begin{aligned}J_i(u_i^*, \mathbf{u}_{-i}^*) &= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \Big[\|x_i^*(t) - \gamma \hat{x}^{(N)}(t)\|_Q^2 + \|u_i^*(t)\|_R^2\Big] dt\bigg],\\ J_i(u_i^*, \gamma v^*) &= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \Big[\|x_i^*(t) - \gamma v^*(t)\|_Q^2 + \|u_i^*(t)\|_R^2\Big] dt\bigg],\\ J_i(u_i, \mathbf{u}_{-i}^*) &= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \Big[\Big\|x_i(t) - \frac{\gamma}{N}\Big(\sum_{j\ne i}^N x_j^*(t) + x_i(t)\Big)\Big\|_Q^2 + \|u_i(t)\|_R^2\Big] dt\bigg],\end{aligned} \tag{22}$$

where $\hat{x}^{(N)} = \frac{1}{N}\sum_{j=1}^N x_j^*$ and x_i refers to the closed-loop solution corresponding to some control $u_i \in \mathcal{U}_{glo,i}$.

To derive the asymptotic equilibrium,we need firstly to prove several lemmas, whose proofs are left to Appendix.

Lemma 3. *For the system (1) and the cost function (2), if (A1)–(A4) hold, then*

$$\begin{aligned}&\sup_{N\ge 1}\max_{1\le i\le N} J_i(u_i^*, \mathbf{u}_{-i}^*) \le C_2,\\ &\sup_{N\ge 1}\max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \Big\|(\gamma/N)\sum_{j\ne i}^N x_j^*(t)\Big\|_Q^2 dt \le C_3.\end{aligned}$$

Lemma 4. *For the system (1) and the cost function (2), if (A1)–(A4) hold, then*

$$J_i(u_i^*, \mathbf{u}_{-i}^*) \le J_i(u_i^*, \gamma v^*) + O\Big(\frac{1}{N} + \epsilon_N\Big).$$

Lemma 5. *For the system (1) and the cost function (2), if (A1)–(A4) hold, then*

$$J_i(u_i^*, \gamma v^*) \le \inf_{u_i \in \mathcal{U}_{glo,i}} J_i(u_i, \mathbf{u}_{-i}^*) + O\Big(\frac{1}{N} + \epsilon_N\Big).$$

Now we give the result of asymptotic optimality of the strategies (16).

Theorem 2. *For the system (1) and the cost function (2), if (A1)–(A4) hold, then the series of strategies* $\{u_i \in \mathcal{U}_{loc,i}, 1 \le i \le N\}$ *is an* ϵ*-Nash equilibrium with respect to the corresponding series of cost functions* $\{J_i,\ 1 \le i \le N\}$, *where* $\epsilon = O(\frac{1}{N} + \epsilon_N)$.

Proof. From Lemma 2, it follows that $\lim_{N\to\infty} \epsilon_N = 0$. By Lemmas 4 and 5, we have

$$J_i(u_i^*, \mathbf{u}_{-i}^*) \le \inf_{u_i \in \mathcal{U}_{glo,i}} J_i(u_i, \mathbf{u}_{-i}^*) + O\left(\frac{1}{N} + \epsilon_N\right).$$

Then, this theorem holds by the definition of ϵ-Nash equilibrium. □

5. The scalar model

Suppose that a pharmaceutical company plans to cultivate a batch of bacteria (such as penicillium) for drug production. For a large number of bacterial strains scattered randomly in the culture medium, the position for each bacterial strain can be regarded as a realization of a random variable with a uniform distribution. Besides, there exists a tiny difference of nutrient profile in different place of the culture medium, and this will lead to a tiny difference of the relative rates of growth among bacteria strains. Based on the statement above, let $\theta_i = (\theta_i^1, \theta_i^2) \in \Theta \triangleq [\underline{\theta}, \overline{\theta}] \times [\underline{\vartheta}, \overline{\vartheta}] \subseteq \mathbb{R}^2$, which is a closed rectangle. For the ith bacteria strain, its population growth model is described by Øksendal [18]

$$dx_i(t) = [a(\theta_i)x_i(t) + bu_i(t)]dt + \sigma x_i(t)dw_i(t), \tag{23}$$

where $x_i(t)$, $u_i(t)$ are its size of the population and the control at time t respectively. $a(\theta_i) + \sigma\widetilde{\omega}_i(t)dt$ is the relative rate of growth at time t, where $\widetilde{\omega}_i(\cdot)$ is the white noise corresponding with the Brownian motion $w_i(\cdot)$.

In order to achieve the average of overall population for each bacterial strain, the performance target of the ith bacteria strain is given by

$$J_i(u_i, \mathbf{u}_{-i}) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \Big[q[x_i(t) - \gamma x^{(N)}(t)]^2 + ru_i^2(t)\Big]dt,$$

where $q > 0$ is constant.

Following the notation in Section 3 and Eq. (16), we can get the optimal control and its closed-loop system of the ith bacteria strain

$$\begin{aligned} u_i^*(t) &= -\frac{b}{r}\Big[P_i x_i^*(t) + g_i^*(t)\Big], \\ dx_i^*(t) &= \left[k_i x_i^*(t) - \frac{b^2}{r} g_i^*(t)\right]dt + \sigma x_i^*(t)dw_i(t), \end{aligned}$$

where P_i and $g_i^*(t)$ are given by

$$\begin{aligned} &(2a_i + \sigma^2)P_i - \frac{b^2}{r}P_i^2 + q = 0, \\ &g_i^*(t) = -\gamma q \int_t^\infty \exp\{k_i(s-t)\}v^*(s)ds, \\ &k_i = a_i - \frac{b^2}{r}P_i. \end{aligned}$$

From the closed-loop system of the ith bacteria strain, we see that for one thing, the optimal control of the ith bacteria strain is used to adjust its relative rate of growth; for another, to track a signal involving the average level of all bacteria strain population.

Next, we deal with the algebraic Riccati equation and the fixed point equation. Taking the positive root of the Riccati equation above, we have

$$P_i = \frac{r(2a_i + \sigma^2) + \sqrt{r^2(2a_i + \sigma^2)^2 + 4qrb^2}}{2b^2}.$$

Under the condition

$$\frac{4\gamma qb^2}{\{r^{1/2}\sigma^2 + \sqrt{4qb^2}\}^2} < 1,$$

we can check that the contraction condition (15) holds, and $v^*(\cdot)$ satisfies the following fixed point integral-equation:

$$v^*(t) = \int_\Theta \int_0^t \int_{s_1}^\infty \exp\{k_\theta(t + s_2 - 2s_1)\} v^*(s_2) ds_2 ds_1 dF(\theta) + \int_\Theta \exp\{k_\theta t\} x_0 dF(\theta), \tag{24}$$

where $k_\theta = a_\theta - \frac{b^2}{r} P_\theta$.

Remark 4. In regard to the fixed point equation above, we ask the system-parameters to satisfy some statistical properties. For detail, we consider the case of the continuum-parameters and ask that the prior distributions weakly converge. Without this assumption, the mean field state is hard to calculate. Recently, [29,30] discussed the mean filed games of finite number of agents and provided other effective methods to avoid the fixed point problem. However, the computational complexity is high, once the number of agents is large.

Remark 5. Notice that the mean field term satisfies an integral-equation above under some contractive conditions. Comparing with the classic linear quadratic optimal tracking, the linear quadratic mean field control needs solving a fixed point integral equation in advance. In addition, the linear quadratic mean field control involves finding a decentralized control for each agent, and the decentralized controls are proved having an ϵ-Nash equilibrium property. However, the classic optimal control involves finding a centralized control which is optimal.

6. Conclusion

This paper studies mean field LQ games for continuum-parameterized multi-agent systems. The decentralized control problem for LPMS with coupled stochastic cost functions is investigated, including the control design and the closed-loop analysis. We show that the decentralized control laws designed is asymptotically optimal with respect to the coupled cost function.

Appendix A

A1. Proof of Proposition 1

Proof. The proof of (a) can be found in [31] or [25], and (b) holds by Theorem 3.7 of [31].

(c). The linear ordinary differential equation

$$\dot{g}(t)+G'g(t)+\sum_{j=1}^{n_w}D^{j'}PE^j-\gamma Qh(t)=0.$$

admits an explicit solution

$$g(t)=\exp\{-G't\}g(0)+\int_0^t\exp\{-G'(t-s)\}\left[\sum_j D^{j'}PE^j-\gamma Qh(s)\right]ds.$$

Since all the eigenvalues of G have negative real parts, there exist $r,\ k>0$ such that $\|\exp\{Gt\}\|\le r\exp\{-kt\}, t\ge 0$. Taking

$$g(0)=\int_0^\infty\exp\{G's\}\left[\gamma Qh(s)-\sum_j D^{j'}PE^j\right]ds,$$

and we get the formula of $g(t)$. To show $g(\cdot)\in\mathcal{C}_n^b([0,\infty);\|\cdot\|_\infty)$ is not difficult, thus we omit it. □

(d). Define the cost index of finite horizon

$$J^T(u)=\int_0^T[\|x(t)-\gamma h(t)\|_Q+\|u(t)\|_R]dt,$$

and we deal with the case of the 1-dimensional Brownian motion, the n_w-dimensional situation is similar.

By using Itô's formula, we have

$$\begin{aligned}x'(T)Px(T)-x'(0)Px(0)&=\int_0^T\Big[x'A'Px+u'BPx+x'PAx+x'PBu\\&\qquad+x'D'PDx+2x'D'PE+E'PE\Big]dt,\\ g'(T)x(T)-g'(0)x(0)&=\int_0^T\Big[-g'Ax+g'BR^{-1}B'Px-E'PDx+\gamma hQx+g'Ax+g')Bu\Big]dt.\end{aligned}$$

Via the algebraic Riccati equation and the ordinary differential equation $g(\cdot)$, we can check

$$\begin{aligned}J^T(u)=&\int_0^T\|u+R^{-1}B'(Px+g)\|_R^2+\gamma^2h'Qh-g'BRB'g\Big]dt\\&+E'PE+2g'(0)x(0)-2g'(T)x(T)+x'(0)Px(0)-x'(T)Px(T).\end{aligned}$$

The conclusion depends on the facts (whose proofs are similar with our Lemma 1)

$$\mathbb{E}\|x(T)\|=o(\sqrt{T}),\ \mathbb{E}\int_0^T\|x(t)\|^2dt=O(T),\quad T\to\infty. \tag{25}$$

Thus, (d) is proved.

A2. Proof of Lemma 2

By the integral form of Eqs. (10) and (11), we know

$$\mathbb{E}x_i(t) = \mathbb{E}x_{\theta_i}(t) = \mathbb{E}x_\theta(t)|_{\theta=\theta_i}. \tag{26}$$

Via the definition of $F_N(\theta)$ and Eq. (26), and substituting v with v^*, we have

$$(1/N)\sum_{i=1}^{N}\mathbb{E}x_i^*(t) = \int_\Theta \mathbb{E}x_\theta^*(t)dF_N(\theta). \tag{27}$$

Thus

$$\begin{aligned}
\mathbb{E}\left\|(1/N)\sum_{i=1}^{N}x_i^* - v^*\right\|^2 &\le 2\mathbb{E}\left\|(1/N)\sum_{i=1}^{N}x_i^*(t) - (1/N)\sum_{i=1}^{N}\mathbb{E}x_i^*(t)\right\|^2 \\
&\quad +2\left\|(1/N)\sum_{i=1}^{N}\mathbb{E}x_i^*(t) - v^*\right\|^2 \\
&= 2\mathbb{E}\left\|(1/N)\sum_{i=1}^{N}x_i^*(t) - (1/N)\sum_{i=1}^{N}\mathbb{E}x_i^*(t)\right\|^2 \\
&\quad +2\left\|\int_\Theta \mathbb{E}x_\theta^*(t)dF_N(\theta) - \int_\Theta \mathbb{E}x_\theta^*(t)dF(\theta)\right\|^2.
\end{aligned} \tag{28}$$

Noticing that $\{x_i^*, i = 1, \ldots, N\}$ are independent with each other, which along with Lemma 1, yield

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N}x_i^*(t) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}x_i^*(t)\right\|^2 &\le \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\|x_i^*(t) - \mathbb{E}x_i^*(t)\|^2 \\
&\le \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\|x_i^*(t)\|^2 \\
&\le \frac{C_5}{N}.
\end{aligned} \tag{29}$$

Let

$$\begin{aligned}
\Gamma_N(t) &\triangleq \int_\Theta \mathbb{E}x_\theta^v(t)dF_N(\theta), \\
\Gamma(t) &\triangleq \int_\Theta \mathbb{E}x_\theta^v(t)dF(\theta), \\
\epsilon_N &\triangleq \sup_{t\in[0,\infty)} \|\Gamma_N(t) - \Gamma(t)\|.
\end{aligned}$$

From Eq. (12), we can check the following statements hold:

$$\begin{aligned}
&\sup_{t\in[0,\infty)}\sup_{\theta\in\Theta}\|\mathbb{E}x_\theta^v(t)\| < \infty, \\
&\sup_{t\in[0,\infty)}\sup_{\theta_1,\theta_2\in\Theta}\|\mathbb{E}x_{\theta_1}^v(t) - \mathbb{E}x_{\theta_2}^v(t)\| < L\|\theta_1 - \theta_2\|,
\end{aligned}$$

where L is a positive constant which is independent of the parameters t, θ_1, θ_2.

Thus, under the basic Assumption (A2), we have

$$\lim_{N\to\infty}\sup_{t\in[0,\infty)}\|\Gamma_N(t)-\Gamma(t)\|=0.$$

So the lemma holds.

A3. Proof of Lemma 3

We only prove the first inequality, another is similar with the first one. From Eq. (20), we have

$$\begin{aligned}J_i(u_i^*,\mathbf{u}_{-i}^*)&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left[\|x_i^*(t)-\gamma\hat{x}^{(N)}(t)\|_Q^2+\|u_i^*(t)\|_R^2\right]dt\\&\le\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left[\|Q\|\|x_i^*(t)-\gamma\hat{x}^{(N)}(t)\|^2+\|R\|\|u_i^*(t)\|^2\right]dt\\&\le\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left[2\|Q\|(\|x_i^*(t)\|^2+\|\gamma\hat{x}^{(N)}(t)\|^2)+\|R\|\|u_i^*(t)\|^2\right]dt.\end{aligned}$$

Then the first inequality of this lemma holds by Lemma 1.

A4. Proof of Lemma 4

Because

$$\begin{aligned}J_i(u_i^*,\mathbf{u}_{-i}^*)&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left[\|x_i^*(t)-\gamma v^*(t)+\gamma v^*(t)-\gamma\hat{x}^{(N)}(t)\|_Q^2+\|u_i^*(t)\|_R^2\right]dt\\&=J_i(u_i^*,\gamma v^*)+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\Big[\|\gamma v^*(t)-\gamma\hat{x}^{(N)}(t)\|_Q^2+2(x_i^*(t)-\gamma v^*(t))'\\&\quad\times Q(\gamma v^*(t)-\gamma\hat{x}^{(N)}(t))\Big]dt,\end{aligned}$$

then the lemma holds by Lemmas 2 and 3.

A5. Proof of Lemma 5

Since $\inf_{u_i\in\mathcal{U}_{glo,i}}J_i(u_i,\mathbf{u}_{-i}^*)\le J_i(u_i^*,\mathbf{u}_{-i}^*)$, thus we only consider such $u_i\in\mathcal{U}_{glo,i}$ that satisfies $J_i(u_i,\mathbf{u}_{-i}^*)\le J_i(u_i^*,\mathbf{u}_{-i}^*)$. From Lemma 3, together with noticing that

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left\|\left(1-\frac{\gamma}{N}\right)x_i(t)-\frac{\gamma}{N}\sum_{j\neq i}^N x_j^*(t)\right\|_Q^2dt\le J_i(u_i,\mathbf{u}_{-i}^*)\le C_6,$$

it follows that

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left\|\left(1-\frac{\gamma}{N}\right)x_i(t)\right\|_Q^2dt\\&\quad\le\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\left\|\left(1-\frac{\gamma}{N}\right)x_i(t)-\frac{\gamma}{N}\sum_{j\neq i}^N x_j^*(t)\right\|_Q^2dt\end{aligned}$$

$$+2\mathbb{E}\int_0^T \left\|\frac{\gamma}{N}\sum_{j\neq i}^{N} x_j^*(t)\right\|_Q^2 dt \leq C_7.$$

From the following inequality,

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \left\|\left(1-\frac{\gamma}{N}\right)x_i(t)\right\|_Q^2 dt = \left(1-\frac{\gamma}{N}\right)^2 \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \|x_i(t)\|_Q^2 dt$$
$$\geq \left(1-2\frac{\gamma}{N}\right)\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \|x_i(t)\|_Q^2 dt,$$

and letting $N\to\infty$, we have

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \|x_i(t)\|_Q^2 dt \leq C_8. \tag{30}$$

Notice that

$$\begin{aligned} J_i(u_i,\mathbf{u}_{-i}^*) &= \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T \left[\|x_i(t)-\frac{\gamma}{N}\left(\sum_{j\neq i}^{N} x_j^*(t)+x_i(t)\right)\|_Q^2 + \|u_i(t)\|_R^2\right]dt \\ &\geq J_i(u_i^*,\gamma v^*) + \liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T 2\gamma\Bigg[v^*(t)-\hat{x}^{(N)}(t)+\hat{x}^{(N)}(t) \\ &\quad -\frac{1}{N}\left(\sum_{j\neq i}^{N} x_j^*(t)+x_i(t)\right)\Bigg]' Q(x_i(t)-\gamma v^*(t))dt \\ &= J_i(u_i^*,\gamma v^*) + I_1^N + I_2^N. \end{aligned}$$

where

$$I_1^N = \liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T 2\gamma[v^*(t)-\hat{x}^{(N)}(t)]'Q(x_i(t)-\gamma v^*(t))dt,$$
$$I_2^N = \liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T 2\gamma\left[\frac{1}{N}(x_j^*(t)-x_j(t)\right]' Q(x_i(t)-\gamma v^*(t))dt.$$

From Lemma 2, we may obtain

$$\max\{|I_1^N|,|I_2^N|\} = O\left(\frac{1}{N}+\epsilon_N\right).$$

Thus the lemma is true.

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