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Robustness Bounds of Hurwitz and Schur Polynomials

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Abstract. In this paper, the problem of robustness bounds of Hurwitz and Schur polynomials is addressed. For weighted L_2-norm perturbations of a Hurwitz polynomial $p(s)$ or a Schur polynomial $p(z)$, a new method is developed for calculating the maximal perturbation bound under which stability is preserved. We show that such a robustness bound is related to the minimum of a rational function. The new method is superior to the previous one developed by Soh, Berger, and Dabke in Ref. 1. Our approach also provides solutions for the perturbation polynomial $\delta p(s)$ or $\delta p(z)$ with minimal coefficient norm which cause $p(s)+\delta p(s)$ or $p(z)+\delta p(z)$ to be unstable.

Key Words. Robust stability, robustness, optimization, uncertain systems.

1. Introduction

Robustness considerations play an important role in control system designs; see Refs. 1-5 and their bibliographies. A number of techniques have been developed recently for estimating robustness bounds of Hurwitz and Schur polynomials. For a Hurwitz polynomial, Barmish (Ref. 2), using a theorem of Kharitonov (Ref. 3), showed that the maximal weighted intervals of the coefficients which can be perturbed while preserving the Hurwitz property are determined by the maximal stability bounds of four extreme polynomials with unidirectional perturbations in the coefficients. A closed-form solution of the maximal allowable perturbations for these polynomials is given recently by Fu and Barmish (Ref. 4). The type of perturbations considered in Ref. 2 is referred to as L_1-norm perturbations. However, the result in Ref. 2 is not applicable to Schur polynomials. In

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order to handle both continuous-time and discrete-time systems, Soh, Berger, and Dabke (Ref. 1) introduced L_2-norm perturbations. They showed that the maximal L_2-norm perturbation bound of a Hurwitz or Schur polynomial is associated with the minimum value of a function in a quadratic form. However, the evaluation of such a function involves numerical computation of the pseudo-inverse of a matrix function. In the case of continuous polynomials, a simpler solution is provided by Chapellat and Bhattacharyya in their recent paper (Ref. 5). It is shown in Ref. 5 that the maximal L_2-norm perturbation bound of a Hurwitz polynomial can be calculated by minimizing a rational function associated with the coefficients of the polynomial.

From the point of view of robust control design, we may not only be interested in how to calculate robustness bounds, but also want to see whether the technique used can give indications of how to improve the robustness. In particular, we would like to know the direction(s) of perturbation which easily lead to instability.

Based on the motivations above, we reformulate the problem studied in Ref. 1 and show that the same robustness bound of a given stable polynomial can be related to the minimum value of a rational function. Consequently, the computation is much simplified. For continuous polynomials, our result is similar to the one in Ref. 5, except that we use a monic polynomial while in Ref. 5 the leading coefficient of the polynomial is forced to have perturbations. Noticing that the coefficients of the polynomial may be unevenly perturbed in practical applications, we allow a weighted L_2-norm in order to reduce conservatism. Moreover, we also describe the minimal destabilizing polynomial(s) to the nominal stable polynomial. These minimal destabilizing polynomials may be useful in adjusting the feedback controller to improve the robustness of the system. After introducting notation and preliminaries in Section 2, we provide the main results in Section 3 and an illustrating example in Section 4.

2. Notation and Preliminaries

In this section, we first define the maximal weighted L_2-norm perturbation bound and the associated minimal destabilizing polynomial(s) for a stable polynomial. Then, the problems of finding the maximal perturbation bound and the minimal destabilizing perturbation polynomials will be converted to a set of quadratic minimization problems with linear constraints. These reformulated problems will be solved in Section 3.

Throughout this paper, every polynomial is assumed to be real and its order is at least one. The coefficient vector of nth order polynomial

$$p(\xi) = a_0 + a_1\xi + \cdots + a_{n-1}\xi^{n-1} + \xi^n \tag{1}$$

is denoted by

$$a=[a_0, a_1, \ldots, a_{n-1}]^T. \tag{2}$$

The perturbed polynomial takes the form $p(\xi)+\delta p(\xi)$, where

$$\delta p(\xi)=\delta a_0+\delta a_1\xi+\cdots+\delta a_{n-1}\xi^{n-1}, \tag{3}$$

with its coefficient vector given by

$$\delta a=[\delta a_0, \delta a_1, \ldots, \delta a_{n-1}]^T. \tag{4}$$

Definition 2.1. Consider a given nth order strictly Hurwitz polynomial

$$p(s)=a_0+a_1 s+\cdots+a_{n-1}s^{n-1}+s^n \tag{5}$$

and a weighting matrix

$$\Gamma=\operatorname{diag}\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}, \qquad \gamma_i>0,\ i=0,1,\ldots,n-1. \tag{6}$$

The maximal weighted L_2-norm perturbation bound for $p(s)$ is defined by

$$d_c \doteq \min\{\sqrt{(\delta a^T\Gamma\delta a)}: p(s)+\delta p(s) \text{ is not strictly Hurwitz}\}. \tag{7}$$

Furthermore, $\delta p(s)$ is called a minimal destabilizing polynomial (MDP) if

$$\delta a^T\Gamma\delta a=d_c^2 \tag{8}$$

and $p(s)+\delta p(s)$ is not strictly Hurwitx.[2]

Definition 2.2. Consider a given nth order strictly Schur polynomial

$$p(z)=a_0+a_1 z+\cdots+a_{n-1}z^{n-1}+z^n \tag{9}$$

and a weighted matrix Γ as in (6). The maximal weighted L_2-norm perturbation bound for $p(z)$ is defined by

$$d_d \doteq \min\{\sqrt{(\delta a^T\Gamma\delta a)}: p(z)+\delta p(z) \text{ is not strictly Schur}\}. \tag{10}$$

Furthermore, $\delta p(z)$ is called a minimal destabilizing polynomial (MDP) if

$$\delta a^T\Gamma\delta a=d_d^2 \tag{11}$$

and $p(z)+\delta p(z)$ is not strictly Schur.

In order to calculate the maximal perturbation bounds for $p(s)$ and $p(z)$ and their associated MDPs, we first notice the following property of an MDP: if $\delta p(s)$ is an MDP of $p(s)$, then $p(s)+\delta p(s)$ has at least one zero on the imaginary axis; i.e., there exists some $\omega\geq 0$ such that

$$p(j\omega)+\delta p(j\omega)=0.$$

[2] It should be noted that there may be more than one MDP.

Similarly, if $\delta p(z)$ is an MDP of $p(z)$, then there exists some $0 \leq \theta \leq \pi$ such that

$$p(e^{j\theta}) + \delta p(e^{j\theta}) = 0.$$

Therefore, we conclude that

$$d_c = \min_{\delta a \in \mathbb{R}^n, \omega \geq 0} \{\sqrt{(\delta a^T \Gamma \delta a)}: p(j\omega) + \delta p(j\omega) = 0\}, \tag{12}$$

$$d_d = \min_{\delta a \in \mathbb{R}^n, 0 \leq \theta \leq \pi} \{\sqrt{(\delta a^T \Gamma \delta a)}: p(e^{j\theta}) + \delta p(e^{j\theta}) = 0\}. \tag{13}$$

Taking (12) one step further by separating $\omega = 0$ and $\omega > 0$, we obtain

$$d_c^2 = \min\{d_{c_1}^2, d_{c_2}^2\}, \tag{14}$$

where

$$d_{c_1}^2 = \min_{\delta a \in \mathbb{R}^n} \{\delta a^T \Gamma \delta a: a_0 + \delta a_0 = 0\}, \tag{15}$$

$$d_{c_2}^2 = \inf_{\delta a \in \mathbb{R}^n, \omega > 0} \{\delta a^T \Gamma \delta a: p(j\omega) + \delta p(j\omega) = 0\}. \tag{16}$$

Notice that, for any nth order polynomial $p(s)$ in the form of (5), we have

$$p(j\omega) = (j\omega)^n + \sum_{i=0}^{r} a_{2i}(-\omega^2)^i + j\omega \sum_{i=0}^{s} a_{2i+1}(-\omega^2)^i, \tag{17}$$

where r and s are the integer parts of $(n-1)/2$ and $(n-2)/2$, respectively; i.e.,

$$r = \left[\frac{n-1}{2}\right], \qquad s = \left[\frac{n-2}{2}\right]. \tag{18}$$

Now, let

$$\Omega_c(\omega) = \begin{bmatrix} 1 & -\omega & \omega^2 & \cdots & (-\omega)^r & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\omega & \omega^2 & \cdots & (-\omega)^s \end{bmatrix}, \tag{19}$$

$$\psi_c(\omega) = \begin{cases} \begin{bmatrix} (-\omega)^{n/2} \\ 0 \end{bmatrix}, & \text{if } n \text{ is even}, \\ \begin{bmatrix} 0 \\ (-\omega)^{(n-1)/2} \end{bmatrix}, & \text{if } n \text{ is odd}, \end{cases} \tag{20}$$

$$\alpha = [a_0, a_2, \ldots, a_{2r}, a_1, a_2, \ldots, a_{2s+1}]^T, \tag{21}$$

$$\delta\alpha = [\delta a_0, \delta a_2, \ldots, \delta a_{2r}, \delta a_1, \delta a_3, \ldots, \delta a_{2s+1}]^T. \tag{22}$$

It is straightforward to verify that, for $\omega \neq 0$,

$$p(j\omega) + \delta p(j\omega) = 0$$

if and only if

$$\Omega_c(\omega^2)(\alpha+\delta\alpha)+\psi_c(\omega^2)=0. \tag{23}$$

In fact, the first row of the vector in (23) corresponds to the real part of $p(j\omega)+\delta p(j\omega)$, and the second part corresponds to the imaginary part divided by ω. Now, substituting (23) and (21) into (16) and replacing ω^2 by ω, we have

$$d_{c_2}^2=\inf_{\delta\alpha\in\mathbf{R}^n,\omega>0}\{\delta\alpha^T\bar{\Gamma}\delta\alpha\colon \Omega_c(\omega)(\alpha+\delta\alpha)+\psi_c(\omega)=0\}, \tag{24}$$

where

$$\bar{\Gamma}=\operatorname{diag}\{\gamma_0,\gamma_2,\ldots,\gamma_{2r},\gamma_1,\gamma_3,\ldots,\gamma_{2s+1}\}. \tag{25}$$

Similar analysis for (13) yields

$$d_d^2=\min\{d_{d_1}^2, d_{d_2}^2, d_{d_3}^2\}, \tag{26}$$

where d_{d_1}, d_{d_2}, d_{d_3} are obtained from (13) by considering three cases: $\theta=0$, $0<\theta<\pi$, and $\theta=\pi$. That is,

$$d_{d_1}^2=\min_{\delta a\in\mathbf{R}^n}\left\{\delta a^T\Gamma\delta a\colon 1+\sum_{i=0}^{n-1}a_i+\sum_{i=0}^{n-1}\delta a_i=0\right\}, \tag{27}$$

$$d_{d_2}^2=\min_{\delta a\in\mathbf{R}^n}\left\{\delta a^T\Gamma\delta a\colon (-1)^n+\sum_{i=0}^{n-1}(-1)^i a_i+\sum_{i=0}^{n-1}(-1)^i\delta a_i=0\right\}, \tag{28}$$

$$d_{d_3}^2=\inf_{\delta a\in\mathbf{R}^n,0<\theta<\pi}\{\delta a^T\Gamma\delta a\colon p(e^{j\theta})+\delta p(e^{j\theta})=0\}. \tag{29}$$

Defining

$$\Omega_d(\theta)=\begin{bmatrix}1 & \cos\theta & \cos(2\theta)\ldots\cos((n-1)\theta)\\ 0 & \sin\theta & \sin(2\theta)\ldots\sin((n-1)\theta)\end{bmatrix}, \tag{30}$$

$$\psi_d(\theta)=\begin{bmatrix}\cos(n\theta)\\ \sin(n\theta)\end{bmatrix}, \tag{31}$$

it is straightforward to verify that

$$p(e^{j\theta})+\delta p(e^{j\theta})=0$$

if and only if

$$\Omega_d(\theta)(a+\delta a)+\psi_d(\theta)=0. \tag{32}$$

Therefore, we can rewrite (29) as follows:

$$d_{d_3}^2=\inf_{\delta a\in\mathbf{R}^n,0<\theta<\pi}\{\delta a^T\Gamma\delta a\colon \Omega_d(\theta)(a+\delta a)+\psi_d(\theta)=0\}. \tag{33}$$

3. Main Results

In this section, we provide the main results of this paper by solving the problems associated with (15), (24), (27), (28), and (33). Theorems 3.2 and 3.3 give solutions for d_c and d_d and the associated MDPs, respectively. Notice that the problems mentioned above all have the same form: minimize a quadratic function with linear constraints (for fixed ω and θ). Based on this observation, we first mention a well-known result for the classical quadratic minimization problem (see, for example Ref. 6, for proof).

Lemma 3.1. Consider the following problem:

$$\text{minimize } J(x) = x^T \Gamma x, \tag{34a}$$

$$\text{s.t. } x \in \mathrm{R}^n,\ Ax + b = 0, \tag{34b}$$

where $\Gamma \in \mathrm{R}^{n \times n}$ is a positive-definite symmetric matrix, $A \in \mathrm{R}^{m \times n}$, with $m < n$ and $\operatorname{rank}(A) = m$. Then, the minimum J^* and the unique minimizer x^* are given by

$$J^* = b^T (A \Gamma^{-1} A^T)^{-1} b, \tag{35}$$

$$x^* = -\Gamma^{-1} A^T (A \Gamma^{-1} A^T)^{-1} b. \tag{36}$$

The application of Lemma 3.1 to the problems associated with (15), (24), (27), (28), and (33) is central to the main results of this paper (Theorems 3.1 and 3.2).

Theorem 3.1. Suppose that $p(s)$ is a strictly Hurwitz polynomial as in (5) and Γ is a weighting matrix as in (6). Then, the maximal weighted L_2-norm perturbation bound for $p(s)$ defined in (7) is given by

$$d_c^2 = \min\{d_{c_1}^2, d_{c_2}^2\}, \tag{37}$$

where

$$d_{c_1}^2 = \gamma_0 a_0^2, \tag{38}$$

$$d_{c_2}^2 = \min_{\omega > 0} \frac{\left[\sum_{i=0}^{[n/2]} a_{2i}(-\omega)^i\right]^2}{\sum_{i=0}^{[(n-1)/2]} \gamma_{2i}^{-1} \omega^{2i}} + \frac{\left[\sum_{i=0}^{[(n-1)/2]} a_{2i+1}(-\omega)^i\right]^2}{\sum_{i=0}^{[(n-2)/2]} \gamma_{2i+1}^{-1} \omega^{2i}}. \tag{39}$$

Furthermore, if $d_c = d_{c_1}$, then $\delta p^*(s)$, with

$$\delta a^* = [-a_0, 0, 0, \ldots, 0]^T, \tag{40}$$

is a minimal destabilizing polynomial. If $d_c = d_{c_2}$, then $\delta p^*(s)$, with

$$\delta a_{2i}^* = -\frac{\sum_{j=0}^{[n/2]} a_{2j}(-\omega^*)^j}{\sum_{j=0}^{[(n-1)/2]} \gamma_{2j}^{-1}(\omega^*)^{2j}} \gamma_{2i}^{-1}(-\omega^*)^i, \qquad i = 0, 1, \ldots, (n-1)/2, \tag{41}$$

$$\delta a_{2i+1}^* = -\frac{\sum_{j=0}^{[(n-1)/2]} a_{2j+1}(-\omega^*)^j}{\sum_{j=0}^{[(n-2)/2]} \gamma_{2j+1}^{-1}(\omega^*)^{2j}} \gamma_{2i+1}^{-1}(-\omega^*)^i, \quad i = 0, 1, \ldots, (n-2)/2, \tag{42}$$

is a minimal destabilizing polynomial for any ω^* minimizing (39).

Proof. Equations (38) and (40) are obtained simply by applying Lemma 3.1 to (15) with

$$x = \delta a, \qquad m = 1, \qquad A = [1, 0, 0, \ldots, 0], \qquad b = a_0.$$

The detailed manipulation is straightforward and therefore omitted. To show (39) and (41)-(42), we first rewrite (24) as

$$d_{c2}^2 = \inf_{\omega > 0} \min_{x \in \mathbb{R}^n} J_c(\omega)$$

$$= \inf_{\omega > 0} \min_{x \in \mathbb{R}^n} \{x^T \bar{\Gamma} x : \Omega_c(\omega)(\alpha + x) + \psi_c(\omega) = 0\}.$$

Now, for each fixed ω, we apply Lemma 3.1 to $J_c(\omega)$ with

$$m = 2, \qquad A = \Omega_c(\omega), \qquad b = \psi_c(\omega) + \Omega_c(\omega)\alpha.$$

Notice that

$$\operatorname{rank} \Omega_c(\omega) = 2, \qquad \text{for any } \omega.$$

It turns out that

$$J_c^* = (\psi_c(\omega) + \Omega_c(\omega)\alpha)^T (\Omega_c(\omega)\bar{\Gamma}^{-1}\Omega_c(\omega))^{-1} (\psi_c(\omega) + \Omega_c(\omega)\alpha), \tag{43}$$

$$x^* = -\bar{\Gamma}\Omega_c^T(\omega)(\Omega_c(\omega)\bar{\Gamma}^{-1}\Omega_c^T(\omega))^{-1}(\psi_c(\omega) + \Omega_c(\omega)\alpha). \tag{44}$$

In the remaining part of the proof, the reader may assume $n=$ even or $n=$ odd, for simplicity. From (19), (25), and (18), we have

$$\begin{aligned}
&\Omega_c(\omega)\bar{\Gamma}^{-1}\Omega_c^T(\omega)\\
&=\begin{bmatrix} 1 & -\omega & \cdots & (-\omega)^r & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & -\omega & \cdots & (-\omega)^s \end{bmatrix}\\
&\quad\times \operatorname{diag}\{\gamma_0^{-1},\ldots,\gamma_{2r}^{-1},\gamma_1^{-1},\ldots,\gamma_{2s+1}^{-1}\}\begin{bmatrix} 1 & 0 \\ -\omega & 0 \\ \cdots & \cdots \\ (-\omega)^r & 0 \\ 0 & 1 \\ 0 & -\omega \\ \cdots & \cdots \\ 0 & (-\omega)^s \end{bmatrix}\\
&=\begin{bmatrix} \sum_{i=0}^{[(n-1)/2]} \gamma_{2i}^{-1}\omega^{2i} & 0 \\ 0 & \sum_{i=0}^{[(n-2)/2]} \gamma_{2i+1}^{-1}\omega^{2i} \end{bmatrix}; \qquad (45)
\end{aligned}$$

and, from (19), (20), and (21),

$$\begin{aligned}
\psi_c(\omega)+\Omega_c(\omega)\alpha &= \psi_c(\omega)+\begin{bmatrix} \sum_{i=0}^{[(n-1)/2]} a_{2i}(-\omega)^i \\ \sum_{i=0}^{[(n-2)/2]} a_{2i+1}(-\omega)^i \end{bmatrix}\\
&= \begin{bmatrix} \sum_{i=0}^{[n/2]} a_{2i}(-\omega)^i \\ \sum_{i=0}^{[(n-1)/2]} a_{2i+1}(-\omega)^i \end{bmatrix}. \qquad (46)
\end{aligned}$$

Substituting (45) and (46) into (43), we conclude that

$$J_c^*(\omega)=\frac{\left(\sum_{i=0}^{[n/2]} a_{2i}(-\omega)^i\right)^2}{\sum_{i=0}^{[(n-1)/2]} \gamma_{2i}^{-1}\omega^{2i}}+\frac{\left(\sum_{i=0}^{[(n-1)/2]} a_{2i+1}(-\omega)^i\right)^2}{\sum_{i=0}^{[(n-2)/2]} \gamma_{2i+1}^{-1}\omega^{2i}}, \qquad (47)$$

$$d_{c_2}=\inf_{\omega>0} J_c^*(\omega). \qquad (48)$$

In order to obtain (39), we need to replace the infimum by minimum by showing that the minimum of $J_c^*(\omega)$ is attainable in $(0, \infty)$. This is established by noticing the fact that, for all sufficient small $\omega > 0$,

$$J_c^*(\omega) \approx \gamma_0(a_0 - 2a_2\omega) + \gamma_1(a_1 - 2a_3\omega) < J_c^*(0)$$

and that

$$\lim_{\omega\to\infty} J_c^*(\omega) \to \infty.$$

Hence, (39) follows. To show (41) and (42), we substitute (19), (25), (45), and (46) into (44) and we obtain

$$x^* = -\begin{bmatrix} y^* \\ z^* \end{bmatrix},$$

where

$$y^* = \frac{\sum_{j=0}^{[n/2]} a_{2j}(-\omega)^j}{\sum_{j=0}^{r} \gamma_{2j}^{-1}\omega^{2j}} \begin{bmatrix} \gamma_0^{-1} \\ -\gamma_2^{-1}\omega \\ \cdots \\ \gamma_{2r}^{-1}(-\omega)^r \end{bmatrix},$$

$$z^* = \frac{\sum_{j=0}^{[(n-1)/2]} a_{2j+1}(-\omega)^j}{\sum_{j=0}^{s} \gamma_{2j+1}^{-1}\omega^{2j}} \begin{bmatrix} \gamma_1^{-1} \\ -\gamma_3^{-1}\omega \\ \cdots \\ \gamma_{2s+1}^{-1}(-\omega)^s \end{bmatrix}.$$

Hence, (41) and (42) follow. □

Theorem 3.2. Suppose that $p(z)$ is a strictly Schur polynomial in the form of (5) and Γ is a weighting matrix as in (6). Then, the maximal weighted L_2-norm perturbation bound, for $p(z)$ defined in (10), is given by

$$d_d^2 = \min\{d_{d_1}^2, d_{d_2}^2, d_{d_3}^2\}, \tag{49}$$

where

$$d_{d_1}^2 = \left[\min_{0\le i\le n-1} \gamma_i\right]\left[1 + \sum_{i=0}^{n-1} a_i\right]^2, \tag{50}$$

$$d_{d_2}^2 = \left[\min_{0\le i\le n-1} \gamma_i\right]\left[(-1)^n + \sum_{i=0}^{n-1} (-1)^i a_i\right]^2, \tag{51}$$

$$d_{d_3}^2 = \inf_{0<\theta<\pi} 2 \times \frac{(f-g(\theta))R^2(\theta) + (f+g(\theta))I^2(\theta) - 2h(\theta)R(\theta)I(\theta)}{f^2 - g^2(\theta) - h^2(\theta)}, \tag{52}$$

with

$$f=\sum_{i=0}^{n-1} \gamma_i^{-1}, \qquad g(\theta)=\sum_{i=0}^{n-1} \gamma_i^{-1}\cos(2i\theta),$$

$$h(\theta)=\sum_{i=0}^{n-1} \gamma_i^{-1}\sin(2i\theta), \tag{53}$$

$$R(\theta)=\sum_{i=0}^{n} a_i\cos(i\theta), \qquad I(\theta)=\sum_{i=0}^{n} a_i\sin(i\theta). \tag{54}$$

Furthermore, if $d_d = d_{d_1}$, then $\delta p^*(z)$, with

$$\delta a_i^* = \begin{cases} 0, & i \neq k, \\ -\left(1+\sum_{j=0}^{n-1} a_j\right), & i=k, \end{cases} \tag{55}$$

is a minimal destabilizing polynomial for any k minimizing γ_k, $k = 0, 1, \dots, n-1$. Similarly, if $d_d = d_{d_2}$, then $\delta p^*(z)$, with

$$\delta a_i^* = \begin{cases} 0, & i \neq k, \\ -(-1)^i\left((-1)^n+\sum_{j=0}^{n-1} (-1)^j a_j\right), & i=k, \end{cases} \tag{56}$$

is a minimal destabilizing polynomial for any k minimizing γ_k, $k = 0, 1, \dots, n-1$. Finally, if $d_d = d_{d_3}$, then $\delta p^*(z)$, with

$$\delta a_i^* = \gamma_i^{-1}(k_R(\theta^*)\cos(i\theta^*)+k_I(\theta^*)\sin(i\theta^*)), \tag{57}$$

is a minimal destabilizing polynomial for any θ^* minimizing (52), where

$$k_R(\theta) = -2\frac{(f-g(\theta))R(\theta)-h(\theta)I(\theta)}{f^2-g^2(\theta)-h^2(\theta)}, \tag{58}$$

$$k_I(\theta) = -2\frac{(f+g(\theta))I(\theta)-h(\theta)R(\theta)}{f^2-g^2(\theta)-h^2(\theta)}. \tag{59}$$

Proof. Equations (50) and (55) are derived simply by applying Lemma 3.1 to (27) with

$$x=\delta a, \qquad m=1, \qquad A=[1, 1, \dots, 1],$$

$$b = a_0 + a_1 + \cdots + a_{n-1} + 1.$$

The detailed manipulation is trivial and therefore omitted. Similarly, (51) and (56) are derived by applying Lemma 3.1 to (28) with

$$x=\delta a, \qquad m=1, \qquad A=[1, -1, \dots, (-1)^{n-1}],$$

$$b = a_0 - a_1 + \cdots + (-1)^{n-1}a_{n-1} + (-1)^n.$$

The details are omitted too. To show (52) and (57), we first rewrite (29) as

$$\begin{aligned} d_{d_3}^2 &= \inf_{0<\theta<\pi} \min_{x\in R^n} J_d(\theta) \\ &= \inf_{0<\theta<\pi} \min_{x\in R^n} \{x^T\Gamma x \colon \Omega_d(\theta)(a+x)+\psi_d(\theta)=0\}. \end{aligned} \tag{60}$$

Now, for each fixed θ, we apply Lemma 3.1 to $J_d(\theta)$, with

$$A=\Omega_d(\theta) \quad \text{and} \quad b=\psi_d(\theta)+\Omega_d(\theta)a.$$

Notice that

$$\operatorname{rank} \Omega_d(\theta)=2, \qquad \text{for any } 0<\theta<\pi.$$

It turns out that

$$\begin{aligned} J_d^*(\theta) &= (\psi_d(\theta)+\Omega_d(\theta)a)^T(\Omega_d(\theta)\Gamma^{-1}\Omega_d^T(\theta))^{-1} \\ &\quad \times(\psi_d(\theta)+\Omega_d(\theta)a), \end{aligned} \tag{61}$$

$$x^*=-\Gamma^{-1}\Omega_d^T(\theta)(\Omega_d(\theta)\Gamma^{-1}\Omega_d^T(\theta))^{-1}(\psi_d(\theta)+\Omega_d(\theta)a). \tag{62}$$

On the other hand,

$$\begin{aligned} &\Omega_d(\theta)\Gamma^{-1}\Omega_d^T(\theta) \\ &= \begin{bmatrix} 1 & \cos\theta \ldots \cos((n-1)\theta) \\ 0 & \sin\theta \ldots \sin((n-1)\theta) \end{bmatrix} \\ &\times \operatorname{diag}\{\gamma_0^{-1}, \gamma_1^{-1}, \ldots, \gamma_{n-1}^{-1}\} \\ &\times \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \\ \ldots & \ldots \\ \cos((n-1)\theta) & \sin((n-1)\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^{n-1} \gamma_i^{-1}\cos^2(i\theta) & \sum_{i=0}^{n-1} \gamma_i^{-1}\sin(i\theta)\cos(i\theta) \\ \sum_{i=0}^{n-1} \gamma_i^{-1}\cos(i\theta)\sin(i\theta) & \sum_{i=0}^{n-1} \gamma_i^{-1}\sin^2(i\theta) \end{bmatrix} \\ &= (1/2)\begin{bmatrix} f+g(\theta) & h(\theta) \\ h(\theta) & f-g(\theta) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} &(\Omega_d(\theta)\Gamma^{-1}\Omega_d^T(\theta))^{-1} \\ &= \frac{2}{f^2-g^2(\theta)-h^2(\theta)} \begin{bmatrix} f-g(\theta) & -h(\theta) \\ -h(\theta) & f+g(\theta) \end{bmatrix}. \end{aligned} \tag{63}$$

Furthermore, it is easy to verify that

$$\psi_d(\theta)+\Omega_d(\theta)a=\begin{bmatrix} R(\theta) \\ I(\theta) \end{bmatrix}. \tag{64}$$

Finally, we obtain (52) and (57) by substituting (63), (64), and (30) into (61) and (62). Detailed manipulation is omitted. □

4. Illustrative Example

To illustrate the simplicity of our new method, we consider the maximal weighted L_2-norm perturbation bound for the strictly Hurwitz polynomial studied in Refs. 1 and 2,

$$p(s)=s^4+5s^2+8s^2+8s+3, \tag{65}$$

with two different weighting matrices.

Case 1. $\Gamma=\text{diag}\{1, 1, 1, 1\}$. Using Theorem 3.2, we find that

$$d_{c_1}^2=\gamma_0 a_0^2=9,$$

$$d_{c_s}^2=\inf_{\omega>0}[(3-8\omega+\omega^2)^2/(1+\omega^2)]+[(8-5\omega)^2/(1+\omega^2)]=12.36.$$

Therefore, the maximal weighted L_2-norm perturbation bound is given by

$$d_c^2=\min\{d_{c_1}^2, d_{c2}^2\}=9,$$

and

$$\delta a^*=[-3, 0, 0, 0]^T$$

is an MDP (in this example, unique).

Case 2. $\Gamma=\text{diag}\{1, 1/3, 1/3, 1/2\}$. Similarly, using (38) and (39), we find that

$$d_{c_1}^2=\gamma_0 a_0^2=9,$$

$$d_{c_2}^2=\inf_{\omega>0}[(3-8\omega+\omega^2)^2/(1+3\omega^2)]+[(8-5\omega)^2/(3+\omega^2)]=5.68,$$

with the unique minimizer $\omega^*=1.1775$. Hence, from (37), (41), and (42), we have

$$d_c^2=\min\{d_{c_1}^2, d_{c_2}^2\}=5.68,$$

and

$$\delta a^* = \begin{bmatrix} -(3-8\omega^*+\omega^{*2})/(1+3\omega^{*2}) \\ -3(8-5\omega^*)/(3+3\omega^{*2}) \\ -3(-\omega^*)(3-8\omega^*+\omega^{*2})/(1+3\omega^{*2}) \\ -2(-\omega^*)(8-5\omega^*)/(3+3\omega^{*2}) \end{bmatrix} = \begin{bmatrix} 0.9756 \\ -1.0980 \\ -3.4461 \\ 0.8618 \end{bmatrix}$$

is an MDP (again, in this example, unique).

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