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ROBUST SYNTHESIS OF TIME-DELAY SYSTEMS

Vladimir L. Kharitonov

Dept. Applied Mathematics and Control Theory
St. Petersburg State University, Russia

Minyue Fu

Dept. Electrical and Computer Engineering
University of Newcastle, NSW 2308, Australia

Abstract. This paper considers the problem of robust stabilization for a class of time-delay systems which involve affine parametric perturbations. We first provide a result for testing the robust stability of such a system, and subsequently apply it in the robust stabilization problem for a family of interval plants with time-delays.

1. Introduction

In this paper, we consider a family of time-delay plants whose transfer function coefficients are subject to affine perturbations, i.e., the coefficients belong to a given polytope. It is known that the exponential stability of the family of plants can be determined from that of the edges of the polytope [1]. We point out that the test usually does not involve all edges, and the elimination of unnecessary edges is simple. In fact, this can be done by checking whether a given edge is a so-called "convex dicrection" [2]. We devise a simple sufficient condition for determining the convex directions. This result is then applied to the robust synthesis problem for a family of interval plants with time-delays and a compensator of first order or with a special structure.

2. Robust Stability Analysis

Consider a single-input-single-output time-delay system with its transfer function described by

$$G(s) = \frac{\sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}s^{n-k}e^{\tau_i s}}{d_0(s)e^{\tau_0 s} + \sum_{k=1}^{n}\sum_{i=1}^{m} a_{ki}s^{n-k}e^{\tau_i s}} \tag{1}$$

where a_{ki} and b_{ki} are real constants, $d_0(s)$ is a monic nth order polynomial and $\tau_0 \geq \tau_1 > \cdots > \tau_m = 0$. The characteristic quasipolynomial of $G(s)$ is given by

$$f(s) = d_0(s)e^{\tau_0 s} + \sum_{k=1}^{n}\sum_{i=1}^{m} a_{ki}s^{n-k}e^{\tau_i s} \tag{2}$$

We consider the robust stability problem where the characteristic quasipolynomial is contained in a polytope defined as follows:

$$P = \{f(s) = \sum_{j=1}^{N}\mu_j f_j(s) | \mu_j \geq 0, \sum_{j=1}^{N}\mu_j = 1\} \tag{3}$$

where $f_j(s)$ are quasipolynomials in form of (2). A subset P_1 of P is called a *testing set* for P if the stability of all the quasipolynomials in P_1 implies that in P. Then, an important problem is how to reduce P to a minimal testing set.

The first useful reduction was done in [1] which provides a generalization of the Edge Theorem [3] to the quasipolynomial case. The result of [1] shows that a polytope of quasipolynomials P in (3) is robustly stable if and only if all the edges of P are robustly stable. Therefore, the key problem now is how to test the robust stability of an edge. One possibility is to use a graphical test proposed in [1]. Alternatively, one can first use a recent result in [2] to test whether the robust stability of the edge can be deduced from the stability of its vertices. Note that an edge quasipolynomial takes the following general form:

$$f(s) = \mu f_i(s) + (1-\mu)f_j(s) = f_j(s) + \mu g(s), \ \mu \in [0, 1] \tag{4}$$

where $g(s) = f_i(s) - f_j(s)$. Then, the result in [2] can be stated as follows:

Lemma 1. *Given any quasipolynomials $f_i(s)$ and $f_j(s)$ in form of (2), suppose the following inequality holds for all $\omega > 0$ where the derivative of $\arg g(j\omega)$ is well defined:*

$$\frac{d \arg g(j\omega)}{d\omega} \leq \frac{\tau_0}{2} + \left|\frac{\sin(2\arg g(j\omega) - \tau_0\omega)}{2\omega}\right|. \tag{5}$$

Then, the stability of $f_i(s)$ and $f_j(s)$ implies that of every convex combination of them.

The following theorem derived from Lemma 1 is important in the robust synthesis problem to be studied later (see [4] for proof).

Theorem 2. *Suppose in Lemma 1, $g(s) = g_0(s)e^{\tau s}$, where $g_0(s)$ is a convex direction for polynomials, i.e., $g_0(s)$ is a polynomial satisfying [5]*

$$\frac{d \arg g_0(j\omega)}{d\omega} \leq \left|\frac{\sin(2\arg g_0(j\omega))}{2\omega}\right|. \tag{6}$$

for all $\omega > 0$, where the derivative of $\arg g_0(j\omega)$ is well defined. Then, $g(s)$ satisfies (5) for all $\omega > 0$, where the derivative of $\arg g(j\omega)$ is well defined, if and only if $\tau \leq \tau_0/2$.

Remark 1. It is known that the condition (6) can be tested using Routh-like tables [6]. In particular, all first order polynomials are convex directions [5]. This point is important because the condition (5) cannot be tested as easily.

3. Robust Synthesis

Consider an "interval plant family" as follows:

$$\begin{aligned} \mathcal{G} &= \{G(s) : a_{ki}^- \le a_{ki} \le a_{ki}^+; b_{ki}^- \le b_{ki} \le b_{ki}^+, \\ &\quad 1 \le i \le m; 1 \le k \le n\} \end{aligned} \quad (7)$$

where $G(s)$ is given in (1) and the bounds a_{ki}^+, a_{ki}^-, b_{ki}^+, b_{ki}^- are prescribed. Our task is to find a linear time-invariant dynamic output feedback compensator

$$u(s) = G_c(s)y(s) = \frac{n_c(s)}{d_c(s)}y(s) \quad (8)$$

such that the closed-loop system is robustly stable. Here, $n_c(s)$ and $d_c(s)$ are coprime polynomials. To this end, we define four extreme polynomials:

$$\begin{aligned} n_i^{(1)}(s) &= b_{ni}^- + b_{(n-1)i}^- s + b_{(n-2)i}^+ s^2 + b_{(n-3)i}^+ s^3 + \cdots \\ n_i^{(2)}(s) &= b_{ni}^- + b_{(n-1)i}^+ s + b_{(n-2)i}^+ s^2 + b_{(n-3)i}^- s^3 + \cdots \\ n_i^{(3)}(s) &= b_{ni}^+ + b_{(n-1)i}^- s + b_{(n-2)i}^- s^2 + b_{(n-3)i}^+ s^3 + \cdots \\ n_i^{(4)}(s) &= b_{ni}^+ + b_{(n-1)i}^+ s + b_{(n-2)i}^- s^2 + b_{(n-3)i}^- s^3 + \cdots \end{aligned}$$

and $d_i^{(j)}(s), j = 1,2,3,4$, in a similar way.

Given the interval plant family in (7), let $1 \le \ell \le m$ be the least integer with $2\tau_\ell \le \tau_0$ and define

$$\hat{G}_1(s) = \frac{\sum_{i=1}^{\ell-1} n_i(s,\alpha_i,\beta_i)e^{\tau_i s} + \sum_{i=\ell}^{m} n_i^{(\nu_i)}(s)e^{\tau_i s}}{d_0(s)e^{\tau_0 s} + \sum_{i=1}^{m} d_i^{(\mu_i)}(s)e^{\tau_i s}} \quad (9)$$

$$\hat{G}_2(s) = \frac{\sum_{i=1}^{m} n_i^{(\nu_i)}(s)e^{\tau_i s}}{d_0(s)e^{\tau_0 s} + \sum_{i=1}^{\ell-1} d_i(s,\alpha_i,\beta_i)e^{\tau_i s} + \sum_{i=\ell}^{m} d_i^{(\mu_i)}(s)e^{\tau_i s}} \quad (10)$$

where $\nu_i, \mu_i = 1,2,3,4$, $0 \le \alpha_i, \beta_i \le 1$,

$$n_i(s,\alpha_i,\beta_i) = n_i^{(1)} + \alpha_i(n_i^{(2)} - n_i^{(1)}) + \beta_i(n_i^{(3)} - n_i^{(1)}) \quad (11)$$

and $d(s,\alpha_i,\beta_i)$ are similarly defined. We further denote by E the four edges of the box $\{(x,y) : 0 \le x, y \le 1\}$. Define two subfamilies of $\mathcal{G}$:

$$\mathcal{G}_{\mathrm{sub},k} = \{\hat{G}_k(s) : \nu_i, \mu_i = 1,2,3,4; (\alpha_i,\beta_i) \in E\} \quad (12)$$

for $k = 1,2$. By applying Theorem 2, we obtain the following robust synthesis result (see [4] for proof):

Theorem 3. *Given an interval plant faimily $\mathcal{G}$ in (7) and a compensator $G_c(s)$ in (8), suppose both $n_c(s)$ and $d_c(s)$ are convex directions for polynomials. Then, $G_c(s)$ robustly stabilizes the family $\mathcal{G}$ if and only if it robustly stabilizes the following subfamily:*

$$\mathcal{G}_{\mathrm{sub}} = \mathcal{G}_{\mathrm{sub},1} \cup \mathcal{G}_{\mathrm{sub},2}. \quad (13)$$

In the special case when $m = 1$, i.e.,

$$G(s) = \frac{n(s)}{d_0(s)e^{\tau s} + d(s)} = \frac{\sum_{k=1}^{n} b_k s^{n-k}}{d_0(s)e^{\tau s} + \sum_{k=1}^{n} a_k s^{n-k}}, \quad (14)$$

where $\tau \ge 0$, then Theorem 3 reduces to a generalization of an extreme point result in [7] (see [4] for proof):

Corollary 4. *Given an interval plant faimily*

$$\mathcal{G} = \{G(s) : a_k^- \le a_k \le a_k^+; b_k^- \le b_k \le b_k^+, 1 \le k \le n\} \quad (15)$$

with $G(s)$ in (14), and compensator $G_c(s)$, suppose both $n_c(s)$ and $d_c(s)$ are convex directions for polynomials. Then, $G_c(s)$ robustly stabilizes the family $\mathcal{G}$ if and only if it robustly stabilizes the following sixteen plants:

$$G_{ij}(s) = \frac{n^{(i)}(s)}{d_0(s)e^{\tau s} + d^{(j)}(s)}, \quad i,j = 1,2,3,4$$

where $n^{(i)}(s)$ and $d^{(j)}(s)$ are the extreme polynomials of $n(s)$ and $d(s)$. Furthermore, if

$$G_c(s) = K\frac{s+a}{s^k(s+b)}, \quad K,a,b > 0; k = 0,1,\cdots, \quad (16)$$

with $a < b$ (lead compensator) (resp. $a > b$ (lag compensator)), then $G_c(s)$ robustly stabilizes the family $\mathcal{G}$ if and only if it robustly stabilizes the following eight plants: $G_{11}, G_{13}, G_{21}, G_{22}, G_{33}, G_{34}, G_{42}, G_{44}$ (resp. $G_{11}, G_{12}, G_{22}, G_{24}, G_{31}, G_{33}, G_{43}, G_{44}$).

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