Robust $\mathcal{H}_\infty$ analysis and control of linear systems with integral quadratic constraints*

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Abstract

In this paper, we consider a class of uncertain linear systems which are subject to a general type of integral quadratic constraints (IQCs). Two problems are addressed: 1) robust $\mathcal{H}_\infty$ analysis and 2) robust $\mathcal{H}_\infty$ control. In the first problem, we determine if the system satisfies a desired $\mathcal{H}_\infty$ performance for all admissible uncertainties subject to the IQCs. In the second problem, we seek for a dynamic output feedback controller to achieve a desired robust $\mathcal{H}_\infty$ performance. We apply the well-known $\mathcal{S}$-procedure and show that these two problems can be effectively solved using linear matrix inequalities (LMIs).

1 Introduction

This paper addresses two problems: robust $\mathcal{H}_\infty$ analysis and robust $\mathcal{H}_\infty$ control of a class of linear systems which are subject to an energy bounded (or L_2 bounded) exogenous input and several uncertainties involving the so-called integral quadratic constraints (IQCs). In the robust $\mathcal{H}_\infty$ analysis problem we determine the worst-case energy (or the induced L_2 norm) at an output, while for the $\mathcal{H}_\infty$ control problem a feedback controller is sought for such that the worst-case energy at a controlled output is less than some desired level. These problems and variations of them have been studied in a number of papers recently; see [9, 14, 7, 10, 1] and references thereof.

There has been a lot of advancement since the $\mathcal{H}_\infty$ control problem was initially proposed by Zames [17]. The landmark paper [3] (known as DGKF paper) provides a simple algebraic Riccati equation (ARE) approach to the problem. Recently, the linear matrix inequality (LMI) approach has attracted a lot of attention; see [6] and [8], for example. The LMI approach is computationally advantageous because of the recent progress in linear programming, i.e., the powerful interior point algorithm proposed in [11]; also see [2]. Another advantage of the LMI approach is its simplicity for treating the singular cases. However, all the works mentioned here require that the system to be controlled or analyzed does not have any uncertainty in the model.

For systems with structural uncertainties, one method popularly used is the so-called μ analysis and synthesis; see [4]. This method is applicable to systems involving linear time-invariant dynamical uncertainties. Recently, several papers have been written about $\mathcal{H}_\infty$ analysis and control of systems with time-varying uncertainties, see [9, 15, 14, 7, 1], for example. The type of uncertainties treated by these papers are all norm-bounded, as illustrated in (4)-(7) in section 2. On the other hand, a more general type of uncertainties described by IQCs have also been used in $\mathcal{H}_\infty$ analysis and control; see [10, 13] for example.

The aim of this paper is to show that the robust $\mathcal{H}_\infty$ analysis and control problems can be solved by using the so-called $\mathcal{S}$-procedure [16, 10] and the linear matrix inequality (LMI) approach.

The type of IQCs used in this paper are very general, allowing uncertainties in the state, exogenous input, control input, controlled output and measured output.

The rest of the paper is outlined as follows: Section 2 studies the $\mathcal{H}_\infty$ analysis problem; section 3, the control problem; and the concluding remarks are given in section 4.

2 Robust $\mathcal{H}_\infty$ analysis

Consider the following linear uncertain system:

$$\dot{x}(t) = Ax(t) + Bw(t) + \sum_{i=1}^{p} H_{1i}\xi_i(t) \tag{1}$$

$$z(t) = Cx(t) + Dw(t) + \sum_{i=1}^{p} H_{2i}\xi_i(t) \tag{2}$$

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where $\dot{x}(t) = Ax(t)$ is asymptotically stable, $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^q$ the exogenous inputs, $z(t) \in \mathbf{R}^r$ the output, and $\xi_i(t) \in \mathbf{R}^{k_i}$ the uncertain variables satisfying the following IQCs:

$$\int_0^T ||\xi_i(t)||^2 dt \le \int_0^T ||E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)||^2 dt, \quad \text{as } T \to \infty,\ i = 1, \cdots, p \tag{3}$$

with

$$\xi(t) = [\xi_1^T(t) \cdots \xi_p^T(t)]^T.$$

Also, $A, B, C, D, H_{1i}, H_{2i}, E_{1i}, E_{2i}$ and E_{3i} are constant matrices of appropriate dimensions.

Remark 1. The IQCs have been used for a long time in Russia; see [16]. They have also been used in recent literature to deal with robust control, see [10, 13] for example.

To understand the generality of the IQCs in (3), let us look at a special class of uncertain systems which have been treated in a number of papers (see, e.g., [9, 15, 14, 7, 1]):

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)w(t) \tag{4}$$
$$z(t) = (C + \Delta C)x(t) + (D + \Delta D)w(t) \tag{5}$$

where

$$\begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)[E_1 \quad E_2] \tag{6}$$

with

$$F^T(t)F(t) \le I,\ \forall t \ge 0 \tag{7}$$

Obviously, this example above corresponds to the case $p = 1$, and (7) is more restrictive than (3).

Remark 2. Note that the following quadratic constraints

$$\|\xi_i(t)\|^2 \le \|E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)\|^2, \quad i = 1, 2, \ldots, p \tag{8}$$

precisely describe the norm-bounded uncertainty (6) -(7). Both (3) and (8) can effectively represent dynamic uncertain structure. However, the significant difference between (3) and (8) is that (8) are local constraints while (3) are weaker "global" constraints. It is obvious that (3) are less conservative than (8) in describing system uncertainties.

We make the following assumption:

(A0) (Zero state detectability) Let $w(t) \equiv 0$. Then $\int_0^T \|z(t)\|^2 dt$ is bounded as $T \to \infty$ implies $x(T) \to 0$ as $T \to \infty$.

The problem of robust $\mathcal{H}_\infty$ analysis is as follows: *Given $\gamma > 0$ and the system (1)-(3) satisfying Assumption (A0), determine if the system is asymptotically stable and that the following condition is satisfied:*

$$\int_0^T ||z(t)||^2 dt < \gamma^2 \int_0^T ||w(t)||^2 dt, \quad \int_0^T ||w(t)||^2 dt > 0 \quad \text{as } T \to \infty, x(0) = 0 \tag{9}$$

for all admissible uncertainties.

Before proceeding further, we need some short-hand notation:

$$H_1 = [H_{11} \cdots H_{1p}]; \quad H_2 = [H_{21} \cdots H_{2p}] \tag{10}$$
$$E_i^T = [E_{i1}^T \cdots E_{ip}^T], \quad i = 1, 2, 3 \tag{11}$$
$$\tau = (\tau_1, \cdots, \tau_p) \tag{12}$$
$$J = \text{diag}\{\tau_1 I_{k_1}, \cdots, \tau_p I_{k_p}\} \tag{13}$$

where $\tau_1, \cdots, \tau_p$ are scalars and k_i are the numbers of columns of H_i. The vector $\tau > 0$ if every component of τ is positive.

Applying the well-known $\mathcal{S}$-procedure[16, 10], we have the following result:

Lemma 1. *Given the system (1)-(3), condition (9) holds if there exist a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and scaling parameters $\tau_1, \cdots, \tau_p > 0$ such that the following condition holds:*

$$2x^T P(Ax + Bw + \sum_{i=1}^p H_{1i}\xi_i) + \sum_{i=1}^p \tau_i(||E_{1i}x + E_{2i}w + E_{3i}\xi||^2 - ||\xi_i||^2) + ||Cx + Dw + \sum_{i=1}^p H_{2i}\xi_i||^2 - \gamma^2||w||^2 < 0, \quad \forall x \in \mathbf{R}^n, w \in \mathbf{R}^q, \xi_i \in \mathbf{R}^{k_i}, i = 1, \cdots, p \tag{14}$$

Proof. (Stability) Set $w(t) = 0$. Integrating the left hand side of the inequality in (14) along any trajectory of the system (1)-(2), we have:

$$x^T(T)Px(T) - x^T(0)Px(0) + \sum_{i=1}^p \tau_i \left\{ \int_0^T ||E_{1i}x + E_{3i}\xi||^2 dt - \int_0^T ||\xi||^2 dt \right\} + \int_0^T ||z(t)||^2 dt < 0$$

It is clear from (3) that

$$x^T(T)Px(T) - x^T(0)Px(0) + \int_0^T ||z(t)||^2 dt < 0$$

If $x(T) \neq 0$ as $T \to \infty$, we will have

$$\int_0^T ||z(t)||^2 \to \infty, \quad \text{as} \quad T \to \infty$$

by Assumption (A0), which is clearly impossible.

($\mathcal{H}_\infty$ performance) Given any $w(t) \in L_2[0, \infty)$. Integrating the left hand side of the inequality in (14) along any trajectory of the system (1)-(2), as $T \to \infty$ and letting $x(0) = 0$, we obtain (9). □

The following theorem establishes several equivalent conditions to (14):

Theorem 1. *Given the uncertain system (1)-(3), the following conditions, all guaranteeing the solution to the associated robust $\mathcal{H}_\infty$ analysis problem, are equivalent:*

(i) *There exist $P = P^T > 0$ and $\tau > 0$ such that (14) holds;*

(ii) *There exist $P = P^T > 0$ and $\tau > 0$ solving the following LMI:*

$$\mathcal{L}_1 = \begin{bmatrix} A^TP + PA + E_1^TJE_1 + C^TC & PB + E_1^TJE_2 + C^TD & PH_1 + C^TH_2 + E_1^TJE_3 \\ B^TP + E_2^TJE_1 + D^TC & -\gamma^2 I + D^TD + E_2^TJE_2 & D^TH_2 + E_2^TJE_3 \\ H_1^TP + H_2^TC + E_3^TJE_1 & H_2^TD + E_3^TJE_2 & -J + H_2^TH_2 + E_3^TJE_3 \end{bmatrix} < 0 \tag{15}$$

(iii) *There exist $P = P^T > 0$ and $\tau > 0$ solving the following LMI:*

$$\mathcal{L}_2 = \begin{bmatrix} A^TP + PA & PB & PH_1 & C^T & E_1^TJ \\ B^TP & -\gamma^2 I & 0 & D^T & E_2^TJ \\ H_1^TP & 0 & -J & H_2^T & E_3^TJ \\ C & D & H_2 & -I & 0 \\ JE_1 & JE_2 & JE_3 & 0 & -J \end{bmatrix} < 0 \tag{16}$$

(iv) *There exists $\tau > 0$ such that the following auxiliary system is asymptotically stable and the $\mathcal{H}_\infty$-norm of the transfer function from $\hat{w}(\cdot)$ to $\hat{z}(\cdot)$ is less than 1:*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + [\gamma^{-1}B \;\; H_1J^{-1/2}]\hat{w}(t) \tag{17}$$

$$\hat{z}(t) = \begin{bmatrix} C \\ J^{1/2}E_1 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} \gamma^{-1}D & H_2J^{-1/2} \\ \gamma^{-1}J^{1/2}E_2 & J^{1/2}E_3J^{-1/2} \end{bmatrix}\hat{w}(t) \tag{18}$$

Moreover, the set of all τ satisfying (iii) is convex.

Proof. The proof consists of straightforward algebraic manipulations:

"(i) <=> (ii)": The inequality (14) can be rewritten as follows:

$$2x^TP(Ax + Bw + H_1\xi) + (x^TE_1^T + w^TE_2^T + \xi^TE_3^T)J\begin{bmatrix} E_1x \\ E_2w \\ E_3\xi \end{bmatrix}$$

$$-\xi^TJ\xi + (x^TC^T + w^TD^T + \xi^TH_2^T)(Cx + Dw + H_2\xi)$$
$$-\gamma^2w^Tw < 0,$$
$$\forall x \in \mathbf{R}^n, w \in \mathbf{R}^q, \xi_i \in \mathbf{R}^{k_i}, i = 1, \cdots, p \tag{19}$$

Equivalently,

$$[x^T \;\; w^T \;\; \xi^T]\mathcal{L}_1\begin{bmatrix} x \\ w \\ \xi \end{bmatrix} < 0,$$
$$\forall x \in \mathbf{R}^n, w \in \mathbf{R}^q, \xi_i \in \mathbf{R}^{k_i}, i = 1, \cdots, p \tag{20}$$

i.e., (15) holds.

"(ii)<=>(iv)": Denote

$$\hat{B} = [\gamma^{-1}B \;\; H_1J^{-1/2}] \tag{21}$$

$$\hat{C}^T = [C^T \;\; E_1^TJ^{1/2}] \tag{22}$$

$$\hat{D} = \begin{bmatrix} \gamma^{-1}D & H_2J^{-1/2} \\ \gamma^{-1}J^{1/2}E_2 & J^{1/2}E_3J^{-1/2} \end{bmatrix} \tag{23}$$

and

$$\hat{\mathcal{L}}_1 = \begin{bmatrix} A^TP + PA + \hat{C}^T\hat{C} & P\hat{B} + \hat{C}^T\hat{D} \\ \hat{B}^TP + \hat{D}^T\hat{C} & -I + \hat{D}^T\hat{D} \end{bmatrix} \tag{24}$$

Note that the system in (18) can be rewritten as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \hat{B}\hat{w}(t) \tag{25}$$
$$\hat{z}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{w}(t) \tag{26}$$

Also, the matrix $\mathcal{L}_1$ in (15) can be alternatively expressed as follows:

$$\mathcal{L}_1 = \mathrm{diag}\{I_n, \gamma^{-1}I_q, J^{-1/2}\}\hat{\mathcal{L}}_1\mathrm{diag}\{I_n, \gamma^{-1}I_q, J^{-1/2}\} \tag{27}$$

That is, $\mathcal{L}_1 < 0$ if and only if $\hat{\mathcal{L}}_1 < 0$. It is well-known (see, for example, [9]) that matrix A is asymptotically stable and $||\hat{D} + \hat{C}(sI - A)^{-1}\hat{B}||_\infty < 1$ if and only if $\hat{\mathcal{L}}_1 < 0$ for some $P = P^T > 0$. Hence, (ii) is equivalent to (iv).

"(ii)<=>(iii)": Since $\mathcal{L}_1 < 0$ if and only if $\hat{\mathcal{L}}_1 < 0$, we need to show that $\mathcal{L}_2 < 0$ if and only if $\hat{\mathcal{L}}_1 < 0$. We first note that $\hat{\mathcal{L}}_1 < 0$ if and only if the following holds:

$$\hat{\mathcal{L}}_2 = \begin{bmatrix} A^TP + PA & P\hat{B} & \hat{C}^T \\ \hat{B}^TP & -I & \hat{D}^T \\ \hat{C} & \hat{D} & -I \end{bmatrix} < 0 \tag{28}$$

which is derived from the well-known fact that

$$\begin{bmatrix} X_1 & X_2^T \\ X_2 & -I \end{bmatrix} < 0 \quad <=> \quad X_1 + X_2^TX_2 < 0 \tag{29}$$

The equivalence between $\hat{\mathcal{L}}_2 < 0$ and $\mathcal{L}_2 < 0$ can be established by similar manipulations used on $\hat{\mathcal{L}}_1$ and $\mathcal{L}_1$. The details are omitted. □

Remark 3. Some discussions about the equivalent conditions in theorem 1 are in order. First, we note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are jointly linear in P, τ and γ^2. This makes it possible to use the recently developed convex optimization algorithms (see, [11, 2], for example) to search for solutions and even to search for the least γ bound. Secondly, Condition (ii) is obviously more economical to compute than (iii) due to the dimensional difference in $\mathcal{L}_1$ and $\mathcal{L}_2$. However, $\mathcal{L}_2$ is also linear in matrices $B, C, D, H_1, H_2, E_1, E_2$ and E_3, which makes it very attractive in control design when these matrices are linear in the design parameters. The auxiliary system (17)-(18) is useful in understanding the nature and conservatism of the $\mathcal{S}$-procedure. It is particularly interesting to see how the IQCs are easily converted into extra terms in the input and output, and to see the connection between the robust $\mathcal{H}_\infty$ analysis problem and an ordinary but scaled $\mathcal{H}_\infty$ analysis problem. The convexity of the $\mathcal{H}_\infty$-norm of the auxiliary system is somehow nontrivial.

3 Robust $\mathcal{H}_\infty$ Control

Consider the following uncertain system generalized from (1)-(2):

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{i=1}^{p} H_{1i}\xi_i(t) \qquad (30)$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) + \sum_{i=1}^{p} H_{2i}\xi_i(t) \quad (31)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) + \sum_{i=1}^{p} H_{3i}\xi_i(t) \quad (32)$$

where $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^q$ the exogenous inputs, $u(t) \in \mathbf{R}^m$ the control input, $z(t) \in \mathbf{R}^r$ the controlled output, $y(t) \in \mathbf{R}^{r_y}$ is the measured output, and $\xi_i(t) \in \mathbf{R}^{k_i}$ the uncertain variables satisfy the following IQCs:

$$\int_0^T ||\xi_i(t)||^2 dt \leq \int_0^T ||E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t) + E_{4i}u(t)||^2 dt, \quad \text{as } T \to \infty,\ i = 1, \cdots, p \qquad (33)$$

Also, $A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}$, H_{1i}, H_{2i}, H_{3i}, E_{1i}, E_{2i} and E_{3i} are constant matrices with appropriate dimensions.

To this end, we assume the following:

(A1) (A, B_2, C_2) is stabilizable and detectable.

(A2) $D_{22} = 0$.

Remark 4. The necessity of assumption (A1) is obvious, while (A2) is made for technical convenience.

Let a desired controller be of the following form:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \qquad (34)$$

$$u(t) = C_c x_c(t) + D_c y(t) \qquad (35)$$

where $x_c(t) \in \mathbf{R}^{n_c}$ is the state, and A_c, B_c and C_c are constant matrices of appropriate dimensions.

Then, the $\mathcal{H}_\infty$ control problem associated with the uncertain system (30)-(33) satisfying Assumptions (A0)-(A2) is as follows: *Given $\gamma > 0$, find a controller of the form (34)-(35) such that the closed-loop system is asymptotically stable and satisfies the following condition:*

$$\int_0^T ||z(t)||^2 dt < \gamma^2 \int_0^T ||w(t)||^2 dt, \quad \text{as } T \to \infty, x(0) = 0 \qquad (36)$$

for all admissible uncertainties.

Besides the short-hand notation in (10)-(13), we define:

$$H_3 = [H_{31} \cdots H_{3p}]; \qquad (37)$$

$$\bar{A} = \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}; \quad \bar{B} = \begin{bmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{bmatrix} \qquad (38)$$

$$\bar{C} = [C_1 + D_{12} D_c C_2 \quad D_{12} C_c]; \quad \bar{D} = D_{11} + D_{12} D_c D_{21} \qquad (39)$$

$$\bar{H}_{1i} = \begin{bmatrix} H_{1i} + B_2 D_c H_{3i} \\ B_c H_{3i} \end{bmatrix}; \quad \bar{H}_{2i} = H_{2i} + D_{12} D_c H_{3i} \quad (40)$$

$$\bar{E}_{1i} = [E_{1i} + E_{4i} D_c C_2 \ \ E_{4i} C_c]; \ \bar{E}_{2i} = E_{2i} + E_{4i} D_c D_{21}; \\ \bar{E}_{3i} = E_{3i} + E_{4i} D_c H_3 \qquad (41)$$

It is straightforward to verify that the closed-loop system of (30)-(35) is given by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}w(t) + \sum_{i=1}^{p} \bar{H}_{1i}\xi_i(t) \qquad (42)$$

$$z(t) = \bar{C}\bar{x}(t) + \bar{D}w(t) + \sum_{i=1}^{p} \bar{H}_{2i}\xi_i(t) \qquad (43)$$

with

$$\int_0^T ||\xi_i(t)||^2 dt \leq \int_0^T ||\bar{E}_{1i}\bar{x}(t) + \bar{E}_{2i}w(t) + \bar{E}_{3i}\xi(t)||^2 dt, \\ \text{as } T \to \infty,\ i = 1, \cdots, p \qquad (44)$$

We further define:

$$\bar{E}_i^T = [\bar{E}_{i1}^T \ldots \bar{E}_{ip}^T], \quad i = 1, 2, 3 \qquad (45)$$

Applying theorem 1, we know that the robust $\mathcal{H}_\infty$ control problem is solvable using the controller in (34)-(35) if the following system is asymptotically stable and its $\mathcal{H}_\infty$-norm is less than 1:

$$\dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + [\gamma^{-1}\bar{B} \quad \bar{H}_1 J^{-1/2}]\hat{w}(t) \qquad (46)$$

$$\hat{z}(t) = \begin{bmatrix} \bar{C} \\ J^{1/2}\bar{E}_1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} \gamma^{-1}\bar{D} & \bar{H}_2 J^{-1/2} \\ \gamma^{-1} J^{1/2}\bar{E}_2 & J^{1/2}\bar{E}_3 J^{-1/2} \end{bmatrix} \hat{w}(t) \quad (47)$$

It can be verified straightforwardly that (46)-(47) above is the closed-loop system of the controller (34)-(35) together with the auxiliary system defined below:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + [\gamma^{-1}B_1 \quad H_1 J^{-1/2}]\hat{w}(t) + B_2 u(t) \qquad (48)$$

$$\hat{z}(t) = \begin{bmatrix} C_1 \\ J^{1/2}E_1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} \gamma^{-1}D_{11} & H_2 J^{-1/2} \\ \gamma^{-1} J^{1/2}E_2 & J^{1/2}E_3 J^{-1/2} \end{bmatrix} \hat{w}(t) + \begin{bmatrix} D_{12} \\ J^{1/2}E_4 \end{bmatrix} u(t) \qquad (49)$$

$$y(t) = C_2\hat{x}(t) + [\gamma^{-1}D_{21} \quad H_3 J^{-1/2}]\hat{w}(t) \qquad (50)$$

for some $\tau > 0$, where τ, J, E_1, E_2, H_1 and H_2 are defined in (10)-(13), E_3 and H_3 are defined similarly.

Consequently, we have the following result:

Theorem 2. *Given the uncertain system (30)-(33), there exists a controller of the form (34)-(35) such that the associated $\mathcal{H}_\infty$ condition (36) is satisfied for a given $\gamma > 0$ if there exists some $\tau > 0$ such that the closed-loop auxiliary system of (48)-(50) and the same controller has $\mathcal{H}_\infty$-norm less than 1.*

Proof. It is a simple consequence of theorem 1, as described above. □

We now address the harder problem: how to solve the $\mathcal{H}_\infty$ control problem associated with (48)-(50) using LMIs.

Our mission now is to find $\tau > 0$ and a controller of the form (34)-(35) such that the closed-loop system of (48)-(50) is asymptotically stable and has $\mathcal{H}_\infty$-norm less than 1. We will show that this problem can be solved using LMIs. To this end, we need the result below:

Lemma 2. *[6] Consider the following system:*

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{51}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{52}$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \tag{53}$$

satisfying assumptions (A1)-(A2). Let N_R (resp. N_S) be any matrix whose columns form a basis of the null space of $[B_2^T \quad D_{12}^T]$ (resp. $[C_2 \quad D_{21}]$). Then, there exists a controller of the form (34)-(35) such that the closed-loop system has $\mathcal{H}_\infty$ norm less than 1 if and only if there exist symmetric matrices R and S satisfying the following LMIs:

$$\left[\begin{array}{c|c} N_R^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -I & D_{11} \\ \hline B_1^T & D_{11}^T & -I \end{array}\right] \left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{54}$$

$$\left[\begin{array}{c|c} N_S^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} A^T S + AS & SB_1 & C_1^T \\ B_1^T S & -I & D_{11}^T \\ \hline C_1 & D_{11} & -I \end{array}\right] \left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{55}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{56}$$

Using lemma 2 and theorem 2, we obtain the following result:

Theorem 3. *Given $\gamma > 0$ and $\tau > 0$ and the auxiliary system (48)-(50) satisfying assumptions (A1)-(A2), the following two conditions are equivalent:*
(a) *There exists a controller of the form (34)-(35) such that the closed-loop system of (48)-(50) is asymptotically stable and has $\mathcal{H}_\infty$-norm less than 1;*
(b)*: Let $\mathcal{N}_R$ (resp. $\mathcal{N}_S$) be any matrix whose columns form a basis of the null space of $[B_2^T \quad D_{12}^T \quad E_4^T]$ (resp. $[C_2 \quad D_{21} \quad H_3]$). There exist symmetric matrices $R, S \in \mathbf{R}^{n\times n}$ such that the following LMIs hold:*

$$\left[\begin{array}{c|c} \mathcal{N}_R^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|cc} AR + RA^T & RC_1^T & RE_1^T & B_1 & H_1 J^{-1} \\ C_1 R & -I & 0 & D_{11} & H_2 J^{-1} \\ E_1 R & 0 & -J^{-1} & E_2 & E_3 J^{-1} \\ \hline B_1^T & D_{11}^T & E_2^T & -\gamma^2 I & 0 \\ J^{-1} H_1^T & J^{-1} H_2^T & J^{-1} E_3^T & 0 & -J^{-1} \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{57}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|cc} A^T S + SA & SB_1 & SH_1 & C_1^T & E_1^T J \\ B_1^T S & -\gamma^2 I & 0 & D_{11}^T & E_2^T J \\ H_1^T S & 0 & -J & H_2^T & E_3^T J \\ \hline C_1 & D_{11} & H_2 & -I & 0 \\ JE_1 & JE_2 & JE_3 & 0 & -J \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{58}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{59}$$

Proof. The proof is a direct application of lemma 2 to the auxiliary system (48)-(50). We first note that the columns of the matrix

$$\mathrm{diag}\{I, I, J^{-1/2}\}\mathcal{N}_R \ (\text{resp. } \mathrm{diag}\{I, \gamma I, J^{1/2}\}\mathcal{N}_S)$$

form a basis of the null space of $[B_2^T \quad [D_{12}^T \quad E_3^T J^{1/2}]]$ (resp. $[C_2 \quad [\gamma^{-1} D_{21} \quad H_3 J^{-1/2}]]$). Then, it is tedious but straightforward to verify that the LMI in (57) is the version of (54) for the auxiliary system, but pre- and post-multiplied by the following matrix:

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \mathrm{diag}\{\gamma I, J^{-1/2}\} \end{array}\right]$$

Similarly, the LMI in (58) is the version of (55) for the auxiliary system, pre-post multiplied by the following matrix:

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \mathrm{diag}\{I, J^{1/2}\} \end{array}\right]$$

□

Remark 5. Note that the LMI in (57) is linear in R, J^{-1} and γ^2; while the LMI in (58) is linear in S, J and γ^2. Also, (57) and (58) are dual to each other. However, they are not jointly linear in J or J^{-1}. That is, for each fixed J, the LMIs are jointly linear in R, S and γ^2. However, in the case of state feedback control, LMI (58) drops out and the result is fully convex. This last point is made precise in the result below. The proof of the following corollary is straightforward using the same technique in [6] and is also implied in [12] for systems without uncertainty, hence is omitted for brevity.

Corollary 1. *Given $\gamma > 0$ and $\tau > 0$ and the uncertain system (30)-(32) with uncertainty satisfying IQCs (33). The following two conditions are equivalent:*
(a) *There exists a controller of the following form*

$$u(t) = K_c x(t) \tag{60}$$

such that the closed-loop system for the auxiliary system (48)-(50) is asymptotically stable and has $\mathcal{H}_\infty$-norm less than 1;
(b)*: Let $\mathcal{N}_R$ be any matrix whose columns form a basis of the null space of $[B_2^T \quad D_{12}^T \quad E_4^T]$. There exist symmetric matrices $R \in \mathbf{R}^{n\times n}$, $R > 0$ and $J > 0$ such that LMI (57) holds:*

4 Conclusion

In this paper, we have studied the problems of robust $\mathcal{H}_\infty$ analysis and robust $\mathcal{H}_\infty$ control for a class of linear system subject to IQCs, as represented by (1)-(3) and (30)-(33), respectively. We have shown that the analysis problem can be solved by using a LMI (either (15) or (16)) which is linear in matrix P, scaling parameters τ (equivalently, J), and $\mathcal{H}_\infty$ performance bound γ, thus a complete LMI solution.

For the robust $\mathcal{H}_\infty$ control problem, we have obtained a set of LMIs (57)-(59); as shown in theorem 3. In the dynamic output feedback control case, one of the LMIs is convex in J and another in J^{-1}. Thus they are not jointly linear in J or J^{-1}. However, in the state feedback control case, one of the LMIs is void and we have a true LMI solution. Further research is needed to see if it is possible to re-parameterize J so that the result for the dynamic output feedback control case is fully convex. When the LMIs (57)-(59) are solved, a robust $\mathcal{H}_\infty$ controller can be constructed by using the procedure given in section 4. This procedure is modified from [5].

Also obtained in the paper are two auxiliary systems, (17)-(18) and (48)-(50), one for the analysis problem and the other for control. These auxiliary systems are transformed from the original uncertain systems and convenient to use, as demonstrated in solving the control problem.

We shall point out that the conditions obtained for the analysis and control problems are all sufficient in general. This is due to the use of $\mathcal{S}$-procedure. Unfortunately, there is no better method available for dealing with IQCs. Further study is needed to analyze this issue.

We also point out that the results in this paper are readily generalizable to discrete-time systems. This will be reported in a separate paper.

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